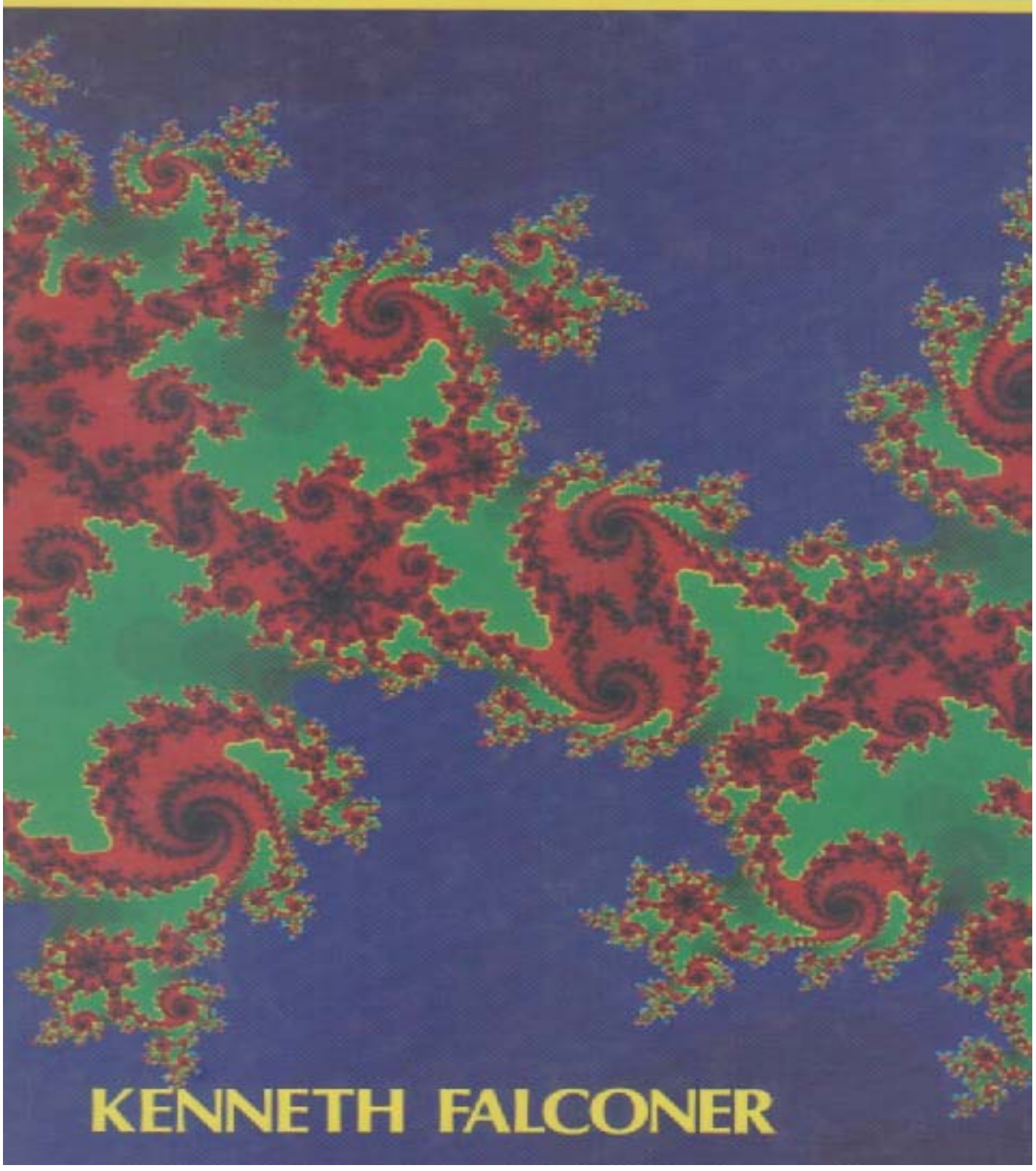
FRACTAL GEOMETRY

Mathematical Foundations and Applications



KENNETH FALCONER

Contents

Preface ix

Introduction xiii

Notes and references xxii

PART I FOUNDATIONS

Chapter 1. Mathematical background 3

1.1 Basic set theory 3
1.2 Functions and limits 6
1.3 Measures and mass distributions 10
1.4 Notes on probability theory 16
1.5 Notes and references 23
Exercises 23

Chapter 2. Hausdorff measure and dimension 25

2.1 Hausdorff measure 25
2.2 Hausdorff dimension 28
2.3 Calculation of Hausdorff dimension—simple examples 31
*2.4 Equivalent definitions of Hausdorff dimension 32
*2.5 Finer definitions of dimension 33
2.6 Notes and references 34
Exercises 34

Chapter 3. Alternative definitions of dimension 36

3.1 Box-counting dimensions 38
3.2 Properties and problems of box-counting dimension 44
*3.3 Modified box-counting dimensions 45
*3.4 Packing measures and dimensions 47
3.5 Some other definitions of dimension 49
3.6 Notes and references 52
Exercises 53

Chapter 4. Techniques for calculating dimensions 54

4.1 Basic methods 54
4.2 Subsets of finite measure 62
4.3 Potential theoretic methods 64
*4.4 Fourier transform methods 66
4.5 Notes and references 67
Exercises 68

Chapter 5. Local structure of fractals 69

5.1 Densities 69
5.2 Structure of 1-sets 73
5.3 Tangents to s-sets 77
5.4 Notes and references 81
Exercises 81

Chapter 6. Projections of fractals 83

6.1 Projections of arbitrary sets 83
6.2 Projections of s-sets of integral dimension 86
6.3 Projections of arbitrary sets of integral dimension 88
6.4 Notes and references 90
Exercises 90

Chapter 7. Products of fractals 92

7.1 Product formulae 92
7.2 Notes and references 100
Exercises 100

Chapter 8. Intersections of fractals 101

8.1 Intersection formulae for fractals 102
8.2 Sets with large intersection 104
8.3 Notes and references 109
Exercises 110

PART II APPLICATIONS AND EXAMPLES

Chapter 9. Fractals defined by transformations—self-similar and self-affine sets 113

9.1 Iterated function schemes 113
9.2 Dimensions of self-similar sets 117
9.3 Some variations 123
9.4 Self-affine sets 126
9.5 Applications to encoding images 132
9.6 Notes and references 136
Exercises 136

Chapter 10. Examples from number theory 138

10.1 Distribution of digits of numbers 138
10.2 Continued fractions 139
10.3 Diophantine approximation 141
10.4 Notes and references 144
Exercises 144

Chapter 11. Graphs of functions 146

11.1 Dimensions of graphs 146
*11.2 Autocorrelation of fractal functions 155
11.3 Notes and references 158
Exercises 159

Chapter 12. Examples from pure mathematics 161

12.1 Duality and the Kakeya problem 161
12.2 Vitushkin's conjecture 164
12.3 Convex surfaces 165
12.4 Groups and rings of fractional dimension 167
12.5 Notes and references 168
Exercises 169

Chapter 13. Dynamical systems 170

13.1 Repellers and iterated function schemes 171
13.2 The logistic map 173
13.3 Stretching and folding transformations 177
13.4 The solenoid 182
13.5 Continuous dynamical systems 184
*13.6 Small divisor theory 188
*13.7 Liapounov exponents and entropies 191
13.8 Notes and references 194
Exercises 195

Chapter 14. Iteration of complex functions—Julia sets 197

14.1 General theory of Julia sets 197
14.2 Quadratic functions—the Mandelbrot set 204
14.3 Julia sets of quadratic functions 208
14.4 Characterisation of quasi-circles by dimension 217
14.5 Newton's method for solving polynomial equations 219
14.6 Notes and references 222
Exercises 223

Chapter 15. Random fractals 224

15.1 A random Cantor set 224
15.2 Fractal percolation 231
15.3 Notes and references 235
Exercises 235

Chapter 16. Brownian motion and Brownian surfaces 237

16.1 Brownian motion 237
16.2 Fractional Brownian motion 245
16.3 Stable processes 248
16.4 Brownian surfaces 250
16.5 Notes and references 252
Exercises 253

Chapter 17. Multifractal measures 254

17.1 A multifractal formalism 255
17.2 Notes and references 264
Exercises 264

Chapter 18. Physical applications 265

18.1 Fractal growth 267
18.2 Singularities of electrostatic and gravitational potentials 272
18.3 Fluid dynamics and turbulence 273
18.4 Notes and references 276
Exercises 277

References 278

Index 284

FRACTAL GEOMETRY

Mathematical Foundations and Applications

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This book provides an accessible treatment of the mathematics of fractals and their dimensions. It is aimed at those wanting to use fractals in their own areas of mathematics or science. The first part of the book covers the general theory of fractals and their geometry. Results are stated precisely, but technical measure theoretic ideas are avoided and difficult proofs are sketched or omitted. The second part contains a variety of examples and applications in mathematics and physics.

About the author

Kenneth Falconer took his MA and PhD degrees at Cambridge University and was Research Fellow at Corpus Christi College, Cambridge, from 1977 to 1980. He then joined Bristol University where he is now a Reader. He was Visiting Professor at Oregon State University, USA, in 1985–6 and has lectured widely in Britain and abroad. He has published *The Geometry of Fractal Sets* (Cambridge University Press 1985) as well as more than 40 research papers, largely on fractals, geometric measure theory, and the geometry of convex sets.

Cover photograph. Part of the Julia set of $f(z) = z^2 + 0.233 + 0.5378i$, with the Fatou set coloured according to escape rate.

ISBN 0-471-92287-0

9 780471 922872

JOHN WILEY & SONS

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