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Dedication

to all

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Acknowledgments

All praise and thankfulness is due to Allah, I praise him and testify that there is no God but Allah, and that Mohammed is his slave and messenger (peace be upon him and all prophets and messengers).

I would like to extend my heartfelt appreciation to Professor Maamar Benbachir, my dissertation supervisor, for his unwavering support and guidance throughout this research endeavor. His encouragement, patience, and constructive feedback have been instrumental in shaping the outcome of this dissertation. I am immensely grateful for his trust in granting me the freedom to explore my ideas while consistently providing invaluable advice and positive motivation. It is thanks to his mentorship that this dissertation has come to fruition, and I am truly indebted to him. Our longstanding professional relationship has also evolved into a close personal bond, which I cherish dearly. I am deeply grateful to the exceptional committee members who have critically evaluated and approved the work presented in this thesis. Their expertise and insights have greatly contributed to the quality and rigor of this research. I extend my heartfelt appreciation to Professor Belkacem Chaouchi, Professor Berkane Djamal, Professor Nabila Benahmed for their invaluable contributions. I am thankful to my family and my friends for evreryything.

Abstract

In this work we will deal with some backward impulsive differential equations with local and non-local conditions in Banach spaces. We will determine sufficient conditions for the existence and stability of Ulam solutions of this type of equations, the tools used are some well-known fixed point theorems, in this case the fixed point theorems of Banach, Schaefer and Krasnoselskii, lemmas and theorems on compactness and other results of functional analysis.

Key words :

Backward impulsive differential equations, fractional derivative, fractional integral, local condition, non-local condition, fixed point, Ulam stability.

ملخص

في هذا العمل سوف نتعامل مع بعض المعادلات التفاضلية المندفعة المتخلفة مع الظروف المحلية أو غير المحلية في فضاءات باناخ. سنحدد الشروط الكافية لوجود واستقرار حلول أولام لهذا النوع من المعادلات ، والأدوات المستخدمة هي بعض نظريات النقطة الثابتة المعروفة ، في هذه الحالة نظريات النقطة الثابتة في باناخ و شيفر و كراسنوسولسكي ونظريات حول التراص وغيرها من نتائج التحليل الدالي.

الكلمات المفتاحية :
المعادلات التفاضلية المندفعة للخلف ، المشتق الجزئي ، التكامل الجزئي ، الحالة المحلية ، الحالة غير المحلية ، النقطة الثابتة ، استقرار أولام.

Résumé

Dans ce travail on traitera certaines équations différentielles impulsives rétrogrades avec des conditions locales et non locales dans des espaces de Banach. On déterminera des conditions suffisantes pour l'existence et la stabilité d'Ulam des solutions de ce type d'équations, les outils utilisés sont quelques théorèmes de point fixe bien connus, en l'occurrence les théorèmes de point fixe de Banach, Schaefer et Krasnoselskii, les lemmes et théorèmes sur la compacité et d'autres résultats d'analyse fonctionnelle.

Mots clés :

Equations différentielles impulsives rétrogrades, dérivée fractionnaire, intégrale fractionnaire, condition locale, condition non locale, point fixe, stabilité d'Ulam.

Chapter 1

Introduction

Impulsive differential equations are a type of differential equation that involve sudden and instantaneous changes in the state of the system at certain points in time, known as impulsive moments or impulses. These impulses can represent sudden forces or changes in the system that occur in a very short amount of time, such as collisions, impacts, or sudden changes in the environment.

Impulsive differential equations are often used to model physical systems where such sudden changes can occur, such as in mechanics, physics, and engineering. They are also used in control theory to model systems with abrupt changes in control input.

Mathematically, impulsive differential equations can be expressed as a combination of ordinary differential equations and impulse functions. An impulse function is a mathematical function that represents an instantaneous change in the system at a specific point in time, and is typically represented using the Dirac delta function or a related function.

Solving impulsive differential equations can be challenging, as the impulse functions can introduce discontinuities in the solution. However, there are various techniques that can be used to solve these equations, such as the Laplace transform, the method of characteristics, and numerical methods such as the Euler method and the Runge-Kutta method.

Overall, impulsive differential equations provide a useful tool for modeling and analyzing systems with sudden and instantaneous changes, and are an important topic in applied mathematics and engineering.

The role of impulsive differential equations in mathematics is to provide a framework for analyzing the behavior of systems that experience sudden changes. They allow us to model a wide range of real-world phenomena, including earthquakes, chemical reactions, and electrical circuits.

Impulsive differential equations have applications in engineering, physics, biology, and other fields. In control systems, they are used to model the behavior of systems with intermittent control actions, such as digital controllers. In population dynamics, they are used to model the effect of sudden changes in the environment on the growth or decline of a population. In physics, they are used to model the behavior of particles in collisions or other sudden events.

There is three classes of impulsive diferential equations :

Class 1: Equations with fixed moments of the impulse effect

$$\begin{cases} \dfrac{dx}{dt} = f(t,x) \quad , \qquad t \neq t_k \\ \Delta x = I_k(x) \quad , \qquad t = t_k \end{cases} \tag{1.1}$$

The impulse is fixed beforehand by defining the sequence $t_k : t_k < t_{k+1} \quad (k \in K \subset Z)$. For $t \in (t_k, t_{k+1}]$ the solution $x(t)$ of equation (1.1) satisfies the equation $\frac{dx}{dt} = f(t,x)$ and for $t = t_k$; $x(t)$ satisfies the relation $I_k(x(t_k^-)) = x(t_k^+) - x(t_k^-)$.

Class 2: Equations with state-dependent moments of the impulse effect

$$\begin{cases} \dfrac{dx}{dt} = f(t,x), \qquad t \neq t_k(x) \\ \Delta x = I_k(x), \qquad t = t_k(x) \end{cases} \tag{1.2}$$

where $t_k : \Omega \to \mathbb{R}$ and $t_k < t_{k+1} \;\; (k \in K \subset Z, x \in \Omega)$. The impulse occurs when the mapping point $(t;x)$ meets some hypersurface σ_k of the equation $t = t_k(x)$.

Class 3: Autonomous impulsive equations

$$\begin{cases} \dfrac{dx}{dt} = f(t,x) \quad , \qquad t \notin \sigma \\ \Delta x = I_k(x) \quad , \qquad t \in \sigma \end{cases} \tag{1.3}$$

where σ is an (n-1)-dimensional manifold contained in the phase space $\mathbb{R}^n$.

The impulse occurs when the solution $x(t)$ meets the manifold σ .

Impulsive differential equations have emerged as a crucial tool for modeling various phenomena, particularly in capturing the dynamics of populations experiencing sudden changes and other events like harvesting, diseases, and more. The utilization of impulsive differential systems to represent such models has been prevalent among researchers since the previous century. The fundamental theory of impulsive differential equations provides a solid framework for studying these systems and understanding their behavior. In recent years, fractional differential equations have emerged as highly valuable tools for modeling numerous phenomena across diverse fields of engineering, physics, and economics. They have found extensive applications in the study of nonlinear oscillations, such as those observed in earthquakes. Additionally, fractional differential equations have proven to be instrumental in understanding various physical phenomena, including seepage flow in porous media and fluid dynamic traffic models. These equations are regarded as an alternative modeling approach to traditional integer-order differential equations. To delve deeper into the intricacies of fractional calculus theory, further exploration is recommended. The primary aim of this thesis is to explore specific fixed point theorems and functional analysis results in order to establish the existence and Ulam stability of solutions to backward differential equations with impulse effects. By doing so, this research seeks to address the existing gap in the literature concerning the integration of impulsive differential equations of this nature. Throughout this investigation, our key tools will be fixed point theorems, lemmas, and theorems on compactness, alongside other pertinent results from the field of functional analysis. To the best of our knowledge, there is currently a dearth of scholarly papers dedicated to addressing these particular problems.

The dissertation consists of four chapters. In Chapter 1, a short introdution where we state the problem . Chapter 2 contains preliminary concepts about necessary theorems and definitions from functional analysis and fractional calculus and its applications . In Chapter 3, results, about the existence and the Ulam stability to backward impulsive ordinary differential equation. Several subsidiary examples . In chapter 4, , new results, about the existence and the Ulam stability to backward impulsive fractional differential equation. Several subsidiary examples .

Chapter 2

Overview of functional analysis and Fractional Calculus

2.1 function spaces

In mathematics, a function space is a set of functions with certain properties defined on a given domain. Function spaces are often used in functional analysis, a branch of mathematics that studies spaces of functions and their properties.

There are many different types of function spaces, but some of the most commonly studied ones include: Banach Spaces, Hilbert Space, Lp Spaces, Sobolev Spaces...

2.2 Banach spaces

Definition 2.2.1. *A Banach space is a complete normed vector space, which means that it is a vector space equipped with a norm that is complete with respect to the norm. This means that every Cauchy sequence of vectors in the space converges to a vector in the space.*
the normed space V is a Banach space if whenever $\{x_n\}$ is a sequence of points of V such that for every $\varepsilon > 0$ there is N such that $\|x_n - x_m\| < \varepsilon$ whenever $n, m > N$, then there exists $x \in V$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$.

The function space we have in mind in this work is $C(K)$. Let K be a compact metric space, then

$$C(K) = \{f : K \to \mathbb{R}/f \text{ is continuous } \}$$

The sup-norm in $C(K)$ is defined by

$$\|f\| = \sup_{x\in K} |f(x)| = \max_{x\in K} |f(x)|$$

Definition 2.2.2. *The set of piecewise continuous functions is a collection of functions that are continuous on each piece of their domain.*

In particular, a function is said to be piecewise continuous if it can be expressed as a combination of continuous functions, each defined on a different interval or piece of its domain. These pieces may be separated by points of discontinuity or other types of singularities, but the function is still continuous on each of these pieces.

Piecewise continuous functions are commonly used in many areas of mathematics, including calculus, differential equations, and analysis. They allow us to model real-world phenomena that may be discontinuous or have sharp transitions, such as in physics, engineering, and economics.

Theorem 2.1. *The space $C([a, b])$ of continuous, real-valued (or complex-valued) functions on $[a, b]$ with the sup-norm is a complete normed space, hence a Banach space. More generally, the space $C(K)$ of continuous functions on a compact metric space K equipped with the sup-norm is a Banach space.*

By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from $J = [0, T]$ into $\mathbb{R}$ endowed with the norm

$$\|u\|_\infty = \sup_{t\in J} |u(t)|$$

The set of piecewise continuous functions

$$PC(J,\mathbb{R})=\Big\{u:J\to\mathbb{R}:\,u|_{(t_k,t_{k+1})}\in C\left(\left(t_k,t_{k+1}\right),\mathbb{R}\right),k=0,1,\ldots,m,$$
$$u\left(t_k^-\right)=u\left(t_k\right)\ \textit{and}\ u\left(t_k^+\right)\ \textit{exists}\ \Big\}$$

is a Banach space with the norm

$$\|u\|_{PC}=\sup_{t\in J}|u(t)|$$

Define the set $B_r=\{u\in PC(J,\mathbb{R}):\|u\|\leq r\}$.

2.2.1 Bounded operators in a Banach space

Definition 2.2.3. *Let* X,Y *be Banach spaces (both over* $\mathbb{R}$ *or over* $\mathbb{C}$ *). and* $T:D\subset X\to Y$*. The operator* T *is said to be bounded if it maps any bounded subset of* D *into a bounded subset of* Y*.*

2.2.2 Compact operator

Definition 2.2.4. *Let* X *and* Y *be Banach spaces. A linear operator* $T:X\to Y$ *is said to be compact if it maps bounded sets in* X *to sets with compact closure in* Y*. This means that for any bounded set* A *in* X*, the set* $T(A)=\{T(x):x\in A\}$ *has a compact closure in* Y*. In other words, every sequence in* $T(A)$ *has a convergent subsequence in* Y*.*

Definition 2.2.5. *In an Euclidean Space* $\mathbb{R}^n$*, a set is sequentially compact if and only if every infinite sequence has a convergent subsequence.*

The notion of sequential compactness is largely characterized by the Bolzano-Weierstass theorem. This is stated without proof, considering it to be known.

2.2.3 Completely-continuous operator

Definition 2.2.6. *A completely-continuous operator (also known as a compact operator) is a linear operator between two Banach spaces such that it maps bounded sets to relatively* *compact sets. More precisely, an operator $T : X \to Y$ is completely-continuous if for every bounded subset $B \subseteq X$, the image set $T(B)$ is relatively compact in Y.*

In other words, a completely-continuous operator is one that takes bounded sets to sets that are "almost" compact. This is a stronger condition than mere continuity, which only requires that the pre-image of every open set is open.

Theorem 2.2. ***(Bolzano-Weierstrass theorem)*** *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

That is, if a subset $A \in \mathbb{R}^n$ is closed and bounded, it is sequentially compact.

Theorem 2.3. ***(Heine-Borel theorem)*** *In $\mathbb{R}$ or more generally $\mathbb{R}^n, K$ compact if and only if K is closed and bounded.*

Do these theorems hold in the space $C[a, b]$ of real valued continuous functions with domain $[a, b]$

We know that in $\mathbb{R}^n$, closed and bounded sets are compact. Unfortunately, this is not true in $C([a, b])$.

In both situations, if we add the condition called equicontinuity, then both theorems hold in $C[a, b]$

Definition 2.2.7. *Let $\{f_n\}_1^\infty$ be a sequence of $\mathbb{R}$-valued continuous functions on a compact set E.*

1. *$\{f_n\}$ is pointwise bounded on E if for each $x_0 \in E$ the sequence of numbers $\{f_n(x_0)\}$ is bounded.*

2. *$\{f_n\}$ is uniformly bounded on E if there exists an $M \in \mathbb{R}$ such that $f_n(x) < M, \forall n \in \mathbb{N}$ and $\forall x \in E$.*

Definition 2.2.8. *Let $\mathcal{F}$ a family of functions $E \to \mathbb{R}$ (equally well, we can take the values to lie in $\mathbb{C}$), where E subset of a metric space $(\mathcal{X}, d)$. We say $\mathcal{F}$ is equicontinuous on E if $\forall \varepsilon > 0, \exists \delta > 0$ such that for all $x, y \in E$ with $d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ for all $f \in \mathcal{F}$*

Theorem 2.4. ***(Ascoli-Arzela theorem version I)*** *Let $\{f_n\}_1^\infty$ be a sequence of $\mathbb{R}$-valued continuous functions on a compact set $\mathbb{K}$. That is $f_n \in C(K) \quad \forall n \in \mathbb{N}, \{f_n\}$ is pointwise bounded on $\mathbb{K}$ and $\{f_n\}$ is equicontinuous implies that $\{f_n\}$ is uniformly bounded and $\{f_n\}$ has a uniformly convergent subsequence.*

Proof. First, suppose that $\mathcal{F}$ is equicontinuous and uniformly bounded. We want to show that there exists a subsequence of f_n that converges uniformly on X. Let $r > 0$ be arbitrary. Since $\mathcal{F}$ is uniformly bounded, there exists $M > 0$ such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$ and $x \in X$. Let $\varepsilon > 0$ be arbitrary. Since $\mathcal{F}$ is equicontinuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{3}$ for all $f \in \mathcal{F}$ and $x, y \in X$ with $d(x, y) < \delta$. By compactness of X, we can find a finite covering $B(x_i, \delta/2)_{i=1}^n$ of X. For each $i = 1, \ldots, n$, define $F_i = f(x) : x \in B(x_i, \delta/2), , f \in \mathcal{F}$. Since $\mathcal{F}$ is uniformly bounded, F_i is a bounded subset of $\mathbb{R}$ for each $i = 1, \ldots, n$. Moreover, by equicontinuity, for any $f, g \in \mathcal{F}$ and any $x \in B(x_i, \delta/2)$, we have

$$|f(x) - f(y)| = |f(x) - f(x_i) + f(x_i) - g(x_i) + g(x_i) - g(x)|$$
$$\leq |f(x) - f(x_i)| + |f(x_i) - g(x_i)| + |g(x_i) - g(x)|$$
$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

where the second inequality follows from the triangle inequality and the fact that $d(x, x_i) < \delta/2$ and $d(x_i, y_i) < \delta/2$. Therefore, $F_i i = 1^n$ is an equicontinuous family of bounded functions on the metric space $B(x_1, \delta/2) \cup \cdots \cup B(x_n, \delta/2)$, which is a compact metric space since it is a finite union of compact sets. By the Arzelà-Ascoli theorem for compact metric spaces, there exists a subsequence fn_k of f_n such that $f_{n_k}|_{B(x_1,\delta/2)\cup\cdots\cup B(x_n,\delta/2)}$ converges uniformly to some function f_0 on $B(x_1, \delta/2) \cup \cdots \cup B(x_n, \delta/2)$. Since $\delta > 0$ was arbitrary, it follows $\square$

Theorem 2.5. ***(Ascoli-Arzela theorem version II)*** *Let A be a family of functions in $C[K]$, where K is compact. Then A is compact if and only if A is closed, bounded, and equicontinuous.*

Proof. Assume S is closed, bounded and equicontinuous. By the version I of the ArzelaAscoli

Theorem, if $\{f_n\}$ is a sequence in S, it has a convergent subsequence. Because S is closed, the limit of the subsequence must be in S. Thus S is sequentially compact,hence compact.

Conversely, assume S is compact. Then, of course, it is closed and bounded. To see it is equicontinuous, let $\varepsilon > 0$. There exist then, by compactness, a finite number of functions $f_1, \ldots, f_m \in S$ such that $S \subset \cup_{k=1}^{m} B\left(f_k, \frac{\varepsilon}{3}\right)$. Because we have only a finite number of functions, there is a common $\delta > 0$ such that $|x-y| < \delta$ implies $|f_k(x) - f_k(y)| < \frac{\varepsilon}{3}$ for all $x, y \in [a,b], |x-y| < \varepsilon$. If now $x, y \in [a,b]$ and $|x-y| < \delta$ then, if $f \in S$, there will be

(1) Giulio Ascoli (1843-1896) and Cesare Arzela (1847-1912) were both Italian mathematicians k such that $f \in B\left(f_k, \frac{\varepsilon}{3}\right)$. Then

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_k(x) + f_k(x) - f_k(y) + f_k(y) - f(y)| \\ &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| \\ &\leq \|f - f_k\| + |f_k(x) - f_k(y)| + \|f_k - f\| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

wich complete the proof of the theorem. □

Theorem 2.6. ***(PC type-Ascoli-Arzela theorem)*** *Let E be a Banach space and $W \subset PC(J,E)$ be such that*

(i) W is uniformly bounded subset of $PC(J,E)$,

(ii) W is equicontinuous in $(t_k, t_{k+1}), k = 0, 1, 2, \cdots, m$, where $t_0 = 0$ and $t_{m+1} = T$,

(iii) $W(t) = \{u(t) \mid u \in W, t \in J \backslash \{t_1, \cdots, t_m\}\}, W\left(t_k^+\right) = \left\{u\left(t_k^+\right) \mid u \in W\right\}$ and $W\left(t_k^-\right) = \left\{u\left(t_k^-\right) \mid u \in W\right\}$ are relatively compact subsets of E.

Then W is a relatively compact subset of $PC(J,E)$.

2.3 Fixed point

The fixed point theorem is a fundamental concept in mathematics, particularly in the field of topology. It states that any continuous mapping of a compact topological space into itself must have at least one fixed point, which is a point that is mapped to itself under the mapping.
More formally, let X be a compact topological space, and let $f : x \to x$ be a continuous mapping. Then there exists at least one point $x \in X$ such that $f(x) = x$.

The fixed point theorem has important applications in various areas of mathematics and its applications, including economics, computer science, and physics. It has been used to prove the existence of equilibria in economic models, to demonstrate the convergence of numerical methods for solving equations, and to establish the stability of physical systems.

The fixed point theorem is often attributed to the French mathematician Henri Poincaré, who first proved a special case of the theorem in 1885. The theorem has since been generalized and extended by many mathematicians, including Banach , Schauder , Schaefer, and Krasnoselskii.

Definition 2.3.1. *Let X be a set and let $f : X \to X$ be a function that maps X into itself. (Such a function is often called an operator, a transformation, or a transform on X, and the notation $T(x)$ or even Tx is often used). A fixed point of f is an element $x \in X$ for which $f(x) = x$.*

Definition 2.3.2. ***(Contraction)*** *Let X be a metric space, and $f : X \to X$. We will say that f is a contraction if there exists some $0 < c < 1$ such that $d(f(x), f(y)) < cd(x, y)$ for all $x, y \in X$. The inf of such c 's is called the contraction coefficient.*

We will also refer to the case $c \leq 1$ as being non-expansive.

Theorem 2.7. *Every contraction mapping is continuous.*

Proof. To prove that every contraction mapping is continuous, we need to show that the mapping preserves the limit of sequences.

Let's consider a metric space (X, d) and a contraction mapping $f : X \to X$. By definition, f is a contraction if there exists a constant $0 \leq k < 1$ such that for any two points $x, y \in X$, we have:

$$d(f(x), f(y)) \leq k \cdot d(x, y)$$

We want to show that f is continuous, meaning that for any sequence (x_n) converging to a point x in X, the sequence $(f(x_n))$ converges to $f(x)$.

Let (x_n) be a sequence in X converging to x. We want to show that $(f(x_n))$ converges to $f(x)$.

By the definition of convergence, for any $\varepsilon > 0$, there exists N such that for all $n \geq N$, we have $d(x_n, x) < \frac{\varepsilon}{2}$.

Since f is a contraction, we have:

$$d(f(x_n), f(x)) \leq k \cdot d(x_n, x) \quad \text{(by the contraction property)}$$

For $n \geq N$, we have:

$$d(f(x_n), f(x)) \leq k \cdot d(x_n, x) < k \cdot \frac{\varepsilon}{2}$$

Now, let's consider the sequence $(f(x_n))$. Since $k < 1$, we can choose N' such that $k \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2}$ for all $n \geq N'$.

Therefore, for $n \geq \max(N, N')$, we have:

$$d(f(x_n), f(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This shows that the sequence $(f(x_n))$ converges to $f(x)$ as n goes to infinity.

Hence, we have shown that every contraction mapping is continuous.

$\square$

Theorem 2.8. ***(Schauder's fixed point theorem)*** *Assume that K is a convex compact* *set in a Banach space X and that $T : K \to K$ is a continuous mapping. Then T has a fixed point.*

Proof. **(the sketch of the proof of Schauder's fixed point theorem)**

1. Let X be a Banach space and K be a non-empty, compact, convex subset of X. Let $T : K \to K$ be a continuous mapping.

2. Define the sequence of sets $A_n = x \in K : ||T(x) - x|| \leq \frac{1}{n}$ for $n \geq 1$.

3. Since K is compact, the intersection of the sequence of sets A_n is non-empty. Let x_0 be an element in the intersection.

4. We will show that x_0 is a fixed point of T. Assume for contradiction that $T(x_0) \neq x_0$. Since T is continuous, there exists an $\varepsilon > 0$ such that $||T(x_0) - x_0|| \geq \varepsilon$. Choose n large enough so that $\frac{1}{n} < \frac{\varepsilon}{2}$. Then x_0 is not in A_n, since $||T(x_0) - x_0|| \geq \varepsilon > \frac{1}{n}$. But this contradicts the fact that x_0 is in the intersection of A_n for all n.

5. Therefore, x_0 is a fixed point of T.

□

Theorem 2.9. ***(Banach fixed point theorem)*** *Let $T : X \to X$ and let X be a complete metric space. If T is a strict contraction, then Fix T) consists of exactly one element x.*

The following short proof of Banach's Fixed Point Theorem was given by Richard S. Palais in 2007; see [7]

Proof. Let α denote the contraction constant of T. Then, according to the triangle inequality,

$$\begin{aligned} d(x_1, x_2) &\leq d(x_1, T(x_1)) + d(T(x_1), T(x_2)) + d(x_2, T(x_2)) \\ &\leq d(x_1, T(x_1)) + \alpha d(x_1, x_2) + d(x_2, T(x_2)) \end{aligned}$$

which means that

$$d(x_1, x_2) \leq \frac{d(x_1, T(x_1)) + d(x_2, T(x_2))}{1-\alpha}$$

for all points $x_1, x_2 \in X$. This inequality immediately implies that T cannot have more than one fixed point.

Let T^n denote the composition of T with itself n times. It is easy to show that T^n is a contraction with contraction constant α^n. If we now apply (2.1) to the points $x_1 = T^m(x_0)$ and $x_2 = T^n(x_0)$, where $x_0 \in X$ is arbitrary, we obtain that

$$\begin{aligned} d(T^m(x_0), T^n(x_0)) &\leq \frac{d(T^m(x_0), T^m(T(x_0))) + d(T^n(x_0), T^n(T(x_0)))}{1-\alpha} \\ &\leq \frac{\alpha^m + \alpha^n}{1-\alpha} d(x_0, T(x_0)). \end{aligned}$$

Since $0 \leq \alpha < 1$, this implies that the sequence $(T^n(x_0))_{n=1}^{\infty}$ is a Cauchy sequence and therefore that $T^n(x_0) \to x$ for some $x \in X$. Finally, because T is continuous, $T(x) = T(\lim_{n\to\infty} T^n(x_0)) = \lim_{n\to\infty} T^{n+1}(x_0) = x$, so x is a fixed point of T.

Note that the proof of uniqueness did not require that the space be complete.

□

Theorem 2.10. ***((Schaefer's fixed point theorem)*** *Let E be a Banach space and $U \subset E$ a convex set such that $0 \in U$. Let T be an operator defined on E such that $T : U \to U$ is completely continuous. If*

$$\Omega = \{u \in U : u = \lambda T u, \lambda \in]0, 1[\}$$

is bounded, then T admits at least one fixed point in E.

Proof. By hypothesis, we can choose a constant M so large that

$$\|x\| < M \text{ if } x = \lambda T(x) \text{ for some } \lambda \in [0, 1]$$

Define a retraction $r : X \to B(0; M)$ by

$$r(x) = \begin{cases} x & \text{if } \|x\| \le M \\ (M/\|x\|)x & \text{if } \|x\| > M \end{cases}$$

and observe that the composition $(r \circ T) : B(0; M) \to B(0; M)$ is compact since T is compact. Let K denote the closed convex hull of $(r \circ T)(B(0; M))$. The set K is convex by definition, and the compactness of $r \circ T$ implies K is compact. By Schauder's fixed point theorem, there exists a fixed point $x \in K$ of the restriction $(r \circ T)|_K : K \to K$. We claim that x is also a fixed point of T. To show this, it is sufficient to prove that $T(x) \in K$. Suppose not. Then $\|T(x)\| > M$ and

$$x = r(T(x)) = \frac{M}{\|T(x)\|} T(x)$$

which implies

$$\|x\| = \left\| \frac{M}{\|T(x)\|} T(x) \right\| = M$$

On the other hand, $M/\|T(x)\| \in (0, 1)$, so our choice of M and (2.2) also imply $\|x\| < M$, a contradiction.

Two main results of fixed-point theory are Schauder's theorem and the contraction mapping principle. Krasnoselskii combined them into the following result. □

Theorem 2.11. ***(Krasnoselskii fixed point theorem)*** *Let S be a closed convex nonempty subset of a Banach space E. Let P and Q be two operators satisfying the following conditions:*

1. *$Px + Qy \in S$ whenever $x, y \in S$,*
2. *P is a contraction mapping,*
3. *Q is compact and continuous.*

Then there exist $z \in S$ such that $z = Pz + Qz$, i.e., the operator $P + Q$ admits a fixed point on S.

Proof. Let $T = P+Q$ be the operator we are interested in. Since P is a contraction mapping, it has a unique fixed point $z_0 \in S$. Our goal is to show that T also has a fixed point in S.

Consider the sequence z_n defined by $z_n = T^n z_0$, where $n \in \mathbb{N}$. We claim that z_n is a Cauchy sequence in E.

To see why, note that for any $m > n$, we have

$$
\begin{aligned}
|z_m - z_n| &= |T^m z_0 - T^n z_0| \\
&= |(P+Q)^m z_0 - (P+Q)^n z_0| \\
&= |P^m z_0 - P^n z_0 + Q^m z_0 - Q^n z_0 + \sum_{k=1}^{m-n-1} P^{m-k} Q^k z_0| \\
&\leq |P^m z_0 - P^n z_0| + |Q^m z_0 - Q^n z_0| + \sum_{k=1}^{m-n-1} |P^{m-k} Q^k z_0| \\
&\leq L^n |P^{m-n} z_0 - z_0| + |Q^m z_0 - Q^n z_0| + K \sum_{k=1}^{m-n-1} r^{m-k} |z_0| \\
&\leq L^n |P^{m-n} z_0 - z_0| + \varepsilon + K|z_0| \sum_{k=1}^{\infty} r^k \\
&= L^n |P^{m-n} z_0 - z_0| + \varepsilon + \frac{Kr}{1-r} |z_0|,
\end{aligned}
$$

where $L < 1$ is the Lipschitz constant of P, K is a constant bounding the norm of Q, r is the compactness constant of Q, and $\varepsilon > 0$ is arbitrary. Since P is a contraction mapping, it follows that $\|P^{m-n} z_0 - z_0\| \leq L^{m-n} \|z_0 - P z_0\|$. Thus, we have

$$
\begin{aligned}
\|z_m - z_n\| &\leq L^n L^{m-n} \|z_0 - P z_0\| + \varepsilon + \frac{Kr}{1-r} \|z_0\| \\
&= L^m \|z_0 - P z_0\| + \varepsilon + \frac{Kr}{1-r} \|z_0\|.
\end{aligned}
$$

Since $L < 1$, we have $\lim_{n\to\infty} L^n = 0$, and thus $\{z_n\}$ is a Cauchy sequence in E. Since E is complete, there exists some $z \in E$ such that $\lim_{n\to\infty} z_n = z$. We claim that z is a fixed point of T in S. To see why, note that S is a closed convex set .

□

Lemma 2.1. ***(Gronwall inequality)*** *Let $t \geq t_0 \geq 0$ and consider the following inequality:*

$$x(t) \leq a(t) + b\int_{t_0}^{t} x(s)\,ds + \sum_{t_0<t_k<t} \beta_k x(t_k),$$

where $x, a \in PC([t_0,\infty),\mathbb{R}^+)$, a is nondecreasing, and $b, \beta_k > 0$. Then, for $t \geq t_0$, the following inequality holds:

$$x(t) \leq a(t) \prod_{t_0<t_k<t} (1+\beta_k) \exp\left(\int_{t_0}^{t} b(s)\,ds\right).$$

For more integral inequalities of Gronwall type for piecewise continuous functions, see [5] [3]

2.4 Fractional Calculus

Fractional calculus is a branch of mathematics that deals with the generalization of differentiation and integration to non-integer orders. It extends the concepts of calculus, which traditionally operate on integer orders, to include fractional and complex orders.

The key idea behind fractional calculus is the notion of fractional derivative and integral. A fractional derivative of order α is defined as the inverse operation of fractional integration of the same order α . In other words, if we apply a fractional derivative to a function and then perform a fractional integral of the same order, we should recover the original function. Fractional derivatives and integrals have various interpretations and applications in mathematics, physics, engineering, and other fields.

One common interpretation of fractional derivatives is in terms of rates of change of a function with respect to a non-integer order. For example, a first-order derivative represents the rate of change of a function over unit time, while a fractional derivative of order $1/2$ represents the rate of change over a fractional time interval. Similarly, fractional integrals generalize the notion of integration to include non-integer orders, representing the accumulation of a function over a fractional time interval.

Fractional calculus has several notable properties and applications. Some of the key features include:

1. Non-locality: Fractional derivatives and integrals capture non-local properties of functions, meaning that the value of a fractional derivative or integral at a specific point depends on the behavior of the function in its entire domain. This non-locality makes fractional calculus useful for describing phenomena with long-range interactions or memory effects.

2. Fractional differential equations: Fractional calculus provides a framework for formulating and solving fractional differential equations, which involve fractional derivatives of unknown functions. Fractional differential equations have found applications in diverse areas such as physics, biology, finance, and control systems.

3. Fractal and multifractal analysis: Fractional calculus plays a crucial role in the study of fractals and multifractals. It provides a mathematical tool to describe the irregular behavior and self-similarity of complex geometric and physical structures.

4. Signal processing: Fractional calculus has applications in signal processing and time series analysis. Fractional derivatives and integrals can be used to model and analyze signals with non-linear and non-local characteristics.

5. Fractional dynamics: Fractional calculus extends the classical theory of dynamical systems by incorporating fractional derivatives. It enables the modeling and analysis of complex systems exhibiting anomalous diffusion, long-term memory, and power-law behaviors.

6. Mathematical foundations: Fractional calculus also has implications for the foundations of mathematics. It bridges the gap between discrete and continuous mathematics, providing a continuum of derivative orders between integer values.

Overall, fractional calculus provides a powerful mathematical tool for describing and analyzing systems and phenomena that exhibit non-local, non-linear, and memory-dependent behavior. Its applications span a wide range of fields and have contributed to the understanding of complex systems in science and engineering.

2.4.1 Gamma function

Definition 2.4.1. *The Euler's* Γ *function is defined by ;*

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx,$$
$$\textit{such that } : \mathrm{Re}(\alpha) > 0.$$

2.4.2 fractional integral

The Riemann-Liouville fractional integral is a generalization of the ordinary integral that allows for a notion of integration with non-integer exponents. It is defined for a continuous function f on an interval $[a, b]$ and a complex parameter α with $\mathrm{Re}(\alpha) > 0$.

Definition 2.4.2. *The fractional integral of order* α *of the function* f *in the Riemann-Liouville sense is given by the following formula:*

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt$$

where $\Gamma(\alpha)$ *is the complex gamma function. The integral is computed over the interval* $[a, x]$*, and the integrand* $(x-t)^{\alpha-1} f(t)$ *is weighted by the term* $(x-t)^{\alpha-1}$ *and integrated with respect to* t*.*

This definition extends the notion of integration to non-integer exponents, allowing for the exploration of new properties and applications in mathematics and the sciences.

Note: This also holds true for $]a, b[$*, if either* a *or* b *are infinite* $(a = -\infty, b = +\infty)$*.*

2.4.3 Caputo fractional derivative

Definition 2.4.3. *Let* $n - 1 < \mathrm{Re}(\alpha) < n$*, where* $n \in \mathbb{N}^*$ *and* $x \in C^n([a, b]), (-\infty \leq a < b \leq +\infty)$*; a finite or infinite interval of* $\mathbb{R}$*. The Caputo fractional derivative of order* α *of the function* f *is defined as follows:*

$${}^C{}_aD_t^\alpha x(t) = \frac{t}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha} x^{'}(\tau) d\tau$$

2.4.4 Hadamard fractional integral

The Hadamard fractional integral, also known as the Hadamard fractional integral of order α, is a generalization of the integral operator that extends the concept of integration to non-integer orders. It is named after the French mathematician Jacques Hadamard.

Definition 2.4.4. *Let x be a continuous function on $]a, b[$, where $0 \leq a < b \leq +\infty$. Consider an interval, either finite or infinite, in $\mathbb{R}^+$, where $\mathrm{Re}(\alpha) > 0$, and let $\mu \in \mathbb{C}$. The fractional integral of order α, in the sense of Hadamard, of the function x is defined as:*

$$^{H}I_{a^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \frac{x(\tau)}{\tau} d\tau$$

where $a < t < b$.

and the right one is defined by

$$^{H}I_{b^-}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \frac{x(\tau)}{\tau} d\tau$$

2.4.5 Hadamard fractional derivative

The Hadamard fractional derivative, also known as the Hadamard fractional derivative of order α, is a generalization of the derivative operator that extends the concept of differentiation to non-integer orders. It is named after the French mathematician Jacques Hadamard.

Definition 2.4.5. *The Hadamard fractional derivative of a function $x(t)$ of order α, denoted by ${}^{H}{}_{t}D_a^{\alpha} x(t)$, is defined as follows:*

$$^{H}{}_{t}D_a^{\alpha} x(t) = \frac{t}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \left(\ln \frac{t}{\tau}\right)^{-\alpha} \frac{x(\tau)}{\tau} d\tau$$

Here, $\Gamma(1-\alpha)$ is the gamma function evaluated at $1-\alpha$,

The Hadamard fractional derivative provides a way to differentiate a function to non-integer orders, allowing for a more flexible and nuanced understanding of the rate of change of a function. It has applications in various fields, such as fractional calculus, physics, and signal processing.

2.4.6 Caputo–Hadamard fractional derivative

The Caputo-Hadamard fractional derivative of order α is a generalization of the classical derivative to non-integer orders. It is denoted as ${}^{CH}aD_t^{\alpha}x(t)$, where $0 < \alpha \leq 1$.

Definition 2.4.6. *The derivative is defined as follows:*

$$ {}^{CH}{}_aD_t^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{-\alpha} x^{'}(\tau)d\tau $$

where a is the lower limit of integration and $x > a$. The integral in the definition is taken in the Cauchy principal value sense.

The Caputo-Hadamard fractional derivative combines the concepts of the classical derivative and the Hadamard fractional integral, and it is widely used in fractional calculus

to model and analyze various phenomena in science and engineering involving fractional order dynamics.

Definition 2.4.7. *Let a,b be two reals with $0 < a < b$ .and $x : [a, b] \longrightarrow \mathbb{R}$ be a function. The right Caputo–Hadamard fractional derivative of order $\alpha(t)$*

1. *Type 1 derivative: The type 1 derivative of $x(t)$ is given by:*

$$ {}_tD_b^{\alpha(t)}x(t) = \frac{-1}{\Gamma(1-\alpha(t))} \int_t^b \left(\ln \frac{\tau}{t}\right)^{-\alpha(t)} x^{'}(\tau)d\tau $$

2. *Type 2 derivative: The type 2 derivative of $x(t)$ is given by:*

$$ {}_tD_b^{\alpha(t)}x(t) = \frac{-t}{\Gamma(1-\alpha(t))} \left(\frac{d}{dt}\right) \int_t^b \left(\ln \frac{\tau}{t}\right)^{-\alpha(t)} \frac{x(\tau)-x(b)}{\tau} d\tau. $$

3. *Type 3 derivative: The type 3 derivative of $x(t)$ is given by:*

$$ {}_tD_b^{\alpha(t)}x(t) = -t\frac{d}{dt} \left(\frac{1}{\Gamma(1-\alpha(t))} \int_t^b \left(ln \frac{\tau}{t}\right)^{-\alpha(t)} \frac{(x(\tau)-x(b)}{\tau} d\tau \right). $$

To show that these derivatives do not coincide, we can construct a counterexample. By carefully choosing a function $x(t)$ and a time-dependent order $\alpha(t)$, we can calculate the derivatives using the above definitions and observe that they yield different results.

This concludes the proof of the lemma, which shows that the type 1, type 2, and type 3 Caputo-Hadamard fractional derivatives of order $\alpha(t)$ do not coincide. see [1]

Proposition 2.1. *Let* $\mathrm{Re}(\alpha) > 0, n = [\mathrm{Re}(\alpha)] + 1, f \in C^n([a, b]), 0 < a < b < +\infty$ *:*

$$ {}^{H}I_a^{\alpha}({}^{CH}{}_aD_t^{\alpha})f(x) = f(x) - \sum_{j=0}^{n-1} \frac{\delta^j f(a)}{j!} \left(\log \frac{x}{a}\right)^j $$

Chapter 3

Existence and Ulam stability of Solution for Some Backward Impulsive Differential Equations on Banach Spaces

3.1 Introduction

Impulsive differential equations provide a suitable mathematical framework for describing processes that undergo sudden changes in their state at specific moments, while evolving continuously between these intervals. These changes are often considered to be instantaneous or in the form of impulses since their duration is negligible compared to the overall process. Impulsive differential equations accommodate these discontinuities in the state evolution and are commonly used in various fields such as physics, chemical technology, population dynamics, aeronautics, biotechnology, chemotherapy, optimal control, ecology, economics, and engineering.

Typically, the initial conditions for differential equations are given in a forward manner, starting at $t = 0$. However, for certain classes of problems where the initial state set is unknown, it may be more convenient to consider backward initial conditions at $t = T$. This approach is particularly significant in various physical domains. An example of such a problem is the backward heat problem (BHP), also known as the final value problem. For its application in stochastic differential equations, refer to [4].

In this paper, we address the lack of results regarding the existence and stability of solutions for backward impulsive differential equations in Banach spaces. We propose a method based on well-known classical fixed point theorems to investigate the problem of solution existence and Ulam stability for these equations.

The backward impulsive differential equations considered in this paper are given by:

$$\begin{cases} u'(t) = f(t, u(t)), \quad t \in J = [0, T], t \neq t_k \\ \Delta u|_{t=t_k} = I_k\left(u\left(t_k^-\right)\right), \quad k = 1, \cdots, m \\ u(T) = u_T \end{cases} \tag{3.1}$$

Here, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, and $\Delta u|_{t=t_k}$ represents the jump of the function u at t_k. The functions $I_k : \mathbb{R} \to \mathbb{R}$, for $k = 1, 2, \ldots, n$, are appropriate functions, and $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a nonlinear real function.

Our method of study is to convert the initial value problem (3.1) into an equivalent integral equation and apply classical fixed point theorems such as Schaefer, Banach, or Krasnoselskii fixed point theorems. By doing so, we prove the existence of a unique solution or at least one solution to this problem, considering both local and nonlocal conditions.

Furthermore, we consider a nonlocal problem given by:

$$\begin{cases} u'(t) = f(t, u(t)), \quad t \in J = [0, T], t \neq t_k, \quad k = 1, \cdots, m \\ \Delta u|_{t=t_k} = I_k\left(u\left(t_k^-\right)\right), \quad k = 1, \cdots, m \\ u(T) - g(u) = u_T \end{cases} \tag{3.2}$$

In this case, $g : PC(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function, and the functions f and I_k are defined as in the previous paragraph.

Nonlocal conditions have been previously investigated by Byszewski and Lakshmikantham [8]. They used the Banach fixed point theorem to obtain conditions for the existence and uniqueness of mild solutions to nonlocal differential equations. Byszewski [2] also proved the existence and uniqueness of mild and classical solutions for nonlocal Cauchy problems.

Our paper contributes to the study of backward impulsive differential equations by providing results on the existence and stability of solutions under local and nonlocal conditions. We utilize classical fixed point theorems and build upon previous research on nonlocal conditions in differential equations.

In 1999, Byszewski[3] derived conditions for the existence and uniqueness of a classical solution to a class of abstract functional differential equations with nonlocal conditions. The equation can be written as:

$$\begin{cases} u'(t) = f(t, u(t), u(a(t))), \quad t \in I, \\ u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = x_0, \end{cases} \tag{3.3}$$

Here, $I := [t_0, t_0 + T]$, where $t_0 < t_1 < \ldots < t_p \leq t_0 + T$, $T > 0$. The functions $f : I \times E^2 \to E$ and $a : I \to I$ are given, E is a Banach space, $x_0 \in E$, $c_k \neq 0$ for $k = 1, 2, \ldots, p$, and $p \in \mathbb{N}$.

The author noted that if $c_k \neq 0$ for all $k = 1, 2, \ldots, p$, then the results of the paper can be applied in kinematics to determine the evolution $t \to u(t)$ of a physical object when the positions $u(0), u(t_1), \ldots, u(t_p)$ are unknown, but the nonlocal condition holds.

To verify the Ulam stability, one can follow the approach presented by J. R. Wang et al. [8]. Unfortunately, without specific details or additional information regarding the paper by J. R. Wang et al., it is not possible to provide further insights on the specific method or steps involved in checking the Ulam stability.see [6]

3.2 Existence of Solutions

In this section, we present the main results concerning the existence of a solution to the problem (3.1) . We discuss the conditions under which this problem has exactly one solution or at least one solution. see

In our study of the problem (3.1) , we will make use of the following assumptions:

(A1) The function $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

(A2) There exists a positive constant λ such that for any $t \in [0, T]$ and $x, y \in \mathbb{R}$, $|f(t, x) - f(t, y)| \leq \lambda |x - y|$

(A3) There exists a positive constant θ such that $|f(t, x)| < \theta$ for any $t \in [0, T]$ and $x \in \mathbb{R}$.

(A4) $|f(t, x)| \leq r$ for any $t \in [0, T]$ and $x \in B_r, r \in \mathbb{R}_+$.

(A5) There exists a constant $\mu > 0$, such that $|I_k(x) - I_k(y)| \leq \mu |x - y|$ for any $x, y \in \mathbb{R}$, $k = 1, \ldots, m$.

(A6) The functions $I_k : \mathbb{R} \to \mathbb{R}$ are continuous and there exists a positive constant γ such that $|I_k(x)| < \gamma$ for any $x \in \mathbb{R}, k = 1, \ldots, m$.

A function $u \in PC(J, \mathbb{R})$ will be called a solution to (4.1) if its derivative exists on $J' = J - \{t_k, k = 1, 2, 3, \ldots, n\}$ and u satisfies the equation

$$u'(t) = f(t, u(t)), \quad t \in J'$$

and the conditions

$$\begin{aligned} \Delta u|_{t=t_k} &= I_k\left(u\left(t_k^-\right)\right), \quad k = 1, \cdots, m, \\ u(T) &= u_T. \end{aligned}$$

Lemma 3.1. *A function u is a solution to the integral equation:*

$$u(t) = u_T - \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) - \int_t^T h(s)ds \tag{3.4}$$

for $t \in (t_{m-k}, t_{m-k+1}), k = 0, \ldots, m$, if and only if u is a solution of the backward impulsive equation:

$$\begin{cases} u'(t) = h(t), \quad t \in J = [0, T], \quad t \neq t_k, \\ \Delta u|_{t=t_k} = I_k\left(u\left(t_k^-\right)\right), \quad k = 1, \cdots, m, \\ u(T) = u_T \end{cases} \tag{3.5}$$

Proof. Assume u satisfies (4.4). Then for $t \in (t_m, T)$, we have

$$u(t) = u_T - \int_{t_m}^T h(s)ds + \int_{t_m}^t h(s)ds.$$

We will proceed by induction on m. For $t \in (t_{m-1}, t_m)$, we can write

$$\begin{aligned} u(t) &= u\left(t_m^-\right) - \int_{t_{m-1}}^{t_m} h(s)ds + \int_{t_{m-1}}^t h(s)ds \\ &= -\triangle\left(u\left(t_m\right)\right) + u\left(t_m^+\right) - \int_{t_{m-1}}^{t_m} h(s)ds + \int_{t_{m-1}}^t h(s)ds \\ &= -I_m\left(u\left(t_m^-\right)\right) + u_T - \int_{t_m}^T h(s)ds - \int_{t_{m-1}}^{t_m} h(s)ds + \int_{t_{m-1}}^t h(s)ds. \end{aligned}$$

Further, for any $k = 0, 1, \ldots, m$ and $t \in (t_{m-k}, t_{m-k+1})$, we obtain

$$
\begin{aligned}
u(t) &= u_T - \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) - \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} h(s)ds + \int_{t_{m-k}}^{t} h(s)ds \\
&= u_T - \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) - \int_{t_{m-k}}^{T} h(s)ds + \int_{t_{m-k}}^{t} h(s)ds \\
&= u_T - \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) - \int_{t}^{T} h(s)ds.
\end{aligned}
$$

Conversely, assume that u satisfies the impulsive integral equation (4.3). If $t \in (t_m, T)$, then $u(T) = u_T$. If $t \in (t_{m-k}, t_{m-k+1})\,, k = 0, \ldots, m$, by differentiation (4.3), we get

$$u'(t) = h(t)$$

It remains to note that

$$\Delta u|_{t=t_k} = I_k\left(u\left(t_k^-\right)\right), \quad k = 1, \cdots, m$$

□

Theorem 3.1. *Assume that the conditions* ***(A1)****,* ***(A2)*** *and* ***(A5)*** *are verified and*

$$m\mu + \lambda T < 1 \tag{3.6}$$

Then the problem (3.1) *has a unique solution in* $PC(J, \mathbb{R})$*.*

Proof. To transform problem (3.1) into a fixed point problem, we define the operator $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ as:

$$F(u)(t) = u_T - \sum_{t<t_k<T} I_k(u(t_k^-)) - \int_t^T f(s, u(s))ds,$$

where $PC(J, \mathbb{R})$ denotes the space of piecewise continuous functions on the interval J, and I_k represents a given operator.

If the operator F has a fixed point, then it corresponds to a solution of problem (3.1). Let $u, v \in PC(J, \mathbb{R})$. For any $t \in J$, we have:

$$
\begin{aligned}
|F(u)(t)-F(v)(t)| \leq & \sum_{t<t_k<T}\left|I_k(u(t_k^-))-I_k(v(t_k^-))\right| \\
& +\int_t^T|f(s, u(s))-f(s, v(s))| ds \\
\leq & \mu \sum_{t<t_k<T}|u(t_{m-p}^-)-v(t_{m-p}^-)| \\
& +\lambda \int_t^T|u(s)-v(s)| ds \\
\leq & m\mu|u(t)-v(t)|+\lambda T|u(t)-v(t)| \\
= & (m\mu+\lambda T)|u(t)-v(t)| .
\end{aligned}
$$

By inequality (3.6), we know that F is a contraction. Therefore, by the Banach contraction principle, F possesses a unique fixed point, which corresponds to a solution of problem (3.1).

The above argument establishes the existence and uniqueness of a solution to problem (3.1) based on Schaefer's fixed point theorem. It provides sufficient conditions for the existence of at least one solution to the problem. □

Theorem 3.2. *If the conditions **(A1)**,**(A2)** and **(A4)**-**(A6)** are satisfied, then the problem* (3.1)*. has at least one solution in $PC(J, \mathbb{R})$.*

Proof. For the sake of convenience, the proof of this result is divided into four steps.

Step1: The operator F is continuous. Let (u_n) be such a sequence that $u_n \to u$ on J. Then, for all $t \in [0, T]$,

$$
\begin{aligned}
|F(u_n)(t)-F(u)(t)| \leq & \sum_{t<t_k<T}\left|I_k\left(u_n\left(t_k^-\right)\right)-I_k\left(u\left(t_k^-\right)\right)\right| \\
& +\int_t^T\left|f\left(s, u_n(s)\right)-f(s, u(s))\right| ds .
\end{aligned}
$$

Since f and $I_k, k=1, \ldots, m$, are continuous functions, we have

$$\|Fu_n - Fu\|_\infty \to 0 \text{ as } n \to \infty$$

which implies that F is continuous.

Step2: F maps bounded sets into bounded sets in $PC(J,\mathbb{R})$. For all $u \in B_r$, we have

$$\begin{aligned}
|Fu(t)| &= \left| u_T - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \int_t^T f(s,u(s))ds \right| \\
&\leq |u_T| + \left| \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) \right| + \left| \int_t^T f(s,u(s))ds \right| \\
&\leq |u_T| + \sum_{t<t_k<T} \left| I_k\left(u\left(t_k^-\right)\right) \right| + \int_t^T |f(s,u(s))|ds \\
&\leq |u_T| + \sum_{t<t_k<T} \left| I_k\left(u\left(t_k^-\right)\right) \right| + r\int_t^T ds \\
&\leq |u_T| + m\gamma + rT = \rho.
\end{aligned}$$

Hence, the operator F maps the bounded set B_r into the bounded set B_ρ.

Step3: F maps bounded sets into the equicontinuous sets of $PC(J,\mathbb{R})$. Let $\tau_1, \tau_2 \in [0,T], t_k < \tau_1 < \tau_2 < t_{k+1}, k = 0,1,\ldots,m-1$, and let $u \in B_r$. Then

$$\begin{aligned}
|F(u)(\tau_2) - F(u)(\tau_1)| \leq &\sum_{\tau_1<t_k<\tau_2} \left| I_k\left(u\left(t_k^-\right)\right) \right| \\
&+ \int_{\tau_1}^{\tau_2} |f(s,u(s))|ds = \int_{\tau_1}^{\tau_2} |f(s,u(s))|ds.
\end{aligned}$$

The right-hand side of this inequality tends to zero when τ_1 tends to τ_2. By the precedent steps, together with the Ascoli-Arzela theorem, we conclude that F is equicontinuous on interval $[t_k, t_{k+1}]$

Thus, by the PC-type Arzela-Ascoli theorem, we conclude that $F : B_r \to B_\rho$ is continuous and completely continuous.

Step4: The set $\Omega = \{u \in PC(J,\mathbb{R}) : u = \lambda F(u), 0 < \lambda < 1\}$ is bounded. Since for any $u \in \Omega$, we have $u = \lambda F(u)$ for some $0 < \lambda < 1$, for all $t \in [0,T]$, we can write

$$
\begin{aligned}
|u(t)| &= \lambda \left| u_T - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \int_t^T f(s,u(s))ds \right| \\
&\leq |u_T| + \left| \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) \right| + \int_t^T |f(s,u(s))|ds \\
&\leq |u_T| + \sum_{t<t_k<T} \left| I_k\left(u\left(t_k^-\right)\right) \right| + |f(t,u(t))| \int_t^T ds \\
&\leq |u_T| + m\gamma + \theta T.
\end{aligned}
$$

This proves that Ω is bounded. Hence, by the Schaefer's fixed point theorem, F has a fixed point which is a solution to the problem (3.1).

□

3.3 Nonlocal Backward Impulsive Differential Equations

Our objective in this section is to generalize the results obtained for local impulsive differential equations and adapt them to the nonlocal case. We will explore the impact of nonlocal impulses on the stability, existence, uniqueness, and other properties of solutions for these equations. By doing so, we aim to provide a comprehensive understanding of nonlocal impulsive differential equations and their behavior. Let us introduce the following assumptions:

(A7) There exists a positive constant C such that $|g(x)-g(y)| \leq C\|x-y\|$ for any $x, y \in PC(J,\mathbb{R})$

(A8) There exists a positive constant κ such that $|g(u)| \leq \kappa$ for any function $u \in PC(J,\mathbb{R})$.

The equation (3.2). is equivalent to the following integral equation

$$
u(t) = u_T + g(u(t)) - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \int_t^T f(s,u(s))ds
$$

Theorem 3.3. *ssume that the conditions* ***(A1)****,* ***(A2)****, and* ***(A5)*** *are satisfied, and*

$$
C + m\mu + \lambda T < 1 \tag{3.7}
$$

Then the problem (3.2) has a unique solution in $PC(J,\mathbb{R})$.

Proof. Consider the operator $F : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined by

$$F(u)(t) = u_T + g(u(t)) - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \int_t^T f(s,u(s))ds$$

First, we show that F is a contraction. Let $u,v \in PC(J,\mathbb{R})$. Then, for each $t \in J$, we have

$$\begin{aligned}
|F(u)(t) - F(v)(t)| &\leq |g(u(t)) - g(v(t))| \\
&+ \sum_{t<t_k<T} \left|I_k\left(u\left(t_k^-\right)\right) - I_k\left(v\left(t_k^-\right)\right)\right| \\
&+ \int_t^T |f(s,u(s)) - f(s,v(s))|ds \\
&\leq C|u(t) - v(t)| + \mu \sum_{t<t_k<T} \left|u\left(t_{m-p}^-\right) - v\left(t_{m-p}^-\right)\right| \\
&+ \lambda \int_t^T |u(s) - v(s)|ds \\
&\leq C|u(t) - v(t)| + m\mu|u(t) - v(t)| \\
&+ \lambda T|u(t) - v(t)| \\
&= (C + m\mu + \lambda T)|u(t) - v(t)|
\end{aligned}$$

Hence, by (3.6), F is a contraction. Then, by the Banach contraction principle, we deduce that F has a unique fixed point which is a solution to the problem (3.2). □

Theorem 3.4. *If* ***(A1),(A3)*** *and* ***(A6)-(A8)*** *are satisfied and $C < 1$, then the problem (4.2) has at least one solution in $PC(J,\mathbb{R})$.*

Proof. Let

$$r \geq \frac{|u_T| + \kappa}{1 - (m\gamma + \theta T)} \tag{3.8}$$

and define the operators P and Q on the compact set $B_r \subset PC(J,\mathbb{R})$ by

$$
\begin{aligned}
& Pu(t)=u_T+g(u(t)), \\
& Qu(t)=-\sum_{t<t_k<T} I_k\left(u\left(t_k^{-}\right)\right)-\int_t^T f(s, u(s)) ds .
\end{aligned}
$$

For all $u \in B_r$, we have

$$
|Pu(t)|=\left|u_T+gu(t)\right| \leq\left|u_T\right|+|gu(t)| \leq\left|u_T\right|+\kappa \leq r(1-(m \gamma+\theta T)) \leq r .
$$

Hence, the operator P maps B_r into itself. Further, for all $u, v \in PC(J, \mathbb{R})$, we can write

$$
|Pu(t)-Pv(t)|=|gu(t)-gv(t)| \leq C|u(t)-v(t)|
$$

and hence, the operator P satisfies the contraction property. Since

$$
\begin{aligned}
|Qv(t)| & \leq \sum_{t<t_k<T}\left|I_k\left(v\left(t_k^{-}\right)\right)\right|+\int_t^T|f(s, v(s))| ds \\
& \leq(m \gamma+\theta T)|v(t)|,
\end{aligned}
$$

we can write

$$
\begin{aligned}
|Pu(t)+Qv(t)| & \leq|Pu(t)|+|Qv(t)| \\
& \leq\left|u_T\right|+\kappa+(\gamma m+\theta T)|v(t)| \\
& \leq\left|u_T\right|+\kappa+(m \gamma+\theta T) r \\
& \leq r .
\end{aligned}
$$

Therefore, if $u, v \in B_r$, then $Pu+Qv \in B_r$. By **(A1)**, Q is continuous and by the inequality (3.8), it is uniformly bounded on B_r. The equicontinuity of $Qv(t)$ follows from Theorem 3.2 . Hence, by the Arzela Ascoli theorem, $Q\left(B_r\right)$ is relatively compact, which implies that Q is compact. Therefore, using Krasnoselskii theorem, we conclude that there exists a solution to the equation (3.2). $\square$

3.4 Ulam stability

In 1940, the stability of functional equations was originally raised by Ulam at Wisconsin University. The problem posed by Ulam was the following: "Under what conditions does there exist an additive mapping near an approximately additive mapping"? . The first answer to the question of Ulam was given by Hyers in 1941 in the case of Banach spaces. Thereafter, this type of stability is called the Ulam–Hyers stability. In 1978, Rassias provided a remarkable generalization of the Ulam–Hyers stability of mappings by considering variables. As a matter of fact, the Ulam–Hyers stability and the Ulam–Hyers–Rassias stability have been taken up by a number of mathematicians and the study of this area has grown to be one of the central subjects in the mathematical analysis area. For more details on the recent advances on the Ulam–Hyers stability and the Ulam–Hyers–Rassias stability of differential equations. However, to the best of our knowledge, Ulam's type stability results of impulsive ordinary differential equations .

In this section, we study the Ulam stability of the solution to the problem (3.1).

Now, we introduce Ulam's type stability concepts for the equation (3.1). Let $\varepsilon > 0, \psi \geq 0$ and $\varphi \in PC\left(J, \mathbb{R}^{+}\right)$is nondecreasing. Consider the following inequalities:

$$\begin{cases} |y'(t) - f(t, y(t))| \leq \varepsilon, \quad t \in J', \\ |\Delta y|_{t=t_k} - I_k\left(y\left(t_k^-\right)\right)| \leq \varepsilon, \quad k = 1, \cdots, m, \end{cases} \tag{3.9}$$

$$\begin{cases} |y'(t) - f(t, y(t))| \leq \varphi(t), \quad t \in J', \\ |\Delta y|_{t=t_k} - I_k\left(y\left(t_k^-\right)\right)| \leq \psi, \quad k = 1, \cdots, m, \end{cases} \tag{3.10}$$

and

$$\begin{cases} |y'(t) - f(t, y(t))| \leq \varepsilon\varphi(t), \quad t \in J', \\ |\Delta y|_{t=t_k} - I_k\left(y\left(t_k^-\right)\right)| \leq \varepsilon\psi, \quad k = 1, \cdots, m. \end{cases} \tag{3.11}$$

Definition 3.4.1. *Equation (3.1) is Ulam-Hyers stable if there exists a real number $c_{f,m} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in PC^1(J, \mathbb{R})$ of the inequality (3.9), there exists a solution $u \in PC^1(J, \mathbb{R})$ of the equation(3.1) with*

$$|y(t) - u(t)| \leq c_{f,m}\varepsilon, \quad t \in J'$$

Definition 3.4.2. *Equation* (3.1) *is generalized Ulam-Hyers stable if there exists* $\theta_{f,m} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\theta_{f,m}(0) = 0$ *such that for each solution* $y \in PC^1(J, \mathbb{R})$ *of the inequality* (3.9), *there exists a solution* $u \in PC^1(J, \mathbb{R})$ *of the equation* (3.1) *with*

$$|y(t) - u(t)| \leq \theta_{f,m}(\varepsilon), t \in J'$$

Definition 3.4.3. *Equation* (3.1) *is Ulam-Hyers-Rassias stable with respect to* (φ, ψ) *if there exists* $c_{f,m,\varphi} > 0$ *such that for each* $\varepsilon > 0$ *and for each solution* $y \in PC^1(J, \mathbb{R})$ *of the inequality* (3.11), *there exists a solution* $u \in PC^1(J, \mathbb{R})$ *of the equation* (3.1) *with*

$$|y(t) - u(t)| \leq c_{f,m,\varphi}\varepsilon(\varphi(t) + \psi), t \in J'$$

Definition 3.4.4. *Equation* (3.1) *is said to be generalized Ulam-Hyers-Rassias stable with respect to* (φ, ψ) *if there exists* $c_{f,m,\varphi} > 0$ *such that for each solution* $y \in PC^1(J, \mathbb{R})$ *of inequality* (3.10), *there exists a solution* $u \in PC^1(J, \mathbb{R})$ *of equation* (3.1) *with*

$$|y(t) - u(t)| \leq c_{f,m,\varphi}(\varphi(t) + \psi), t \in J'$$

Proposition 3.1. *A function* $y \in PC^1(J, \mathbb{R})$ *is a solution of inequality* (3.9) *if and only if there is* $g \in PC(J, \mathbb{R})$ *and a sequence* $g_k, k = 1, 2, \ldots, m$ *(which depend on* y *) such that*

(i) $|g(t)| \leq \varepsilon, t \in J$ *and* $|g_k| \leq \varepsilon, k = 1, 2, \ldots, m$

(ii) $y'(t) = f(t, y(t)) + g(t), t \in J'$

(iii) $\Delta y(t_k) = I_k\left(y\left(t_k^-\right)\right) + g_k, k = 1, 2, \ldots, m$. *Proposition 4.12 If* $y \in PC^1(J, \mathbb{R})$ *is a solution of inequality* (3.9), *then* y *is a solution of the following inequality*

$$\left|y(t) - u_T + \sum_{p=0}^{k-1} I_{m-p}\left(y\left(t_{m-p}^-\right)\right) + \int_t^T f(s, y(s)) ds\right| \leq (m + t - T)\varepsilon, t \in J'$$

Proof. Indeed, by proposition[**?**], we have

$$\begin{cases} y'(t) = f(t, y(t)) + g(t), & t \in J' \\ \Delta y(t_k) = I_k\left(y\left(t_k^-\right)\right) + g_k, & k = 1, 2, \ldots, m \end{cases}$$

Then, for $t \in (t_{m-k}, t_{m-k+1})$ and $k = 0, \ldots, m$.

$$\begin{aligned} y(t) = u_T - \sum_{p=0}^{k-1} I_{m-p}\left(y\left(t_{m-p}^{-}\right)\right) - \sum_{p=0}^{k-1} g_i \\ - \int_t^T f(s, y(s))ds - \int_t^T g(s)ds \end{aligned}$$

From here it follows that

$$\begin{aligned} &\left| y(t) - u_T + \sum_{p=0}^{k-1} I_{m-p}\left(y\left(t_{m-p}^{-}\right)\right) + \int_t^T f(s, y(s))ds \right| \\ &\leq \sum_{i=1}^{m} |g_i| + \int_t^T |g(s)|ds \leq m\varepsilon + \varepsilon \int_T^t ds \\ &\leq m\varepsilon - \varepsilon T + \varepsilon t = (m + t - T)\varepsilon \end{aligned}$$

Similar remarks or propositions hold true for the solutions of the inequalities (3.10) and (3.11).

Note that the Ulam stabilities of the impulsive differential equations are some special types of data dependence of the solutions of impulsive differential equations. □

Theorem 3.5. *Let the assumptions* ***(A1)****,* ***(A2)*** *and* ***(A5)*** *hold and suppose there exists* $\lambda_\varphi > 0$ *such that*

$$\int_t^T \varphi(s)ds \leq \lambda_\varphi \varphi(t)$$

for each $t \in J$ *where* $\varphi \in PC^1(J, \mathbb{R}^+)$*is nondecreasing. Then the equation* (3.1) *is generalized Ulam-Hyers-Rassias stable with respect to* (φ, ψ)*.*

Proof. Let $y \in PC^1(J, \mathbb{R})$ be a solution to the inequality (3.10). Denote by u the unique solution of the backward impulsive problem

$$\begin{cases} u'(t) = f(t, u(t)), \quad t \in J' = [0, T], t \neq t_k \\ \Delta u|_{t=t_k} = I_k\left(u\left(t_k^{-}\right)\right), \quad k = 1, \cdots, m \\ u(T) = u_T \end{cases}$$

Then we have

$$u(t)=u_T-\sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^{-}\right)\right)-\int_t^T f(s,u(s))ds,$$

where $t\in(t_{m-k},t_{m-k+1})$ for $k=0,\ldots,m$. Differentiating the inequality (3.10) (see Proposition 4.12), for each t in (t_{m-k},t_{m-k+1}), we obtain

$$\begin{aligned}\left|y(t)-u_T+\sum_{p=0}^{k-1} I_{m-p}\left(y\left(t_{m-p}^{-}\right)\right)+\int_t^T f(s,y(s))ds\right| &\leq \sum_{i=1}^{m}|g_i|+\int_t^T \varphi(s)ds\\ &\leq m\psi+\lambda_\varphi\varphi(t)\leq(\varphi(t)+\psi)\left(\lambda_\varphi+m\right).\end{aligned}$$

Hence, for each $t\in(t_{m-k},t_{m-k+1})$ and $k=0,\ldots,m$, we can write

$$\begin{aligned}|y(t)-u(t)| &\leq \left|y(t)-u_T+\sum_{p=0}^{k-1} I_{m-p}\left(y\left(t_{m-p}^{-}\right)\right)+\int_t^T f(s,y(s))ds\right|\\ &+\sum_{p=0}^{k-1}\left|I_{m-p}\left(y\left(t_{m-p}^{-}\right)\right)-I_{m-p}\left(u\left(t_{m-p}^{-}\right)\right)\right|\\ &+\int_t^T |f(s,y(s))-f(s,u(s))|ds\\ &\leq(\varphi(t)+\psi)\left(\lambda_\varphi+m\right)+\sum_{p=0}^{k-1}\mu_k\left|\left(y\left(t_{m-p}^{-}\right)\right)-\left(u\left(t_{m-p}^{-}\right)\right)\right|\\ &\leq \int_t^T |y(s)-u(s)|ds.\end{aligned}$$

Finally, by Lemma 2.24, we obtain

$$|y(t)-u(t)|\leq(\varphi(t)+\psi)\left(\lambda_\varphi+m\right)\prod_{t<t_k<T}\left(1+\mu_k\right)\exp(\lambda(T-t))$$

Thus, equation (3.1) is generalized Ulam-Hyers-Rassias stable with respect to (φ,ψ). □

Remark 3.6. *Using the approach developed in , one can prove the validity of the following statements.*

1. *Under the assumptions of Theorem 4.13, if we consider the equation* (3.1) *and inequality* (3.9)*, by the same process we can verify that the equation* (3.1) *is Ulam-Hyers stable.*

2. *Under the assumptions of Theorem 4.13, if we consider the equation* (3.1) *and inequality*(3.11)*, we can use the same process to verify that the equation* (3.1) *is Ulam-HyersRassias stable with respect to* (φ, ψ).

3. *The above results can be extended to the case of the equation*(3.2).

3.5 Applications

Exemple 3.7. *Let's analyze the stability of the given example:*

$$\begin{cases} u'(t) = 0, \quad t \in J' = [0,2] - \{1\}, \\ \Delta u|_{t=1} = \sin(u(1)) \\ u(2) = 3 \end{cases} \tag{3.12}$$

To study the stability, we'll assume that there exists a solution y to the impulsive differential equation with the following inequalities:

$$\begin{cases} |y'(t)| \leq \varepsilon, \quad t \in J' = [0,2] - \{1\}, \\ |\Delta y|_{t=1} - \sin(y(1)) \mid\leq \varepsilon, \quad \varepsilon > 0. \end{cases} \tag{3.13}$$

Now, let's find an expression for $y(t)$. Since $u'(t) = 0$, we have $u(t) = c$ for some constant c. Using the initial condition $u(2) = 3$, we find that $c = 3$. Therefore, $u(t) = 3$ for all $t \in [0,2]$.

Now, let's compare $y(t)$ with $u(t)$:

$$|y(t) - u(t)| = |y(t) - 3|.$$

Since $y(t)$ is a solution to the impulsive differential equation, it satisfies the inequalities:

$$|y'(t)| \leq \varepsilon, \quad t \in J' = [0,2] - \{1\},$$
$$|\Delta y|_{t=1} - \sin(y(1))| \leq \varepsilon.$$

Taking $t = 1$, we have:

$$|y(1) - 3| = |\Delta y|_{t=1} - \sin(y(1))|| \leq \varepsilon.$$

Since ε is a positive constant, we can conclude that $|y(t) - 3| \leq \varepsilon$ for all $t \in [0,2]$. Therefore, the solution $y(t)$ is stable with respect to the given inequalities.

To further analyze the stability, let's consider the function $g(t) = y'(t)$. Since $y'(t) = g(t)$, we have $g(t) = 0$ for $t \in J' = [0, 2] - \{1\}$.

Using the impulsive boundary condition $\Delta y|_{t=1} = \sin(y(1))$, we can write $\sin(y(1)) = g(1)$.

Now, let's integrate the equation $y'(t) = g(t)$ from t to 2:

$$y(t) = y(2) - \int_t^2 g(s)ds.$$

Using the initial condition $y(2) = 3$, we have:

$$y(t) = 3 - \int_t^2 g(s)ds.$$

Next, we consider the solution $u(t)$ of the equation $u'(t) = 0$ with the boundary condition $\Delta u|_{t=1} = \sin(u(1))$.

Since $u'(t) = 0$, we have $u(t) = c$ for some constant c. Using the boundary condition $\Delta u|_{t=1} = \sin(u(1))$, we find:

$$\sin(u(1)) = u(1) - u(1^-),$$

which simplifies to $\sin(u(1)) = u(1) - c$. Rearranging the terms, we get:

$$c = u(1) - \sin(u(1)).$$

Therefore, the solution $u(t)$ is given by:

$$u(t) = u(1) - \sin(u(1)).$$

Now, let's compare $y(t)$ with $u(t)$:

$$|y(t) - u(t)| = \left|3 - \int_t^2 g(s)ds - u(1) + \sin(u(1))\right|.$$

Since $|g(t)| \leq \varepsilon$ for $t \in [0, 2]$, we can bound the integral term as follows:

$$\left|\int_t^2 g(s)ds\right| \leq \varepsilon \int_t^2 ds = \varepsilon(2 - t).$$

Substituting this bound into the expression, we have:

$$|y(t) - u(t)| \leq |3 - u(1) + \sin(u(1))| + \varepsilon(2 - t).$$

Since $|3 - u(1) + \sin(u(1))|$ is a constant, we can denote it as δ:

$$|y(t) - u(t)| \leq \delta + \varepsilon(2 - t).$$

Therefore, we have shown that $|y(t) - u(t)|$ is bounded by $\delta + \varepsilon(2 - t)$ for $t \in [0, 2]$. This indicates that the solution $y(t)$ is stable with respect to the given inequalities.

In conclusion, based on the analysis, the solution $y(t)$ to the impulsive differential equation is stable with respect to

the given inequalities.

To establish the stability of the solution, we need to show that $|y(t) - u(t)| \leq M\varepsilon$ for $t \in [0, 2]$, where M is a constant.

From the previous derivation, we have:

$$|y(t) - u(t)| \leq \delta + \varepsilon(2 - t).$$

To proceed, we need to determine the values of δ and M.

Since $\delta = |3 - u(1) + \sin(u(1))|$ and $u(1) = u(1^-) - \sin(u(1^-))$, we can express δ in terms of $u(1^-)$:

$$\delta = |3 - (u(1^-) - \sin(u(1^-))) + \sin(u(1^-))| = |3 - u(1^-) + \sin(u(1^-))|.$$

Now, let's consider the function $f(x) = 3 - x + \sin(x)$. We can analyze the behavior of $f(x)$ to find a suitable upper bound for δ.

Taking the derivative of $f(x)$ with respect to x, we have:

$$f'(x) = -1 + \cos(x).$$

Since $-1 \leq \cos(x) \leq 1$ for any x, we can conclude that $f'(x) \leq 0$ for all x. Therefore, $f(x)$ is a decreasing function.

To find the maximum value of $f(x)$, we consider its critical points where $f'(x) = 0$.

Solving $-1+\cos(x)=0$, we get $\cos(x)=1$ and $x=2\pi k$ for any integer k.

Since $f(x)$ is decreasing, the maximum value of $f(x)$ occurs at the smallest critical point. Thus, $f(x)$ achieves its maximum value at $x=2\pi$. Evaluating $f(2\pi)$, we have:

$$f(2\pi)=3-2\pi+\sin(2\pi)=3-2\pi.$$

Therefore, we can bound δ as follows:

$$\delta=|3-u(1^-)+\sin(u(1^-))|\leq 3-2\pi.$$

Now, let's determine the value of M to complete the stability proof. From the inequality $|y(t)-u(t)|\leq\delta+\varepsilon(2-t)$, we need to ensure that $\delta+\varepsilon(2-t)\leq M\varepsilon$.

Since δ is a constant, we can choose M such that $M\geq\delta$. This guarantees that $\delta+\varepsilon(2-t)\leq M\varepsilon$ holds for any $t\in[0,2]$.

Therefore, we can choose $M=3-2\pi$ to satisfy the inequality.

In conclusion, for the given impulsive differential equation and inequalities, we have established that $|y(t)-u(t)|\leq(3-2\pi)\varepsilon$ for $t\in[0,2]$. This confirms the

generalized Ulam-Hyers-Rassias stability of the equation.

Exemple 3.8.

$$\begin{cases} u'(t)=\sin(u(t)), & t\in J'=[0,1] \\ u(0)=0 \end{cases}$$

We will study the generalized Ulam-Hyers-Rassias stability of this equation.

To begin the study, we first consider the reference solution $u_r(t)$, which satisfies the differential equation and the initial condition. In this case, we can find the reference solution by solving the differential equation:

$$u_r'(t)=\sin(u_r(t))$$

This is a separable differential equation, which we can solve as follows:

$$\begin{aligned}
\int \frac{du_r(t)}{\sin(u_r(t))} &= \int dt \\
\ln|\csc(u_r(t)) - \cot(u_r(t))| &= t + C_1 \\
\csc(u_r(t)) - \cot(u_r(t)) &= C_2 e^t \\
\frac{1}{\sin(u_r(t))} - \frac{\cos(u_r(t))}{\sin(u_r(t))} &= C_2 e^t \\
\csc(u_r(t)) &= \frac{1}{C_2 e^t} + \cot(u_r(t)) \\
\sin(u_r(t)) &= \frac{C_2 e^t}{1 + C_2 e^t}
\end{aligned}$$

To determine the constant C_2, we use the initial condition $u_r(0) = 0$:

$$\sin(u_r(0)) = \frac{C_2}{1 + C_2} = 0 \implies C_2 = 0$$

Thus, the reference solution is given by $u_r(t) = 0$.

Next, we consider a perturbed solution $u(t)$, which satisfies the inequality:

$$|\Delta u|_{t=0} \leq \varepsilon$$

Here, we assume that $|\Delta u|_{t=0}$ represents the jump in u at $t = 0$. We want to find conditions on $u(t)$ that ensure the stability of the equation.

Using the definition of the jump, we have:

$$|\Delta u|_{t=0} = |u(0^+) - u(0^-)| = |u(0) - u(0^-)| = |u(0)|$$

Therefore, the inequality becomes:

$$|u(0)| \leq \varepsilon$$

To study the stability, we want to find conditions on $u(t)$ such that $|u(t) - u_r(t)| \leq M\varepsilon$, where M is a constant.

For this example, we can see that $u(t)$ must be bounded in the interval $[0, 1]$, since

$\sin(u(t))$ is bounded. Thus, we have $|u(t)| \leq A$ for some constant A. Additionally, we have $|u(0)| \leq \varepsilon$.

Using the mean value theorem for integrals, we can write:

$$\begin{aligned}
|u(t) - u_r(t)| &= \left| \int_0^t u'(\tau) - u_r'(\tau) d\tau \right| \\
&= \left| \int_0^t \sin(u(\tau)) - \sin(u_r(\tau)) d\tau \right| \\
&\leq \int_0^t |\sin(u(\tau)) - \sin(u_r(\tau))| d\tau \\
&\leq \int_0^t |\sin(u(\tau))| + |\sin(u_r(\tau))| d\tau \\
&\leq \int_0^t 1 + 1 d\tau \\
&= 2t \\
&\leq 2
\end{aligned}$$

So we can choose $M = 2$ in the stability condition.

Therefore, for this example, we have established the generalized Ulam-Hyers-Rassias stability condition:

$$|u(t) - u_r(t)| \leq 2\varepsilon$$

This means that if the perturbed solution $u(t)$ satisfies $|u(0)| \leq \varepsilon$ and $|u(t)| \leq A$ for $t \in [0, 1]$, then it is stable with respect to the reference solution $u_r(t)$.

Chapter 4

Existence and Ulam stability of Solutions of Some Backward Impulsive fractional differential equations on Banach space

In this paper, using some well known classical fixed point theorems, we study the problem of the existence of solutions and their Ulam stability for the following backward impulsive differential equations in Banach spaces

$$\begin{cases} {}^{CH}{}_{t}D_{b}^{\alpha}u(t) = f(t, u(t)), \quad t \in J = [0, T], t \neq t_k \\ \Delta u|_{t=t_k} = I_k\left(u\left(t_k^-\right)\right), \quad k = 1, \cdots, m \\ u(T) = u_T \end{cases} \tag{4.1}$$

where ${}^{CH}{}_{t}D_{b}^{\alpha}$ is a the Caputo-Hadamard fractional derivative, $0 < \alpha < 1, 0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $\Delta u|_{t=t_k} = u\left(t_k^+\right) - u\left(t_k^-\right)$represents the jump of the function u at t_k, $I_k : \mathbb{R} \to \mathbb{R}, k = 1, 2, \ldots, n$, are appropriate functions, and $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a nonlinear real function. Our method of study is to convert the initial value problem (4.1) into an equivalent integral equation and apply Schaefer, Banach or Krasnoselskii fixed point theorem. Further, we prove the existence of a unique solution or at least one solution to this problem with local and nonlocal conditions. Consider the following nonlocal problem

$$\begin{cases} {}^{CH}{}_tD_b^{\alpha}u(t) = f(t,u(t)), \quad t \in J = [0,T], t \neq t_k, \quad k = 1, \cdots, m \\ \Delta u|_{t=t_k} = I_k\left(u\left(t_k^-\right)\right), \quad k = 1, \cdots, m \\ u(T) - g(u) = u_T, \end{cases} \tag{4.2}$$

where f and $I_k, k = 1, \ldots, m$, are defined as in the previous paragraph and $g : PC(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function. Nonlocal conditions were first investigated by Byszewski and Lakshmikantham [8]. Using the Banach fixed point theorem, they obtained conditions for the existence and uniqueness of mild solutions to nonlocal differential equations. Byszewski [2] proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problem.

4.1 Existence of Solutions

In this section, our attention is focused on the main results on the existence of a solution to the problem (4.1). We discuss conditions under which this problem has exactly one solution or at least one solution.

In the study of the problem (4.1), we will work with the following assumptions:

(A1) The function $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

(A2) There exists a positive constant λ such that for any $t \in [0,T]$ and $x, y \in \mathbb{R}$, $|f(t,x) - f(t,y)| \leq \lambda|x - y|$

(A3) There exists a positive constant θ such that $|f(t,x)| < \theta$ for any $t \in [0,T]$ and $x \in \mathbb{R}$.

(A4) $|f(t,x)| \leq r$ for any $t \in [0,T]$ and $x \in B_r, r \in \mathbb{R}_+$

(A5) There exists a constant $\mu > 0$, such that $|I_k(x) - I_k(y)| \leq \mu|x - y|$ for any $x, y \in \mathbb{R}$, $k = 1, \ldots, m$.

(A6) The functions $I_k : \mathbb{R} \to \mathbb{R}$ are continuous and there exists a positive constant γ such that $|I_k(x)| < \gamma$ for any $x \in \mathbb{R}, k = 1, \ldots, m$.

A function $u \in PC(J, \mathbb{R})$ will be called a solution to (4.1) if its derivative exists on $J' = J - \{t_k, k = 1, 2, 3, \ldots, n\}$ and u satisfies the equation

$${}^{CH}{}_tD_b^{\alpha}u(t) = f(t,u(t)), \quad t \in J'$$

and the conditions

$$\begin{aligned} &\Delta u|_{t=t_k} = I_k\left(u\left(t_k^-\right)\right), \quad k=1,\cdots,m, \\ &u(T) = u_T. \end{aligned}$$

Lemma 4.1. *A function u is a solution of the integral equation*

$$u(t) = \begin{cases} u_T - \frac{1}{\Gamma(\alpha)} \int_{t_m}^{T} \left(\ln \frac{T}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds, \quad \text{if } t \in (t_m, T) \\ u_T - \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) \\ - \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} \left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds, \quad \text{if } t \in (t_{m-k}, t_{m-k+1}) \end{cases} \tag{4.3}$$

if and only if u is a solution of the backward impulsive differential equation

$$\begin{cases} {}^{CH}{}_{t}D_b^{\alpha} u(t) = h(t), \quad t \in I = [0,T], t \neq t_k \\ \Delta u|_{t=t_k} = I_k\left(u\left(t_k^-\right)\right), \quad k = 1, \cdots, m \\ u(T) = u_T \end{cases} \tag{4.4}$$

Proof. Assume u satisfies (4.4). for $t \in (t_m, t_{m+1})$, where $t_{m+1} = T$; we have

$$u(t) = u_T - \frac{1}{\Gamma(\alpha)} \int_{t_m}^{T} \left(\ln \frac{T}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds$$

For $t \in (t_{m-1}, t_m)$

$$\begin{aligned}
u(t) &= u\left(t_m^-\right) - \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t_m} \left(\ln \frac{t_m}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds + \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\
&= -\left(-u\left(t_m^-\right)\right) - \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t_m} \left(\ln \frac{t_m}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\
&= -\triangle\left(u\left(t_m\right)\right) + u\left(t_m^+\right) - \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t_m} \left(\ln \frac{t_m}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\
&= -I_m\left(u\left(t_m^-\right)\right) + u_T - \frac{1}{\Gamma(\alpha)} \int_{t_m}^{T} \left(\ln \frac{T}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\
&- \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t_m} \left(\ln \frac{t_m}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds + \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds
\end{aligned}$$

For $t \in (t_{m-2}, t_{m-1})$

$$\begin{aligned}
u(t) &= u\left(t_{m-1}^{-}\right) - \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t_{m-1}} \left(\ln \frac{t_{m-1}}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\
&= -\left(-u\left(t_{m-1}^{-}\right)\right) - \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t_{m-1}} \left(\ln \frac{t_{m-1}}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\
&= -\triangle\left(u\left(t_{m-1}\right)\right) + u\left(t_{m-1}^{+}\right) - \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t_{m-1}} \left(\ln \frac{t_{m-1}}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\
&= -I_m\left(u\left(t_m^{-}\right)\right) - I_{m-1}\left(u\left(t_{m-1}^{-}\right)\right) + u_T - \frac{1}{\Gamma(\alpha)} \int_{t_m}^{T} \left(\ln \frac{T}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\
&- \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t_m} \left(\ln \frac{t_m}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds - \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t_{m-1}} \left(\ln \frac{t_{m-1}}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds
\end{aligned}$$

Then, by induction, we obtain : For $t \in \left(t_{m-k}, t_{m-k+1}\right), \quad k = 0, \ldots, m$

$$\begin{aligned}
u(t) &= u_T - \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^{-}\right)\right) - \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} \left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds
\end{aligned}$$

for $k = 1, 2, \ldots, m$. Conversely, assume that u satisfies the impulsive integral equation (5.3). If $t \in (t_m, T)$, then $u(T) = u_T$. If $t \in \left(t_{m-k}, t_{m-k+1}\right), k = 0, \ldots, m$, by differentiation, we get

$$^{CH}{}_{t}D_{b}^{\alpha} u(t) = h(t)$$

Also, we can easily show that

$$\Delta u|_{t=t_k} = I_k\left(u\left(t_k^-\right)\right), \quad k = 1, \cdots, m$$

□

Let's start by discussing the conditions under which problem (4.1) has a unique solution. The following result is based on the Banach fixed point theorem.

Theorem 4.1. *Assume that the function f verifies the conditions **(A1)**,**(A2)** and **(A5)** , and*

$$m\mu + \frac{\lambda(m+1)}{\Gamma(\alpha+1)}(\ln T)^{\alpha} < 1 \tag{4.5}$$

then the problem (4.1) has a unique solution in $PC(J,\mathbb{R})$.

Proof. We transform the problem (4.1) into a fixed point problem. Consider the operator $F : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined by

$$\begin{aligned} F(u)(t) = u_T - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln\frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s} ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s} ds. \end{aligned}$$

Clearly, a fixed point of the operator F is a solution of the problem (4.1). We use the Banach contraction principle to prove that F has a fixed point. We shall show that F is a contraction. Let $u, v \in PC(J,\mathbb{R})$. Then, for each $t \in J$, we have

$$
\begin{aligned}
|F(u)(t)-F(v)(t)| \leq & \sum_{t<t_k<T}\left|I_k\left(u\left(t_k^{-}\right)\right)-I_k\left(v\left(t_k^{-}\right)\right)\right| \\
& +\frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}}\left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{|f(s, u(s))-f(s, v(s))|}{s} d s \\
& +\frac{1}{\Gamma(\alpha)} \int_{t_k}^t\left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{|f(s, u(s))-f(s, v(s))|}{s} d s \\
& \leq \mu \sum_{t<t_k<T}\left|u\left(t_{m-p}^{-}\right)-v\left(t_{m-p}^{-}\right)\right| \\
& +\frac{\lambda}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}}\left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{|u(s)-v(s)|}{s} d s \\
& +\frac{\lambda}{\Gamma(\alpha)} \int_{t_k}^t\left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{|u(s)-v(s)|}{s} d s \\
& \leq m \mu\|u(t)-v(t)\|+\frac{m \lambda(\ln T)^\alpha}{\Gamma(\alpha)}\|u(t)-v(t)\| \\
& +\frac{\lambda(\ln T)^\alpha}{\Gamma(\alpha+1)}\|u(t)-v(t)\| \\
& \leq m \mu\|u(t)-v(t)\|+\frac{m \lambda(\ln T)^\alpha}{\Gamma(\alpha+1)}\|u(t)-v(t)\| \\
& +\frac{\lambda(\ln T)^\alpha}{\Gamma(\alpha+1)}\|u(t)-v(t)\| \\
& =\left[m \mu+\frac{\lambda(m+1)}{\Gamma(\alpha+1)}(\ln T)^\alpha\right]\|u(t)-v(t)\|
\end{aligned}
$$

Hence, by (4.5), F is a contraction. Then, by the Banach contraction principle, we deduce that F has a unique fixed point which is a solution of the problem (4.1). □

The following result provides sufficient conditions for the existence of at least one solution to the problem (4.1). It is based on the Schaefer's fixed point theorem.

Theorem 4.2. *If the conditions* ***(A1)***, ***(A2)***, ***(A3)***, ***(A5)*** *and* ***(A6)***, *are satisfied then the problem* (4.1) *has at least one solution in* $PC(J, \mathbb{R})$.

Proof. The proof of this result is divided in several steps

Step1: The operator F is continuous. Let (u_n) a sequence such that $u_n \to u$ on J. For all $t \in [0,T]$ then

$$\begin{aligned}
|F(u_n)(t) - F(u)(t)| \leq & \sum_{t<t_k<T} \left| I_k\left(u_n\left(t_k^-\right)\right) - I_k\left(u\left(t_k^-\right)\right) \right| \\
& + \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s} \right)^{\alpha-1} \frac{|f(s,u_n(s)) - f(s,u(s))|}{s} ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s} \right)^{\alpha-1} \frac{|f(s,u_n(s)) - f(s,u(s))|}{s} ds
\end{aligned}$$

Since f and $I_k, k = 1, \ldots, m$ are continuous functions, then

$$\|Fu_n - Fu\|_\infty \to 0 \text{ as } n \to \infty$$

which implies that F is continuous.

Step2: F maps bounded sets into the bounded sets in $PC(J,\mathbb{R})$. For all $u(t) \in B_r$ we have

$$
\begin{aligned}
|F(u)(t)| =& \left| u_T - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s} ds \right. \\
& \left. + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s} ds \right| \\
& \leq |u_T| + \left| \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) \right| + \left| \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s} ds \right| \\
& + \left| \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s} ds \right| \\
& \leq |u_T| + \sum_{t<t_k<T} \left| I_k\left(u\left(t_k^-\right)\right) \right| + \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{|f(s,u(s))|}{s} ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{|f(s,u(s))|}{s} ds \\
& \leq |u_T| + \sum_{t<t_k<T} \left| I_k\left(u\left(t_k^-\right)\right) \right| + \frac{1}{\Gamma(\alpha)} \|f(s,u(s))\| \sum_{t<t_k<T}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \frac{\left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1}}{s} ds \\
& + \frac{1}{\Gamma(\alpha)} \|f(s,u(s))\| \int_{t_k}^{t} \frac{\left(\ln \frac{t}{s}\right)^{\alpha-1}}{s} ds \\
& \leq |u_T| + m\gamma + \frac{m\theta\left(\ln T\right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\theta\left(\ln T\right)^{\alpha}}{\Gamma(\alpha+1)} \\
& \leq |u_T| + m\gamma + \frac{m\theta\left(\ln T\right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\theta\left(\ln T\right)^{\alpha}}{\Gamma(\alpha+1)} \\
& = |u_T| + m\gamma + \frac{\theta(m+1)}{\Gamma(\alpha+1)} \left(\ln T\right)^{\alpha} = \rho.
\end{aligned}
$$

Hence, the operator F maps the bounded set B_r into a bounded set B_ρ.

Step3: F maps bounded sets into the equicontinuous sets of $PC(J,\mathbb{R})$. Let $\tau_1, \tau_2 \in [0,T], \tau_1 < \tau_2$ and let $u \in B_r$, then

$$
\begin{aligned}
|F(u)\left(\tau_2\right) - F(u)\left(\tau_1\right)| \leq & \sum_{\tau_1<t_k<\tau_2} \left| I_k\left(u\left(t_k^-\right)\right) \right| \\
& + \int_{\tau_1}^{\tau_2} \left| f\left(s, u_n(s)\right) \right| \frac{ds}{s}
\end{aligned}
$$

which tends to zero when τ_1 tends to τ_2. By the precedent steps, together with the AscoliArzela theorem, therefore F is equicontinuous on interval $[t_k, t_{k+1}]$.

As a consequence of Step 1-3 together with the PC-type Arzela-Ascoli theorem, we conclude that $F : B_r \to B_\rho$ is continuous and completely continuous.

Step4: We show that the set $\Omega = \{u \in PC(J, \mathbb{R}) : u = \lambda F(u), \quad 0 < \lambda < 1\}$ is bounded. Let $u \in \Omega$, then $u = \lambda F(u)$, for some $0 < \lambda < 1$, then for all $t \in [0, T]$ we have

$$
\begin{aligned}
&|u(t)| = \lambda \mid u_T - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s} ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s} ds \mid \\
&\leq |u_T| + \sum_{t<t_k<T} \left|I_k\left(u\left(t_k^-\right)\right)\right| + \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{|f(s,u(s))|}{s} ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{|f(s,u(s))|}{s} ds \\
&\leq |u_T| + \sum_{t<t_k<T} \left|I_k\left(u\left(t_k^-\right)\right)\right| + \frac{1}{\Gamma(\alpha)} \|f(s,u(s))\| \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \|f(s,u(s))\| \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&\quad \leq |u_T| + m\gamma + \frac{m\theta\left(\ln T\right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\theta\left(\ln T\right)^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad \leq |u_T| + m\gamma + \frac{\theta m\left(\ln T\right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\theta\left(\ln T\right)^{\alpha}}{\Gamma(\alpha+1)} \\
&= |u_T| + m\gamma + \frac{\theta(m+1)}{\Gamma(\alpha+1)} \left(\ln T\right)^{\alpha}
\end{aligned}
$$

which prove that Ω is bounded. By the Schaefer's fixed point theorem, F has a fixed point which is a solution of the problem (4.1).

□

4.1.1 Nonlocal Backward Impulsive Fractional Differential Equations

Now, we generalize the results of the previous section to nonlocal impulsive differential equations(4.2) . Let us introduce the following assumptions:

(A7) There exists a positive constant C such that $|g(x)-g(y)| \leq C\|x-y\|$ for any $x, y \in PC(J, \mathbb{R})$

(A8) There exists a positive constant κ such that $|g(u)| \leq \kappa$ for any function $u \in PC(J, \mathbb{R})$.

The equation (4.2) is equivalent to the following integral equation

$$
\begin{aligned}
F(u)(t) = u_T + g(u) - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{f(s, u(s))}{s} ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s, u(s))}{s} ds.
\end{aligned}
$$

Theorem 4.3. *Assume that the function f verifies the conditions **(A2)**,**(A5)**, and*

$$
C + m\mu + \frac{\lambda(m+1)(\ln T)^{\alpha}}{\Gamma(\alpha+1)} < 1 \tag{4.6}
$$

then the problem (4.2) has a unique solution in $PC(J, \mathbb{R})$.

Proof. We transform the problem (4.2) into a fixed point problem. Consider the operator $F : PC(J, R) \to PC(J, R)$ defined by

$$
\begin{aligned}
F(u)(t) = u_T + g(u) - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{f(s, u(s))}{s} ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s, u(s))}{s} ds.
\end{aligned}
$$

It is clear that the fixed points of the operator F are solutions of the problem (4.2). We use the Banach contraction principle to prove that F has a fixed point. We shall show that

F is a contraction. Let $u, v \in PC(J, R)$. Then, for each $t \in J$, we have

$$
\begin{aligned}
|F(u)(t)-F(v)(t)| \leq & |g(u(t))-g(v(t))|+\sum_{t<t_k<T}\left|I_k\left(u\left(t_k^{-}\right)\right)-I_k\left(v\left(t_k^{-}\right)\right)\right| \\
& +\frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}}\left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1}|f(s, u(s))-f(s, v(s))| \frac{ds}{s} \\
& +\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t}\left(\ln \frac{t}{s}\right)^{\alpha-1}|f(s, u(s))-f(s, v(s))| \frac{ds}{s} \\
& \leq C|u(t)-v(t)|+\mu \sum_{t<t_k<T}\left|u\left(t_k^{-}\right)-v\left(t_k^{-}\right)\right| \\
& +\frac{\lambda}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}}\left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1}|u(s)-v(s)| \frac{ds}{s} \\
& +\frac{\lambda}{\Gamma(\alpha)} \int_{t_k}^{t}\left(\ln \frac{t}{s}\right)^{\alpha-1}|u(s)-v(s)| \frac{ds}{s} \\
& \leq C\|u(t)-v(t)\|+m\mu\|u(t)-v(t)\|+\frac{\lambda m\left(\ln T\right)^{\alpha}}{\Gamma(\alpha+1)}\|u(t)-v(t)\| \\
& +\frac{\lambda\left(\ln T\right)^{\alpha}}{\Gamma(\alpha+1)}\|u(t)-v(t)\| \\
& =\left(C+m\mu+\frac{\lambda m\left(\ln T\right)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\lambda\left(\ln T\right)^{\alpha}}{\Gamma(\alpha+1)}\right)\|u(t)-v(t)\| \\
& =\left(C+m\mu+\frac{\lambda(m+1)\left(\ln T\right)^{\alpha}}{\Gamma(\alpha+1)}\right)\|u(t)-v(t)\| .
\end{aligned}
$$

Hence, by (4.6), F is a contraction. Then, by the Banach contraction principle, we deduce that F has a unique fixed point which is a solution of the problem (4.2).

□

Theorem 4.4. *If* ***(A1)***,***(A3)***,***(A6)***,***(A7)*** *and* ***(A8)*** *are satisfied, and if* $C<1$, *and*

$$
\frac{|u_T|}{1-\left(C+m\gamma+\frac{(m+1)\theta(\ln T)^{\alpha}}{\Gamma(\alpha+1)}\right)} \leq r \tag{4.7}
$$

then the Problem (4.2) *has at least a solution in* $PC(J,\mathbb{R})$.

Proof. Define the operators P, and Q on the compact set $B_r = \{y \in X : y \leq r\} \subset X$ by

$$Pu(t) = u_T + g(u(t))$$

and

$$Qu(t) = -\sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s} ds$$
$$+\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s} ds$$

Firstly, P maps B_r into itself i.e. $PB_r \subset B_r$. For all $u(t) \in B_r$ we have

$$|Pu(t)| = |u_T + gu(t)| \leq |u_T| + |gu(t)| \leq r.$$

Hence, the operator P maps B_r into itself. We prove that P is a contraction map. Let $u, v \in PC(J, \mathbb{R})$, then

$$|Pu(t) - Pv(t)| = |gu(t) - gv(t)| \leq C|u(t) - v(t)|$$

then the operator P satisfies the contraction property, and

$$
\begin{aligned}
|Qv(t)| &\leq \sum_{t<t_k<T} \left| I_k \left(v \left(t_k^- \right) \right) \right| + \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s} \right)^{\alpha-1} \frac{|f(s,u(s))|}{s} ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s} \right)^{\alpha-1} \frac{|f(s,u(s))|}{s} ds \\
&\leq \sum_{t<t_k<T} \left| I_k \left(v \left(t_k^- \right) \right) \right| + \frac{1}{\Gamma(\alpha)} \|f(s,u(s))\| \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s} \right)^{\alpha-1} \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \|f(s,u(s))\| \int_{t_k}^{t} \left(\ln \frac{t}{s} \right)^{\alpha-1} \frac{ds}{s} \\
&\leq \left(m\gamma + \frac{m\theta \left(\ln T \right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(m+1)\theta \left(\ln T \right)^{\alpha}}{\Gamma(\alpha+1)} \right) \|v(s)\| \\
&= \left(m\gamma + \frac{(m+1)\theta \left(\ln T \right)^{\alpha}}{\Gamma(\alpha+1)} \right) \|v(s)\|.
\end{aligned}
$$

hence

$$
\begin{aligned}
|Pu(t) + Qv(t)| &\leq |Pu(t)| + |Qv(t)| \\
&\leq |u_T| + C|u(t)| + \left(m\gamma + \frac{(m+1)\theta \left(\ln T \right)^{\alpha}}{\Gamma(\alpha+1)} \right) |v(s)| \\
&\leq |u_T| + Cr + \left(m\gamma + \frac{(m+1)\theta \left(\ln T \right)^{\alpha}}{\Gamma(\alpha+1)} \right) r \\
&= |u_T| + \left(C + m\gamma + \frac{(m+1)\theta \left(\ln T \right)^{\alpha}}{\Gamma(\alpha+1)} \right) r \\
&\leq r.
\end{aligned}
$$

Therefore, if $u, v \in B_r$, then $Pu + Qv \in B_r$. Obviously, in view of the condition **(A1)**, Q is continuous and by the inequality (4.7) , it is uniformly bounded on B_r. Evidently, the equicontinuity of $Qv(t)$ follows from Theorem (4.2). Hence, by the Arzela Ascoli Theorem, $Q\left(B_r\right)$ is relatively compact which implies that Q is compact. Therefore, using Krasnoselkii Theorem, there exists a solution to equation (4.2) .

□

4.1.2 Ulam stability

In this section, we study the Ulam stability of the solution of the problem (4.1)

Let $\varepsilon > 0, \psi \geq 0$ and $\varphi \in PC\left(J, \mathbb{R}^{+}\right)$is nondecreasing. We consider the following inequalities

$$\begin{cases} |D^{\alpha}u(t) - f(t, u(t))| \leq \varepsilon, & t \in J', t \neq t_k \\ |\Delta u|_{t=t_k} - I_k\left(u\left(t_k^-\right)\right)| \leq \varepsilon, & k = 1, \cdots, m \end{cases} \tag{4.8}$$

$$\begin{cases} |D^{\alpha}u(t) - f(t, u(t))| \leq \varphi(t), & t \in J', t \neq t_k \\ |\Delta u|_{t=t_k} - I_k\left(u\left(t_k^-\right)\right)| \leq \psi, & k = 1, \cdots, m \end{cases} \tag{4.9}$$

and

$$\begin{cases} |D^{\alpha}u(t) - f(t, u(t))| \leq \varepsilon\varphi(t), & t \in J', t \neq t_k \\ |\Delta u|_{t=t_k} - I_k\left(u\left(t_k^-\right)\right)| \leq \varepsilon\psi, & k = 1, \cdots, m \end{cases} \tag{4.10}$$

Definition 4.1.1. *Equation (4.1) is Ulam-Hyers stable if there exists a real number $c_{f,m} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in PC^1(J, \mathbb{R})$ of inequality (4.8) there exists a solution $x \in PC^1(J, \mathbb{R})$ of equation (4.1) with*

$$|y(t) - x(t)| \leq c_{f,m}\varepsilon, t \in J$$

Definition 4.1.2. *Equation(4.1) is generalized Ulam-Hyers stable if there exists $\theta_{f,m} \in C\left(\mathbb{R}^{+}, \mathbb{R}^{+}\right)$, $\theta_{f,m}(0) = 0$ such that for each solution $y \in PC^1(J, \mathbb{R})$ of inequality (4.8) there exists a solution $x \in PC^1(J, \mathbb{R})$ of equation (4.1) with*

$$|y(t) - x(t)| \leq \theta_{f,m}(\varepsilon), t \in J$$

Definition 4.1.3. *Equation (4.1) is Ulam-Hyers-Rassias stable with respect to (φ, ψ) if there exists $c_{f,m,\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in PC^1(J, \mathbb{R})$ of inequality (4.10) there exists a solution $x \in PC^1(J, \mathbb{R})$ of equation (4.1) with*

$$|y(t) - x(t)| \leq c_{f,m,\varphi}\varepsilon(\varphi(t) + \psi), t \in J$$

Definition 4.1.4. *Equation (4.1) is generalized Ulam-Hyers-Rassias stable with respect to* (φ, ψ) *if there exists* $c_{f,m,\varphi} > 0$ *such that for each solution* $y \in PC^1(J, \mathbb{R})$ *of inequality (4.9) there exists a solution* $x \in PC^1(J, \mathbb{R})$ *of equation (4.1) with*

$$|y(t) - x(t)| \leq c_{f,m,\varphi}(\varphi(t) + \psi), t \in J$$

Remark 4.5. *A function* $y \in PC^1(J, \mathbb{R})$ *is a solution of inequality (4.8)if and only if there is* $g \in PC(J, \mathbb{R})$ *and a sequence* $g_k, k = 1, 2, \ldots, m$ *(which depend on* y *) such that :*

(i) $|g(t)| \leq \varepsilon, t \in J$ *and* $|g_k| \leq \varepsilon, k = 1, 2, \ldots, m$

(ii) $D^\alpha u(t) = f(t, u(t)) + g(t), t \in J'$

(iii) $\Delta y\left(t_k\right) = I_k\left(y\left(t_k^-\right)\right) + g_k, k = 1, 2, \ldots, m.$

We can have similar remarks for the inequalities(4.9) and (4.10).

So, the Ulam stabilities of the impulsive differential equations are some special types of data dependence of the solutions of impulsive differential equations.

Proposition 4.1. *If* $y \in PC^1(J, \mathbb{R})$ *is a solution of inequality (4.8), then* y *is a solution of the following inequality*

$$\begin{aligned} &| y(t) - u_T + \sum_{t<t_k<T} I_k\left(y\left(t_k^-\right)\right) + \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(\ln \frac{t_{k+1}}{s}\right)^{\alpha-1} f(s, y(s)) \frac{ds}{s} \\ &- \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, y(s)) \frac{ds}{s} \mid \leq \frac{\left(\alpha m \Gamma(\alpha) + m\left(\ln T\right)^\alpha + \left(\ln \frac{t}{T}\right)^\alpha\right)}{\alpha \Gamma(\alpha)} \varepsilon, \quad t \in J. \end{aligned}$$

Proof. Indeed, by remark (4.5), we have that

$$\begin{cases} D^\alpha y(t) = f(t, y(t)) + g(t), t \in J' \\ \Delta y\left(t_k\right) = I_k\left(y\left(t_k^-\right)\right) + g_k, k = 1, 2, \ldots, m. \end{cases}$$

Then, for $t \in (t_{m-k}, t_{m-k+1})$ for $k = 0, \ldots, m$.

$$
\begin{aligned}
&| \ y(t) - u_T + \sum_{p=0}^{k-1} I_{m-p}\left(y\left(t_{m-p}^-\right)\right) + \sum_{p=0}^{k-1} g_i + \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} \left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1} f(s,y(s)) \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} \left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1} g(s) \frac{ds}{s} - \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s,y(s)) \frac{ds}{s} \\
&- \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} g(s) \frac{ds}{s} | \leq |y(t) - u_T + \sum_{p=0}^{k-1} I_{m-p}\left(y\left(t_{m-p}^-\right)\right) \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} \left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1} f(s,y(s)) \frac{ds}{s} - \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s,y(s)) \frac{ds}{s} \ | \\
&+ \sum_{p=0}^{k-1} |g_i| + \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} \left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1} |g(s)| \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} g(s) \frac{ds}{s}.
\end{aligned}
$$

From this it follows

$$
\begin{aligned}
&| \ y(t) - u_T + \sum_{p=0}^{k-1} I_{m-p}\left(y\left(t_{m-p}^-\right)\right) + \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} \left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1} f(s,y(s)) \frac{ds}{s} \\
&- \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s,y(s)) \frac{ds}{s} \ | \leq \sum_{i=1}^{m} \varepsilon + \frac{\varepsilon}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_i}^{t_{i+1}} \left(\ln \frac{t_{i+1}}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&+ \frac{\varepsilon}{\Gamma(\alpha)} \int_{T}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&\leq \sum_{i=1}^{m} \varepsilon - \frac{\varepsilon}{\alpha\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_i}^{t_{i+1}} \left(\ln \frac{t_{i+1}}{s}\right)^{\alpha-1} \frac{ds}{s} + \frac{\varepsilon}{\alpha\Gamma(\alpha)} \int_{T}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&\leq m\varepsilon + \frac{\varepsilon}{\alpha\Gamma(\alpha)} \left(\sum_{i=1}^{m} \left(\ln \frac{t_{i+1}}{t_i}\right)^{\alpha} + \left(\ln \frac{t}{T}\right)^{\alpha} \right) \\
&\leq m\varepsilon + \frac{\varepsilon}{\alpha\Gamma(\alpha)} \left(m(\ln T)^{\alpha} + \left(\ln \frac{t}{T}\right)^{\alpha} \right) \leq \frac{\left(\alpha m\Gamma(\alpha) + m(\ln T)^{\alpha} + \left(\ln \frac{t}{T}\right)^{\alpha}\right)}{\alpha\Gamma(\alpha)} \varepsilon.
\end{aligned}
$$

□

Theorem 4.6. *If the assumptions **(A1)**, **(A2)** and **(A5)** hold. Then equation (4.1) is generalized Ulam-Hyers.*

Proof. Let $u \in PC^1(J,\mathbb{R})$ be a solution of inequality(4.9). Denote by x the unique solution of the backward impulsive problem

$$\begin{cases} D^\alpha u(t) = f(t,u(t)), \quad t \in J=[0,T], t \neq t_k \\ \Delta u|_{t=t_k} = I_k\left(u\left(t_k^-\right)\right), \quad k=1,\cdots,m \\ u(T) = u_T \end{cases}$$

Then we have

$$\begin{aligned} u(t) = u_T &- \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) - \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} \left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1} f(s,u(s)) \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s,u(s)) \frac{ds}{s}, \end{aligned}$$

where $t \in (t_{m-k}, t_{m-k+1})$ for $k=0,\ldots,m$. for each $t \in (t_{m-k}, t_{m-k+1})$, we have

$$\begin{aligned} &| u(t) - u_T + \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) + \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} \left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1} f(s,u(s)) \frac{ds}{s} \\ &- \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s,u(s)) \frac{ds}{s} \mid \\ &\leq \sum_{i=1}^{m} |g_i| + \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \left(\ln \frac{t_{i+1}}{s}\right)^{\alpha-1} \varepsilon \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \varepsilon \frac{ds}{s} \\ &\leq m\varepsilon + \frac{\varepsilon}{\alpha\Gamma(\alpha)} \sum_{i=0}^{m} \left(\ln \frac{t_{i+1}}{t_i}\right)^{\alpha} + \frac{\varepsilon}{\alpha\Gamma(\alpha)} \left(\ln \frac{t}{t_{m-k}}\right)^{\alpha} \\ &\leq m\varepsilon + \frac{\varepsilon}{\Gamma(\alpha+1)} m \left(\ln T\right)^\alpha + \frac{\varepsilon}{\Gamma(\alpha+1)} \left(\ln T\right)^\alpha = \frac{\varepsilon}{\Gamma(\alpha+1)} \left(m\varepsilon + (m+1)\left(\ln T\right)^\alpha\right) \end{aligned}$$

Hence for each $t \in (t_{m-k}, t_{m-k+1})$ for $k=0,\ldots,m$, it follows

$$
\begin{aligned}
|u(t)-x(t)| \leq & \mid u(t)-u_T+\sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^{-}\right)\right) \\
& +\frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}}\left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1} f(s, u(s)) \frac{ds}{s} \\
& -\frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t}\left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, u(s)) \frac{ds}{s} \\
& +\sum_{p=0}^{k-1}\left|I_{m-p}\left(u\left(t_{m-p}^{-}\right)\right)-I_{m-p}\left(x\left(t_{m-p}^{-}\right)\right)\right| \\
& +\frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}}\left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1}|f(s, u(s))-f(s, x(s))| \frac{ds}{s} \\
& +\frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t}\left(\ln \frac{t}{s}\right)^{\alpha-1}|f(s, u(s))-f(s, x(s))| \frac{ds}{s} \\
& \leq \frac{\varepsilon}{\Gamma(\alpha+1)}\left(m \varepsilon+(m+1)(\ln T)^{\alpha}\right)+\sum_{p=0}^{k-1} \mu_k\left|u\left(t_{m-p}^{-}\right)-x\left(t_{m-p}^{-}\right)\right| \\
& +\frac{\lambda}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}}\left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1}|u(s)-x(s)| \frac{ds}{s} \\
& +\frac{\lambda}{\Gamma(\alpha)} \int_{t_{m-k}}^{t}\left(\ln \frac{t}{s}\right))^{\alpha-1}|u(s)-x(s)| \frac{ds}{s} \\
& \leq \frac{\varepsilon}{\Gamma(\alpha+1)}\left(m \varepsilon+(m+1)(\ln T)^{\alpha}\right)+\|u-x\|_{PC} \sum_{p=0}^{k-1} \mu_k \\
& +\frac{\lambda\|u-x\|_{PC}}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}}\left(\ln \frac{t_{m-p+1}}{s}\right)^{\alpha-1} \frac{ds}{s} \\
& +\frac{\lambda\|u-x\|_{PC}}{\Gamma(\alpha)} \int_{t_{m-k}}^{t}\left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
& \leq \frac{\varepsilon}{\Gamma(\alpha+1)}\left(m \varepsilon+(m+1)(\ln T)^{\alpha}\right)+\|u-x\|_{PC} \sum_{p=0}^{k-1} \mu_k \\
& +\frac{\lambda\|u-x\|_{PC}}{\alpha \Gamma(\alpha)} \sum_{p=0}^{k}\left(\ln \frac{t_{m-p+1}}{t_{m-p}}\right)^{\alpha}+\frac{\lambda\|u-x\|_{PC}}{\alpha \Gamma(\alpha)}\left(\ln \frac{t}{t_{m-k}}\right)^{\alpha} \\
& \leq \frac{\varepsilon}{\Gamma(\alpha+1)}\left(m \varepsilon+(m+1)(\ln T)^{\alpha}\right)+\|u-x\|_{PC} \sum_{p=0}^{k-1} \mu_k
\end{aligned}
$$

$$+\frac{\lambda\|u-x\|_{PC}}{\Gamma(\alpha+1)}m\left(\ln T\right)^{\alpha}+\frac{\lambda\|u-x\|_{PC}}{\Gamma(\alpha+1)}\left(\ln\frac{t}{t_{m-k}}\right)^{\alpha}$$

From which we have

$$\begin{aligned}\|u-x\|_{PC} \leq & \frac{\varepsilon}{\Gamma(\alpha+1)}\left(m\varepsilon+(m+1)\left(\ln T\right)^{\alpha}\right)+\|u-x\|_{PC}\sum_{p=0}^{k-1}\mu_k \\ & +\frac{\lambda\|u-x\|_{PC}}{\Gamma(\alpha+1)}m\left(\ln T\right)^{\alpha}+\frac{\lambda\|u-x\|_{PC}}{\Gamma(\alpha+1)}\left(\ln\frac{t}{t_{m-k}}\right)^{\alpha}\end{aligned}$$

which implies that

$$\|u-x\|_{PC} \leq \frac{1}{1-\left[\sum_{p=0}^{k-1}\mu_k+\frac{\lambda}{\Gamma(\alpha+1)}m\left(\ln T\right)^{\alpha}+\frac{\lambda}{\Gamma(\alpha+1)}\left(\ln\frac{t}{t_{m-k}}\right)^{\alpha}\right]}\frac{\varepsilon}{\Gamma(\alpha+1)}\left(m\varepsilon+(m+1)\left(\ln T\right)^{\alpha}\right)$$

Then

$$\|u-x\|_{PC} \leq \frac{\left(m\varepsilon+(m+1)\left(\ln T\right)^{\alpha}\right)\varepsilon}{\Gamma(\alpha+1)-\left[\Gamma(\alpha+1)\sum_{p=0}^{k-1}\mu_k+\lambda m\left(\ln T\right)^{\alpha}+\lambda\left(\ln\frac{t}{t_{m-k}}\right)^{\alpha}\right]}$$

Thus, equation (4.1) is generalized Ulam-Hyers stable. □

4.1.3 Applications

Exemple 4.7. *Consider the backward impulsive fractional differential equation :*

$$\begin{cases} {}^{CH}{}_{t}D_{b}^{\alpha}u(t)=0, \quad t\in J'=[0,1]-\{1/3\}, \\ \Delta u\big|_{t=1/3}=\frac{\left|u\left(\frac{1}{3}^{-}\right)\right|}{1+\left|u\left(\frac{1}{3}^{-}\right)\right|}, \\ u(1)=1, \end{cases} \tag{4.11}$$

where $0<\alpha<1$*, and the inequalities*

$$\begin{cases} \left|{}^{CH}{}_{t}D_{b}^{\alpha}y(t)\right| \leq \varepsilon, \quad t \in J' = [0,1] - \{1/3\}, \\ \left|\Delta y\big|_{t=1/3} - \frac{\left|y\left(\frac{1}{3}^{-}\right)\right|}{1+\left|y\left(\frac{1}{3}^{-}\right)\right|}\right| \leq \varepsilon, \end{cases} \tag{4.12}$$

hold for some $\varepsilon > 0$.

Let $y \in PC^1([0,1],\mathbb{R})$ be a solution to the inequality (4.12). Then there exist $g \in PC^1([0,1],\mathbb{R})$ and $g_1 \in \mathbb{R}$ such that:

$$|g(t)| \leq \varepsilon, \quad t \in [0,1] \tag{4.13}$$

$$D^{\alpha}y(t) = g(t), \quad t \in J' = [0,1] - \{1/3\}, \tag{4.14}$$

$$\Delta y\big|_{t=\frac{1}{3}} = \frac{\left|y\left(\frac{1}{3}^{-}\right)\right|}{1+\left|y\left(\frac{1}{3}^{-}\right)\right|} + g_1. \tag{4.15}$$

Integrating (4.14) from t to 1 via (4.15), we obtain

$$\begin{aligned} y(t) &= y(1) - \left(I_{\frac{1}{3}}\left(y\left(t_{\frac{1}{3}}^{-}\right)\right) + g_1\right) - \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{3}}^{1}\left(\ln\frac{1}{s}\right)^{\alpha-1} g(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{3}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1} g(s)\frac{ds}{s} \\ &= 1 - \left(I_{\frac{1}{3}}\left(y\left(t_{\frac{1}{3}}^{-}\right)\right) + g_1\right) - \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{3}}^{1}\left(\ln\frac{1}{s}\right)^{\alpha-1} g(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{3}}^{t}\left(\ln\frac{1}{s}\right)^{\alpha-1} g(s)\frac{ds}{s}. \end{aligned}$$

Let us consider the solution u of (4.11) given by

$$u(t) = 1 - \frac{\left|u\left(\frac{1}{3}^{-}\right)\right|}{1+\left|u\left(\frac{1}{3}^{-}\right)\right|}.$$

Then we can write

$$
\begin{aligned}
|y(t)-u(t)| =& \left| \frac{\left|u\left(\frac{1}{3}^{-}\right)\right|}{1+\left|u\left(\frac{1}{3}^{-}\right)\right|} - \frac{\left|y\left(\frac{1}{3}^{-}\right)\right|}{1+\left|y\left(\frac{1}{3}^{-}\right)\right|} - g_1 \right.\\
&\left. - \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{3}}^{1}\left(\ln\frac{1}{s}\right)^{\alpha-1} g(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{3}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1} g(s)\frac{ds}{s} \right|\\
&\leq \left| \frac{\left|u\left(\frac{1}{3}^{-}\right)\right|}{1+\left|u\left(\frac{1}{3}^{-}\right)\right|} - \frac{\left|y\left(\frac{1}{3}^{-}\right)\right|}{1+\left|y\left(\frac{1}{3}^{-}\right)\right|} \right| + |g_1| + \left| \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{3}}^{1}\left(\ln\frac{1}{s}\right)^{\alpha-1} g(s)\frac{ds}{s} \right|\\
&+ \left| \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{3}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1} g(s)\frac{ds}{s} \right|\\
&\leq \left\| u\left(\frac{1^{-}}{3}\right) | - |y\left(\frac{1^{-}}{3}\right) \right\| + |g_1| + |g_1| + \left| \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{3}}^{1}\left(\ln\frac{1}{s}\right)^{\alpha-1} g(s)\frac{ds}{s} \right|\\
&+ \left| \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{3}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1} g(s)\frac{ds}{s} \right|\\
&\leq \left| u\left(\frac{1}{3}^{-}\right) - y\left(\frac{1^{-}}{3}\right) \right| + |g_1| + \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{3}}^{1}\left(\ln\frac{1}{s}\right)^{\alpha-1} |g(s)|\frac{ds}{s}\\
&+ \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{3}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1} |g(s)|\frac{ds}{s}\\
&\leq \left| u\left(\frac{1}{3}^{-}\right) - y\left(\frac{1^{-}}{3}\right) \right| + \varepsilon + \frac{\varepsilon}{\Gamma(\alpha)}\int_{\frac{1}{3}}^{1}\left(\ln\frac{1}{s}\right)^{\alpha-1} \frac{ds}{s}\\
&+ \frac{\varepsilon}{\Gamma(\alpha)}\int_{\frac{1}{3}}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{ds}{s}\\
&= \left| u\left(\frac{1}{3}^{-}\right) - y\left(\frac{1}{3}^{-}\right) \right| + \varepsilon + \frac{\varepsilon}{\alpha\Gamma(\alpha)}\left(s(1-\ln(s))\right)^{\alpha}\Big|_{\frac{1}{3}}^{1} + \frac{\varepsilon}{\alpha\Gamma(\alpha)}\left(\ln\frac{t}{s}\right)^{\alpha}\Bigg|_{\frac{1}{3}}^{t}\\
&= \left| u\left(\frac{1}{3}^{-}\right) - y\left(\frac{1}{3}^{-}\right) \right| + \varepsilon + \frac{\varepsilon}{\alpha\Gamma(\alpha)}\left(\frac{2+3\ln(3)}{3}\right)^{\alpha} - \frac{\varepsilon}{\alpha\Gamma(\alpha)}(\ln 3t)^{\alpha}\\
&= \left| u\left(\frac{1}{3}^{-}\right) - y\left(\frac{1}{3}^{-}\right) \right| + \varepsilon + \frac{\varepsilon}{\alpha\Gamma(\alpha)}\left(\frac{2+3\ln(3)}{3}\right)^{\alpha} - \frac{\varepsilon}{\alpha\Gamma(\alpha)}(\ln 3t)^{\alpha}\\
&\leq \left| u\left(\frac{1}{3}^{-}\right) - y\left(\frac{1}{3}^{-}\right) \right| + \varepsilon + \frac{\varepsilon}{\alpha\Gamma(\alpha)}\left(\frac{2+3\ln(3)}{3}\right)^{\alpha} - \frac{\varepsilon}{\alpha\Gamma(\alpha)}(\ln 3t)^{\alpha}, \quad t\in[0,1]\\
&\leq 3\varepsilon + \frac{\varepsilon}{\Gamma(\alpha+1)}\left(\frac{2+3\ln(3)}{3}\right)^{\alpha} - \frac{\varepsilon}{\Gamma(\alpha+1)}(\ln 3t)^{\alpha}
\end{aligned}
$$

$$= \varepsilon \left(3 + \frac{1}{\Gamma(\alpha+1)} \left(\frac{2 + 3\ln(3)}{3} \right)^{\alpha} - \frac{1}{\Gamma(\alpha+1)} \left(\ln 3t \right)^{\alpha} \right), \quad t \in [0,1]$$

Thus, Equation (4.11) is generalized Ulam-Hyers stable, which is a special case of generalized Ulam-Hyers-Rassias stable.

Conclusion

In this thesis, our focus was on investigating the solutions and Ulam stability of backward impulsive differential equations on Banach spaces. We encountered several challenges throughout our research, which we successfully tackled by utilizing various classical fixed point theorems. By imposing suitable conditions on the nonlinear term, we were able to establish the existence of solutions for our problem.

To obtain a unique solution, we employed the Banach contraction principle, while other fixed point theorems such as Schaefer's and Krasnosel'skii's were used to obtain at least one solution. We considered both local and nonlocal conditions in our analysis. Additionally, we derived generalized-Ulam-Hyers-Rassias stability results for our problem, which showcased the robustness and consistency of our theoretical findings.

Furthermore, we extended our study to encompass both ordinary and fractional differential equations. By comparing these two types of differential equations, we aimed to provide insights into their respective characteristics and applications. It is worth noting that backward impulsive differential equations and their stability have wide-ranging practical implications.

Overall, our research contributes to the understanding of backward impulsive differential equations on Banach spaces, their solution existence, and stability properties. The provided examples serve to illustrate the practical relevance and validity of our theoretical results.

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Annex

Bernhard Riemann (1826-1866) , was a German mathematician who made significant contributions to several areas of mathematics, particularly in the field of analysis and geometry. His work had a profound impact on the development of modern mathematics and laid the foundation for many subsequent advancements.

Riemann was born on September 17, 1826, in Breselenz, a village in the Kingdom of Hanover (now part of Germany). From a young age, he showed exceptional mathematical talent and quickly gained recognition for his abilities. He entered the University of Göttingen in 1846, where he studied under mathematicians such as Carl Friedrich Gauss and Ernst Weber.

One of Riemann's most famous works is his 1854 habilitation dissertation, "On the Hypotheses Which Underlie Geometry." In this groundbreaking paper, Riemann introduced the concept of Riemannian geometry, which extends the ideas of Euclidean geometry to curved spaces. His work laid the groundwork for the later development of Einstein's theory of general relativity, which describes the gravitational force as the curvature of spacetime.

Riemann also made significant contributions to complex analysis. He introduced the Riemann surface, a two-dimensional surface that allows for the study of complex functions with multiple values. His insights into complex analysis provided a deeper understanding of the behavior of functions of a complex variable, leading to the development of many important mathematical tools and theorems.

Another important contribution of Riemann is his work on the theory of functions. He introduced the Riemann integral, a generalization of the definite integral that allowed for the integration of a broader class of functions. His ideas laid the foundation for the modern theory of integration, which is fundamental to many branches of mathematics and physics.

Riemann's work also had an impact on number theory. He made significant progress on the Riemann Hypothesis, one of the most famous unsolved problems in mathematics, which concerns the distribution of prime numbers. Although he didn't prove the hypothesis himself, his insights and conjectures have inspired generations of mathematicians in their pursuit of understanding prime numbers.

Tragically, Riemann's life was cut short by tuberculosis, and he passed away at the age of 39 on July 20, 1866, in Selasca, Italy. Despite his short life, his contributions to mathematics were profound and continue to influence the field to this day. Bernhard Riemann's ideas and theorems are regarded as some of the most important and influential in the history of mathematics.

**Stanislaw Ulam (1909-1984)** was a Polish-American mathematician and physicist known for his contributions to a wide range of fields, including mathematics, physics, and computer science. He made significant contributions to number theory, set theory, and the development of the atomic bomb during World War II. Ulam was also instrumental in the early development of computer science and the field of computational mathematics.

Born on April 13, 1909, in Lwów, Poland (now Lviv, Ukraine), Ulam demonstrated exceptional mathematical talent from a young age. He studied mathematics at the Lviv Polytechnic Institute and later pursued his doctoral studies at the University of Warsaw, where he earned his Ph.D. in 1933.

Ulam's work in mathematics covered diverse areas, including number theory, group theory, and set theory. He made significant contributions to the study of prime numbers and worked on problems related to the distribution of prime numbers, which led to advancements in analytic number theory.

During World War II, Ulam played a crucial role in the Manhattan Project, the research and development project that produced the first atomic bombs. He worked on the design of the bomb and made important contributions to the development of the implosion method, a technique used to compress the fissile material in the bomb's core. Ulam's contributions to the project were highly valued and recognized.

In the field of computational mathematics, Ulam was a pioneer. He made important contributions to the field of Monte Carlo simulations, a numerical technique that uses random sampling to solve complex mathematical problems. Ulam recognized the potential of this method and applied it to a wide range of scientific and engineering problems, including neutron transport calculations, the calculation of pi, and the behavior of nuclear particles.

Ulam was also known for his collaboration with John von Neumann, another influential mathematician and computer scientist. They worked together on the development of the Monte Carlo method and other projects, and their collaboration led to significant advancements in computational mathematics.

In addition to his scientific contributions, Ulam was known for his intellectual curiosity and his ability to bridge multiple disciplines. He had a wide range of interests, including biology, economics, and history, and he made important contributions to these fields as well.

Stanislaw Ulam passed away on May 13, 1984, in Santa Fe, New Mexico, USA. His work continues to be influential in various areas of mathematics, physics, and computer science. Ulam's contributions to the Manhattan Project and the development of computational mathematics have had a lasting impact on scientific and technological advancements.

 **J. Hadamard refers to Jacques Hadamard (1865-1963)**, a French mathematician known for his contributions to various fields of mathematics, including analysis, number theory, and mathematical physics. He made significant advancements in the theory of partial differential equations, the theory of functions, and the study of prime numbers.

Born on December 8, 1865, in Versailles, France, Hadamard displayed exceptional mathematical talent from a young age. He studied at the École Normale Supérieure in Paris and later became a professor at the Collège de France.

Hadamard made significant contributions to the theory of functions of a complex variable, particularly in the area of singularities and the growth of entire functions. His research on entire and meromorphic functions led to the development of the Hadamard factorization theorem, which provides a representation of an entire function as a product of exponential factors. This theorem has applications in complex analysis and the theory of partial differential equations.

In the field of number theory, Hadamard worked on the distribution of prime numbers. He made significant progress on the prime number theorem, which establishes the asymptotic behavior of prime numbers. Hadamard, along with Charles Jean de la Vallée-Poussin, independently proved the theorem in 1896.

The prime number theorem is considered one of the most important results in number theory.

Hadamard also made important contributions to mathematical physics, particularly in the area of partial differential equations. He worked on the theory of hyperbolic equations and introduced the concept of characteristic curves, which are fundamental in the study of wave propagation.

Apart from his mathematical achievements, Hadamard was known for his influential work in psychology. He conducted research on perception and memory, exploring the mathematical aspects of these fields. His book "Psychology of Invention in the Mathematical Field" examines the creative process and the mindset of mathematicians.

Jacques Hadamard received numerous honors and awards for his work, including the Bolyai Prize, the Sylvester Medal, and election to the French Academy of Sciences. He passed away on October 17, 1963, in Paris, France, leaving behind a rich legacy of mathematical achievements and interdisciplinary contributions.

Krasnoselskii refers to Mark Aleksandrovich Krasnoselskii (1920-1997), a prominent Soviet mathematician known for his contributions to functional analysis and nonlinear operator theory. He made significant advancements in the study of fixed-point theory, nonlinear equations, and nonlinear functional analysis.

Born on October 27, 1920, in Moscow, Russia, Krasnoselskii studied at Moscow State University, where he later became a professor. He played a key role in the development of the Moscow School of Nonlinear Analysis, which had a profound impact on the field.

Krasnoselskii's most notable contribution is his work on fixed-point theory. He developed the concept of a cone in a Banach space, which led to the development of fixed-point theorems for nonlinear operators. The Krasnoselskii fixed-point theorem, also known as the cone-contraction theorem, is a fundamental result in nonlinear functional analysis and has numerous applications in various areas of mathematics and physics.

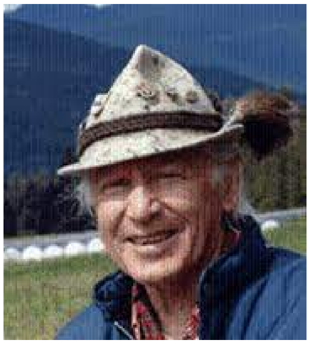
His research on nonlinear equations and functional analysis also contributed to the understanding of bifurcation theory, stability theory, and the study of nonlinear partial differential equations. Krasnoselskii's work had a significant impact on the development of mathematical physics and engineering applications.

Krasnoselskii authored several influential books, including "Topological Methods in the Theory of Nonlinear Integral Equations" and "Positive Solutions of Operator Equations." These books are widely regarded as important references in the field of nonlinear analysis.

Throughout his career, Krasnoselskii received several prestigious awards and honors, including the Lenin Prize, the Order of the Red Banner of Labour, and membership in the Russian Academy of Sciences.

Mark Aleksandrovich Krasnoselskii passed away on October 15, 1997, in Moscow, leaving behind a significant legacy in the field of functional analysis and nonlinear operator theory. His contributions continue to be influential and widely studied by mathematicians and researchers in various branches of mathematics and its applications.

Joseph Liouville (1809-1882): was a French mathematician known for his contributions to a wide range of mathematical fields, including analysis, number theory, and mathematical physics. He made significant contributions to the theory of elliptic functions, complex analysis, and transcendental numbers. Liouville's theorem, named after him, states that every bounded entire function must be constant, which has applications in complex analysis and the theory of differential equations. He also worked on the development of Riemann surfaces and the approximation of algebraic numbers by rational numbers.



Michele Caputo: Michele Caputo is an Italian physicist known for his contributions to the field of condensed matter physics. He has made significant advancements in the study of disordered systems, including the development of theoretical models and methods to understand the behavior of disordered materials.