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Presented by

Temmar Abir

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High order partial differential equations for images restoration

Publicly defended on 20/ 07 / 2023, in front of committee members :

Mr. LAMAMRI Abdelkader	**MCB**	**University of Blida1**	**Chairman**
Mr. HACHAMA Mohammed	**Professor**	**University of Blida1**	**Supervisor**
Mrs. BOUTAOUS Fatiha	**MCA**	**University of Blida1**	**Co-Supervisor**
Mr. TALBI Mohamed el Amine	**MCA**	**University of Blida1**	**Examiner**

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ملخص

في هذا العمل، يتم تقديم دراسة رياضية معمقة لنموذج قائم على الانتشار الغير متجانس الذي يمكن استخدامه لأغراض استعادة الصور. صمم النموذج لمعالجة عوامل تلف الصور الشائعة مثل الضوضاء والطمس. أثبت التحليل الرياضي للنموذج صحته واستقراريته وخاصية تقاربه. بالإضافة إلى ذلك، تم إجراء تجارب رقمية لإظهار فعالية النموذج في استعادة الصور المتدهورة. أعطت هذه الاستقصاءات نظرة قيمة حول الجوانب النظرية والعملية لقدرات النموذج في استعادة الصور. وبهذا، تسهم هذه البحث بشكل كبير في تطوير النماذج الرياضية لمعالجة الصور وتوفر أساسًا قويًا للبحوث المستقبلية في هذا المجال.

الكلمات الرئيسية: استعادة الصور، تخفيف الضوضاء مع الحفاظ على الحدود، الانتشار الغير خطي المتجانس.

Résumé

Dans le présent travail, une étude mathématique approfondie est présentée d'un modèle basé sur la diffusion anisotrope qui peut être utilisé à des fins de restauration d'images. Le modèle est conçu pour traiter les facteurs de corruption d'image courants tels que le bruit et le flou. L'analyse mathématique du modèle a démontré son caractère bien posé, sa stabilité et sa propriétés de convergence. De plus, des expériences numériques ont été menées pour montrer l'efficacité du modèle à restaurer les images dégradées. Ces enquêtes ont donné de précieux un aperçu des aspects théoriques et pratiques de la restauration d'image du modèle capacités. A ce titre, cette recherche contribue de manière significative au développement des mathématiques. modèles pour le traitement d'images et fournit une base solide pour la recherche future dans ce champ.

Mots-clés : Restauration d'images, débruitage préservant les contours, diffusion anisotrope non linéaire.

Abstract

In the present work, a thorough mathematical investigation is presented of a nonlinear anisotropic diffusion-based model that can be utilized for image restoration purposes. The model is designed to address common image corruption factors such as noise and blurring. The mathematical analysis of the model has demonstrated its well-posedness, stability, and convergence properties. Furthermore, numerical experiments have been conducted to show the model's effectiveness in restoring degraded images. These investigations have yielded valuable insights into both the theoretical and practical aspects of the model's image restoration capabilities. As such, this research significantly contributes to the development of mathematical models for image processing and provides a solid foundation for future research in this field.

Keywords: Image restoration, edge-preserving image denoising, nonlinear anisotropic diffusion.

Notations

PDE :partial differential equations.

RGB : Red, green, blue.

CMYK: Cyan, magenta, yellow, black.

HSV : Hue, saturation, value.

HSL : Hue, saturation, lightness.

DPI : Dots per inch.

Bpp : Bits per pixel.

JPEG: Joint Photographic Experts Group.

GIF : Graphics Interchange Format.

PNG : Portable Network Graphics.

BMP : Bitmap.

SVG : Scalable Vector Graphics.

MRI : Magnetic resonance imaging.

TV : Total Variation.

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Introdution

The partial differential equations of high order for image restoration is a fascinating research field aiming to enhance the quality of damaged or degraded images. Images are ubiquitous in our daily lives, whether they are photographs, videos, screenshots, or medical images. However, various factors can degrade the image quality, such as noise, blurriness, compression artifacts, or sensor defects.

In the context of image restoration, high-order partial differential equations (PDEs) play a crucial role. PDEs are powerful mathematical tools used to model and solve complex problems involving quantities varying in space and time. High-order partial differential equations are particularly suited to capture image features at higher levels of detail, enabling more precise restoration.

One commonly used approach for image restoration is to formulate the problem as a high-order PDE. These equations describe the desired properties of the restored image, such as spatial smoothness, contour continuity, fidelity to details, and suppression of unwanted noise. By using high-order PDEs, researchers and engineers can design sophisticated algorithms to estimate missing or corrupted information in an image, leveraging its spatial and structural properties.

The use of high-order PDEs in image restoration offers several advantages. Firstly, it allows for the consideration of complex image characteristics, such as textures, contours, and shapes, which are often challenging to model with simpler approaches. Additionally, high-order PDEs provide a solid mathematical framework to incorporate specific image constraints and regularities, leading to more accurate and consistent results.

However, it is important to note that solving high-order PDEs for image restoration also presents significant challenges. These equations can be complex and require advanced numerical methods for efficient resolution. Furthermore, applying these methods may demand significant computational resources due to the mathematical complexity and the size of processed images.

As part of our work, our main objective is to develop advanced methods and techniques using high-order partial differential equations (PDEs) to improve the quality of damaged or degraded images. We aim to address complex image restoration problems by precisely leveraging spatial and structural properties.

Dissertation Structure:

In order to carry out our work, our dissertation is structured into 3 different chapters:

Chapter 1 : Introduction to Images and their properties. In this chapter, we provide an overview of images, discussing their basic concepts such as digital representation, different image formats, and pixel properties. We also address the common challenges encountered in image restoration.

Chapter 2 : Theoretical study of the denoising and image restoration problem. This chapter is dedicated to the study of Partial Differential Equations (PDEs) and variational models used in our approach. We delve into the theoretical principles of EDPs and their applications in modeling image restoration problems.

Chapter 3 : Numerical resolution and Implementation. This chapter focuses on the practical aspects of numerically solving EDPs and implementing our developed image restoration system. We describe the numerical methods employed, optimization algorithms utilized, and implementation details.

Chapter 1

Preliminaries

1.1 Introduction

This preliminary chapter aims to establish definitions and fundamental concepts related to the theme of image and its characteristics. We will also address two important techniques for image processing, namely image restoration and image noise addition, along with its inverse, image denoising. Finally, we will examine two types of diffusion used in image processing: isotropic diffusion and anisotropic diffusion. Additionally, we will briefly discuss partial differential equations (PDEs), which are essential for understanding certain aspects of image processing.

1.2 The image and its characteristics

1.2.1 Definition of an image

An image can be defined as a visual representation of an object or a scene [1].

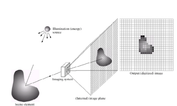


Figure 1.1: representation of a scene

In the context of digital image processing, an image is represented as numerical data organized in a matrix of pixels.

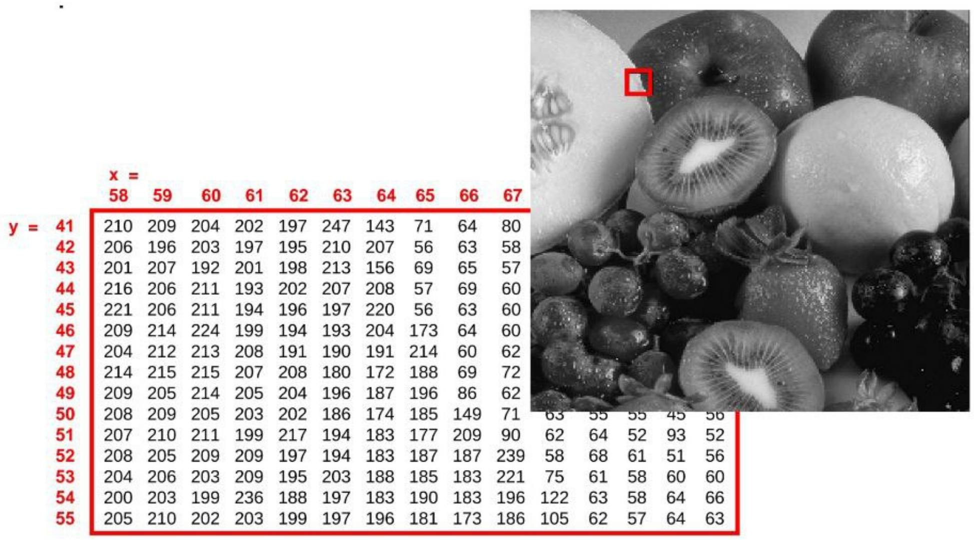


Figure 1.2: Intensity matrix

1.2.2 Types of images

There are different types of images, including grayscale images, binary images, and color images [2].

- **Grayscale images:**

The Grayscale images are composed of pixels that have varying brightness levels, typically ranging from black (minimum brightness) to white (maximum brightness). These images are often represented using a single channel, where each pixel's intensity value represents its brightness. The intensity values are usually represented as grayscale values, such as 0 (black) to 255 (white) in an 8-bit grayscale image. Grayscale images are commonly used in applications where only brightness in.

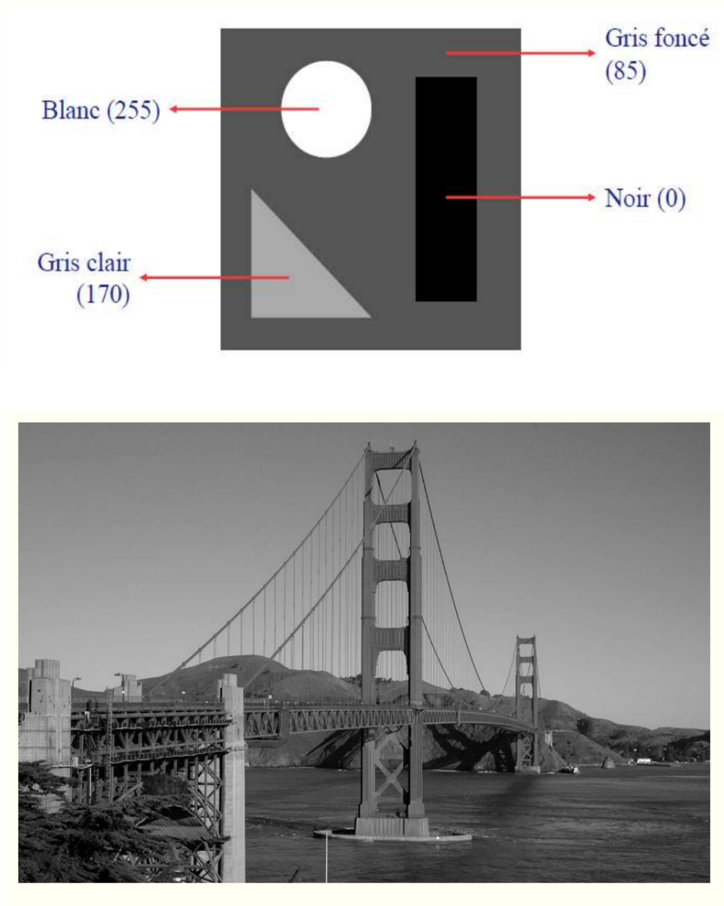


Figure 1.3: grayscale images

- **Binary images:**

The binary images are a type of image that contains only two pixel values, usually black and white. These images are often used to represent objects and backgrounds or to define regions of interest in an image. Each pixel in a binary image is typically represented by a single bit, where 0 represents black and 1 represents white. Binary images are commonly used in image analysis tasks such as edge detection, morphological operations, and object recognition.

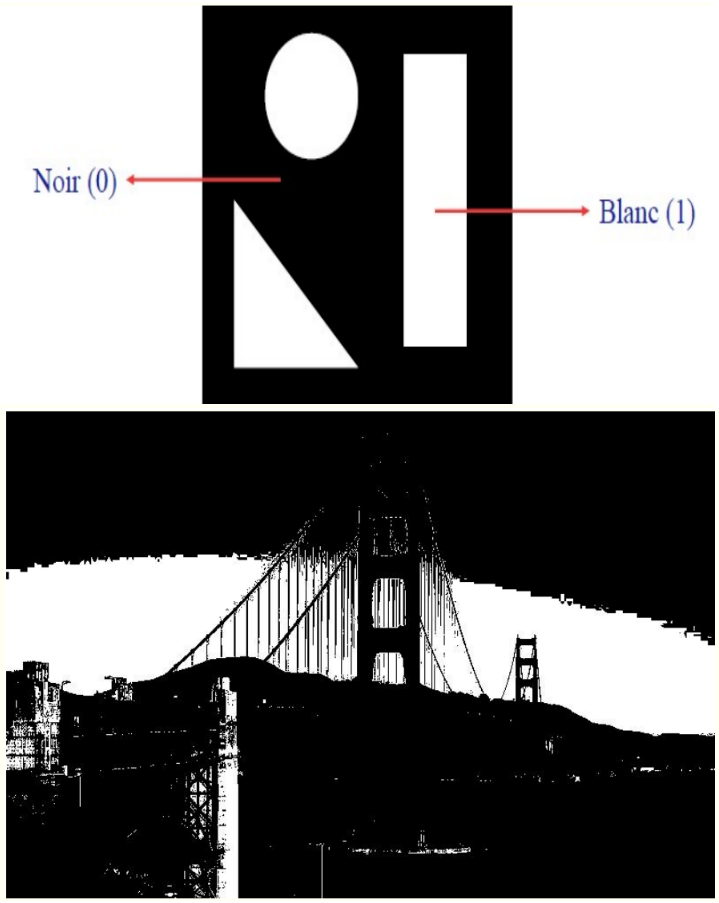


Figure 1.4: binary images

- **Color images:**

The color images contain additional color information beyond just brightness. They are typically represented using three color channels: red, green, and blue (RGB). In an RGB image, each pixel is represented by a combination of red, green, and blue color intensities. The intensities for each color channel can range from 0 to 255 in an 8-bit representation. By combining the intensities of the three color channels, a wide range of colors can be represented. Color images are commonly used In addition to RGB, there are other color models used for representing color images, such as CMYK (cyan, magenta, yellow, black) used in printing, and HSV (hue, saturation, value) and HSL (hue, saturation, lightness) used in color manipulations and image processing. These models provide different ways to represent and manipulate colors based on different properties and characteristics of human perception.

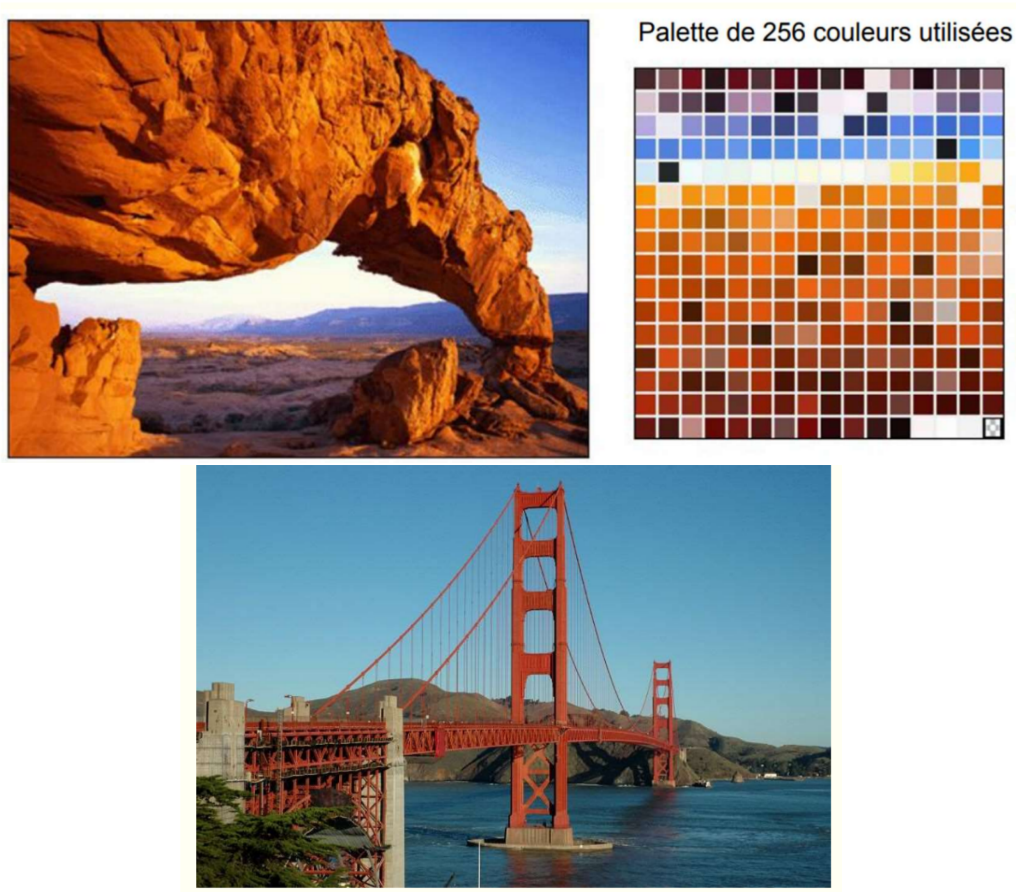


Figure 1.5: color images

1.2.3 Characteristics of an image

Images possess various characteristics that can be used for their analysis and processing [2]. Some common characteristics include :

a-Resolution: It is the number of points within a given length (in inches). It is expressed in dots per inch (DPI). An inch measures 2.54 cm, which is a British unit of measurement. Resolution allows establishing the relationship between the pixel definition of an image and its actual size representation on a physical medium (screen display, paper printing, etc.). **Example:**

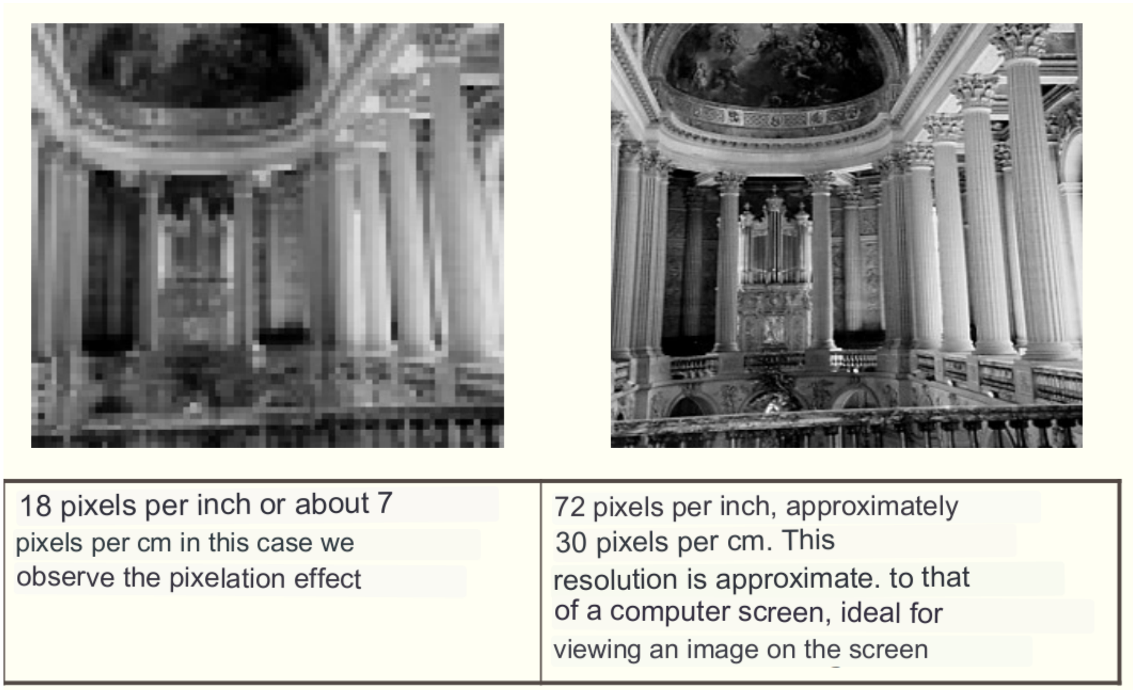
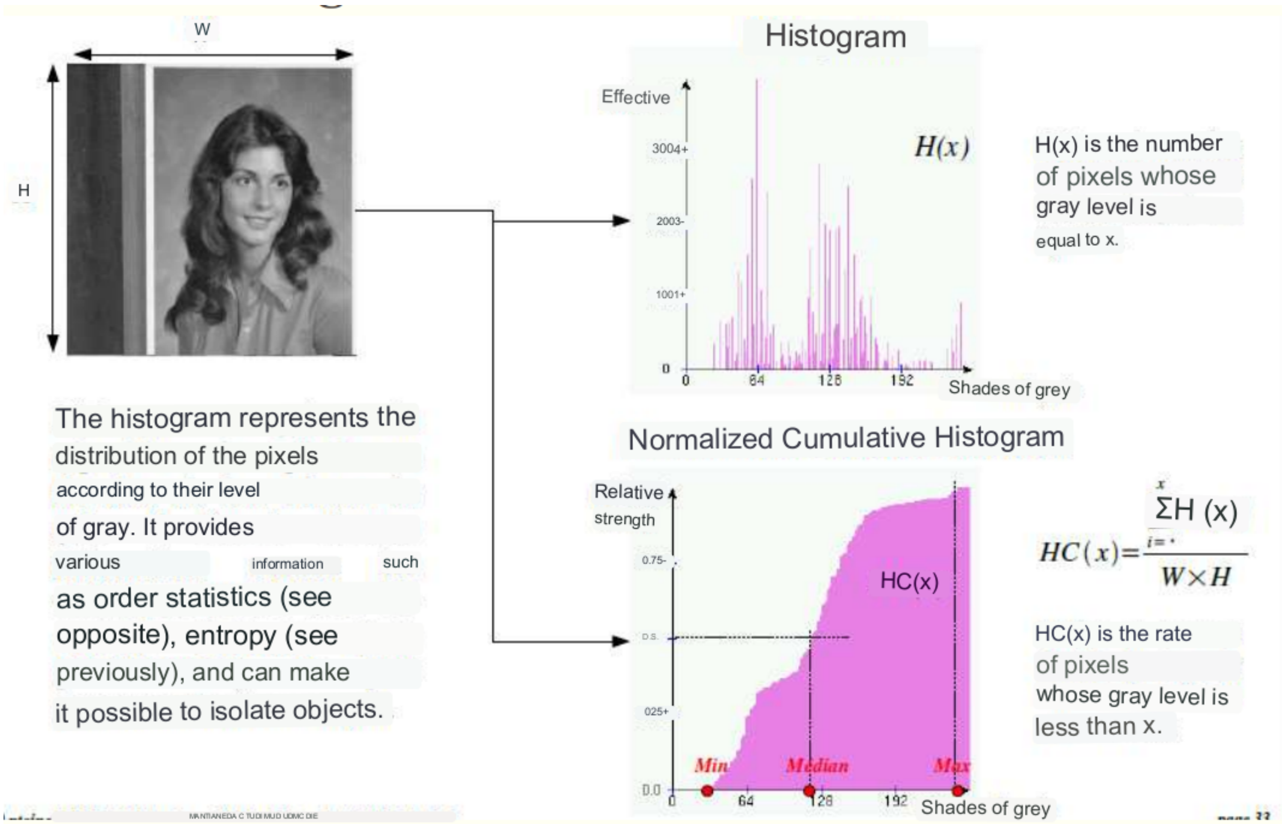


Figure 1.6: resolution of an image

b-Color coding (or color depth) refers to the amount of memory used by a digital image based on the coding of its color information. This is commonly referred to as color coding or color depth, expressed in bits per pixel (bpp): 1, 4, 8, 16 bits, and so on.

c-Histogram of an image:



d- Texture: statistical or geometric distribution of intensities in the image.

e- Contour: boundary between two (or a group of) pixels with significant difference in grayscale (color) values.

f- Region: group of pixels that exhibit similar characteristics (intensity, motion, etc.).

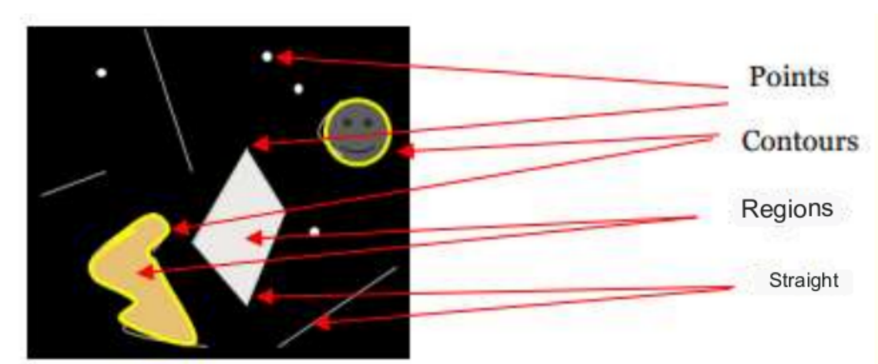


Figure 1.7: different characteristics of an image

1.2.4 Methods of digital image representation

1-Bitmap formats: Bitmap or raster formats represent images as a grid of individual pixels, where each pixel stores color information. Common bitmap formats include JPEG, PNG, BMP, and GIF. These formats are widely used for photographs and complex images as they can preserve fine details and color accuracy. However, they may result in larger file sizes, which can be a limitation for storage and transmission.

2-Vector formats: Vector formats represent images using mathematical formulas to define shapes and lines. They are based on geometric primitives such as points, lines, and curves. Vector formats, such as SVG (Scalable Vector Graphics), are resolution-independent and can be scaled without loss of quality. They are well-suited for logos, icons, and graphics with simple shapes. However, they may not be suitable for representing complex photographs or images with fine details.

3-Compressed formats: Compressed image formats use various algorithms to reduce file size while preserving visual quality. JPEG is a widely used compressed format that allows variable levels of compression, balancing between file size and image quality. It is commonly used for photographs on the web. However, excessive compression can lead to loss of image details and visible artifacts.

Each representation method has its trade-offs, and the choice of format depends on the specific requirements of the application, such as storage constraints, visual fidelity, and processing capabilities.

1.3 Image restoration

1.3.1 Definition of image restoration

Image restoration is a computational technique used in digital image processing to enhance the visual quality of images that have been affected by various degradations or distortions. These degradations may be caused by factors such as noise, blurriness, compression artifacts, or other forms of image corruption [3] .

The main goal of image restoration is to recover the original, undistorted information present in the image. This involves the removal or reduction of unwanted artifacts and the enhancement of important image features. By restoring the image, it becomes sharper, clearer, and more visually appealing, making it easier to interpret and analyze.

The process of image restoration typically involves the use of mathematical and statistical algorithms to estimate and compensate for the degradation effects. These algorithms analyze the image and attempt to reverse the degradation process, based on certain assumptions about the nature of the degradation. Some common methods used in image restoration include filtering, deconvolution, and denoising techniques.

Image restoration is widely used in various applications, including medical imaging, satellite and aerial imagery, forensic analysis, and digital photography. In medical imaging, for example, image restoration techniques can help improve the clarity and accuracy of medical scans, aiding in disease diagnosis and treatment planning. In the field of digital photography, image restoration can remove noise and artifacts from photos, enhancing their overall quality.

It is important to note that image restoration is a challenging task, as it involves dealing with incomplete and corrupted data. The success of the restoration process depends on the accuracy of the degradation model and the effectiveness of the restoration algorithm. Additionally, there is often a trade-off between the level of restoration and the potential introduction of artifacts or loss of fine details.

Overall, image restoration plays a crucial role in enhancing the visual quality of digital images, making them more suitable for various applications and improving their interpretability by both humans and machine-based systems..

1.3.2 Objectives of image restoration

The objectives of image restoration are as follows:

1-Noise removal: One of the main issues encountered in images is the presence of noise, which can be caused by factors such as sensor sensitivity, lighting conditions, or electromagnetic interference. The goal is to reduce or eliminate this unwanted noise to improve the image quality.

2-Blur reduction: Blur is another factor that can degrade the quality of an image. It can be caused by incorrect focus, camera movements, or unfavorable atmospheric conditions. The objective is to restore lost sharpness and details by reducing blur.

3-Contrast and brightness enhancement Sometimes, images may have low contrast or inadequate brightness, making it difficult to visualize important details. Image restoration aims to adjust the contrast and brightness to improve readability and interpretation of the

image.
4-Lost detail restoration: When an image is degraded, certain details may be lost or attenuated. The objective of image restoration is to restore these details to best reconstruct the original appearance of the scene or object.

1.3.3 Image restoration process

The image restoration process typically involves the following steps:
1-Acquisition of the degraded image: The first step is to obtain the degraded image, which can be obtained from a sensor, camera, or other imaging sources.
2-Degradation modeling: To restore the image, it is essential to understand how it has been degraded. This involves modeling the different sources of degradation, such as noise, blur, etc. This step helps determine the necessary processing operations to reverse or reduce these effects.
3-Selection of the restoration algorithm: Depending on the nature of the degradation, different restoration techniques and algorithms can be used. There is a variety of approaches, ranging from Fourier transform-based methods to adaptive filtering techniques or model-based approaches.
4-Application of the restoration algorithm: Once the algorithm is selected, it is applied to the degraded image to perform the restoration. This may involve filtering operations, regularization, or others.

1.4 Image noise addition and image denoising

1.4.1 Introduction to Image Denoising

Image denoising refers to the removal of unwanted disturbances or interference in an image, resulting in an improvement of its quality and readability. Noise can be caused by various factors such as camera sensors, lighting conditions, electromagnetic interference, or transmission errors.

1.4.2 Types of noise

There are different types of noise that can affect an image. Some commonly encountered types of noise include:

Additive Noise: This type of noise is typically caused by external factors such as electronic noise or electromagnetic interference. It manifests as random increases in the brightness levels of image pixels.

Multiplicative Noise: Multiplicative noise is often associated with varying lighting conditions or sensor defects. It affects the overall quality of the image by altering the relationship between pixel brightness levels.

Impulse Noise: Also known as "salt and pepper" noise, impulse noise leads to sporadic occurrence of extremely bright or dark pixels in the image. It can be caused by transmission errors or sensor defects.

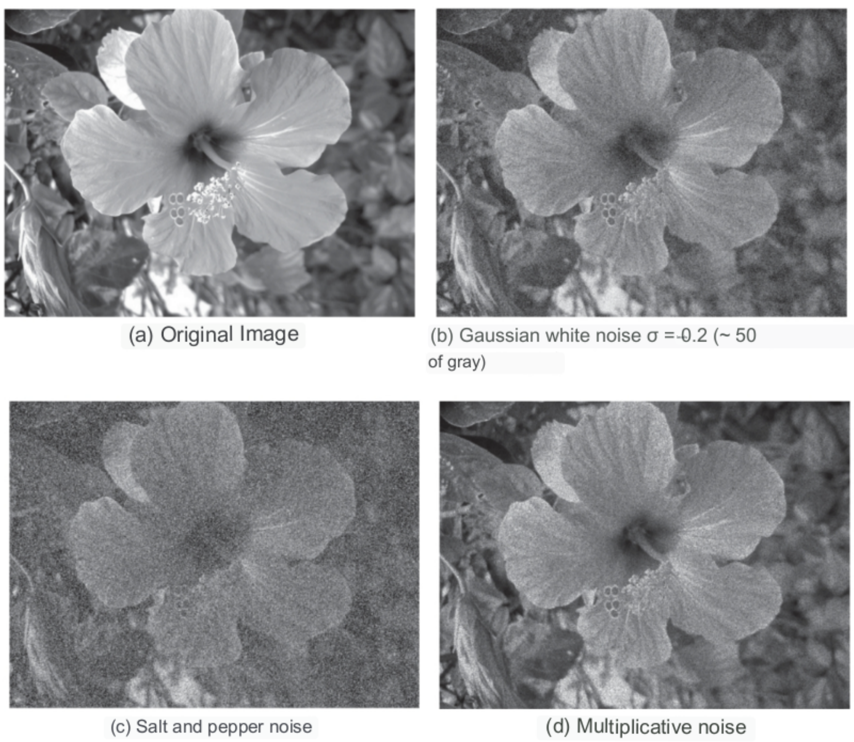


Figure 1.8: different type of noise

1.4.3 Effects of noise on images

Noise in an image can have several negative effects, including:

1. Reduction of clarity and details: Noise in an image can add random variations to pixel values, making it challenging to distinguish fine details and structures in the image. High levels of noise can blur edges and obscure subtle features, leading to a loss of image clarity. As a result, the visual quality of the image is diminished, and it becomes difficult to interpret or analyze the content accurately.

2. Alteration of colors: Different types of noise can affect the color representation of pixels in an image. For example, in additive noise, random variations in brightness levels can alter the intensity of individual color channels, causing color distortion. In multiplicative noise, variations in lighting conditions can impact the relationship between color channels, leading to changes in color fidelity. Consequently, the image may exhibit unnatural colors or lose its original color balance.

3. Undesirable increase in contrast: Noise can introduce random fluctuations in pixel values, resulting in both dark and bright spots throughout the image. This can cause an undesirable increase in contrast, with some areas becoming overly bright (overexposed) and others becoming excessively dark (underexposed). Such variations in contrast can make it difficult to perceive the overall content and may hinder image analysis and interpretation.

4. Reduction in measurement accuracy: In scientific or quantitative applications, images are often used for measurements and analysis. However, noise can introduce inaccuracies in the measured data, leading to reduced measurement precision. The noise can mask true details and introduce false information, which affects the accuracy of any quantitative results obtained from the image. This reduction in measurement accuracy can be particularly problematic in fields such as medical imaging, remote sensing, and scientific research.

Overall, the presence of noise in an image can significantly impact its quality and usability. To mitigate the negative effects of noise, image processing techniques, such as denoising algorithms and restoration methods, are employed to enhance the image's visual quality, recover lost details, and improve its overall fidelity. These techniques aim to effectively reduce noise while preserving essential image features, making the image more suitable for various applications, including visual inspection, computer vision tasks, and accurate data analysis.

1.4.4 Image denoising techniques

Image denoising is an essential process in image processing, as it aims to restore the visual quality of an image by reducing or eliminating undesirable noise effects. Here are more details on some of the image denoising techniques mentioned earlier:

1. Linear Filters:

- Mean Filter: This filter calculates the average of neighboring pixel values to replace the value of the central pixel. It is straightforward to implement, but it may not be very effective in reducing noise if the image contains fine details.

- Median Filter: The median filter replaces the value of the central pixel with the median of neighboring pixel values. It is particularly effective in reducing impulse noise, such as "salt and pepper" noise.

- Gaussian Filter: The Gaussian filter applies convolution with a Gaussian kernel to attenuate noise while preserving image edges. It is commonly used for reducing additive noise.

2. Thresholding denoising:

- Simple Thresholding denoising: This method applies a fixed threshold to pixel values. Pixels with values below or above the threshold are considered noisy and are corrected or replaced.

- Adaptive Thresholding Denoising: Here, the threshold is adjusted based on the local properties of the image, allowing for better adaptation to noise variations in different regions of the image.

3. Non-Local means denoising:

- Non-local means denoising is based on the principle that similar regions in the image will have similar statistics. It involves searching for similar image patches throughout the image and averaging them to estimate the value of the central pixel. This method is highly effective in reducing noise while preserving image edges and structures.

Each image denoising technique has its advantages and disadvantages in terms of denoising performance, preservation of image details, and computational time. The choice of the technique depends on the type of noise present in the image, the complexity of the image, and the specific objectives of the application. In general, a combination of different denoising techniques can be used to achieve the best results based on the use-case scenario. The performance of denoising methods can be evaluated using objective measures such as Signal-to-Noise Ratio (SNR), Structural Similarity Index (SSIM), and the perceived visual quality by human observers.

1.5 Isotropic diffusion

1.5.1 Definition:

Isotropic diffusion is a technique used in image processing to reduce noise and improve image quality through diffusion processes. It is based on the principle of propagating information equally in all directions throughout the image, without favoring any specific direction.

In isotropic diffusion, each pixel in the image exchanges information with its neighboring pixels, and this process is repeated iteratively. The diffusion process acts as a smoothing mechanism, adjusting the pixel values based on the values of nearby pixels. The objective is to reduce the impact of noise on the image while preserving essential structures and features. Unlike anisotropic diffusion, which considers the image's local gradients to guide the diffusion direction, isotropic diffusion treats all directions equally. As a result, isotropic diffusion tends to uniformly blur the image, which can be beneficial for noise reduction and producing smoother images.

The isotropic diffusion process can be mathematically formulated using partial differential equations ($PDEs$), where the diffusion rate is constant in all directions. Various numerical algorithms, such as explicit or implicit finite difference methods, can be used to implement isotropic diffusion in practice.

Overall, isotropic diffusion is a widely used technique in image processing and computer vision applications for denoising, smoothing, and enhancing image quality. Its ability to uniformly reduce noise in all directions makes it particularly useful in scenarios where preserving fine details is not a primary concern, and noise reduction is the main goal.

1.5.2 Principles of isotropic diffusion

Isotropic diffusion involves applying a diffusion operator to the image, which acts locally on each pixel based on its neighbors. This operator modifies the pixel values based on the differences in brightness between neighboring pixels. The diffusion process is iteratively repeated, gradually reducing noise while preserving important contours and structures in the image. The diffusion operator typically uses a partial differential equation (PDE) to model the diffusion. The most commonly used equation is the heat equation, also known as the diffusion equation, which describes the propagation of heat in a medium. This equation is suitable for isotropic diffusion as it ensures equal information propagation in all directions.

1.5.3 Applications of isotropic diffusion in image restoration

Isotropic diffusion has various applications in image restoration, including:

1-Noise reduction: One of the main applications of isotropic diffusion is noise reduction in images. Noise, such as Gaussian or impulse noise, can degrade the visual quality of images by adding random variations in brightness. Isotropic diffusion acts as a filter that attenuates these undesirable variations, thereby improving the clarity and sharpness of the image.

2. Contour Smoothing: Isotropic diffusion is also used to smooth object contours in an image. The contours of an image can be noisy or irregular, making object analysis and recognition challenging. By applying isotropic diffusion, the contours are regularized, making the image more coherent and facilitating further processing.

3. Enhancement of visual quality: By eliminating noise and smoothing contours, isotropic diffusion contributes to enhancing the overall quality of the image. Important details are preserved while undesirable artifacts are eliminated, making the image more visually appealing and easier to interpret.

4. Preprocessing for other operations: Isotropic diffusion can be used as a preprocessing step before applying other image processing operations. For example, in image segmentation, isotropic diffusion can improve the image quality before applying contour detection or object separation algorithms.

5. Restoration of old or damaged Images: Isotropic diffusion can be used to restore old or damaged images by reducing imperfections and improving the overall quality of the image. This can be valuable in the field of cultural heritage conservation or for restoring old photographs.

It is important to note that isotropic diffusion may also have limitations, particularly in terms of loss of fine details in the image. Therefore, it is essential to carefully select the diffusion parameters to achieve a balance between noise reduction and preservation of important image features. Depending on the specific characteristics of the image and the requirements of the application, other denoising or image restoration techniques may also be considered to achieve the best results.

1.6 Anisotropic diffusion

1.6.1 anisotropic diffusion:

Anisotropic diffusion is a powerful image processing technique that addresses the limitations of isotropic diffusion by taking into account the directional characteristics of an image . It is particularly effective in preserving and enhancing the edges and structures in the image while reducing noise.

In isotropic diffusion, information is propagated equally in all directions, leading to a smoothing effect that can blur edges and fine details in the image. However, in real-world images, edges and structures often have a preferred direction, and smoothing them uniformly can lead to loss of important information [4].

Anisotropic diffusion overcomes this limitation by using diffusion processes that consider the local image gradient. The diffusion rate is modulated based on the gradient magnitude, allowing for higher diffusion along smooth regions and lower diffusion along edges and contours. This adaptive diffusion process preserves important features while reducing noise in areas where image variations are less significant.

The anisotropic diffusion equation can be expressed as:

$$\frac{\partial u}{\partial t} = div(c(|\nabla u|)\nabla u)$$

where u is the image, t is the time, ∇u represents the image gradient, and $c(|\nabla u|)$ is a diffusion coefficient function that depends on the gradient magnitude. The diffusion coefficient function determines the degree of anisotropy and controls the diffusion process in different image regions.

Anisotropic diffusion is particularly useful in denoising images with complex structures, such as medical images, textured images, or images with intricate patterns. It effectively preserves the structural details while reducing noise, leading to visually appealing and more informative results.

However, like isotropic diffusion, anisotropic diffusion may also have some drawbacks. If the diffusion coefficient is not appropriately selected, it can lead to over-smoothing or under-smoothing of certain regions. Finding the right balance between noise reduction and preservation of image features requires careful parameter tuning.

In summary, anisotropic diffusion is a valuable tool in image restoration and denoising, especially in scenarios where preserving structural information and fine details are crucial. Its ability to adaptively diffuse information in different directions makes it a powerful technique for various applications in image processing and computer vision.

1.6.2 Principles of anisotropic diffusion

Anisotropic diffusion is based on the use of a diffusion operator that considers the local characteristics of the image, such as brightness gradients. This operator is designed to limit diffusion along the image's contours to preserve important details and structures. It promotes diffusion in homogeneous regions of the image while limiting diffusion along edges and contours. The anisotropic diffusion operator typically employs partial differential equations (PDEs) tailored to model diffusion. One of the most commonly used equations is the **Perona-Malik** anisotropic diffusion equation, which introduces a conductivity coefficient that controls diffusion based on the local characteristics of the image.

1.6.3 Comparison with isotropic diffusion

Anisotropic diffusion differs from isotropic diffusion primarily in its consideration of the directional characteristics of the image [5, 6, 7]. While isotropic diffusion propagates information equally in all directions, anisotropic diffusion adapts diffusion based on the image's structures and contours. This allows for better preservation of details and contours, which can be advantageous in certain image restoration scenarios.

1.6.4 Use of anisotropic diffusion in image restoration

Anisotropic diffusion finds applications in various areas of image restoration, including:

- **Noise reduction with contour preservation :** By limiting diffusion along contours, anisotropic diffusion effectively reduces noise while preserving important details and contours of the image.
- **Image segmentation :** Anisotropic diffusion can be used as a preliminary step in image segmentation, favoring diffusion in homogeneous regions and limiting diffusion between regions with high contrasts.
- **Enhancement of textured images:** Anisotropic diffusion is effective in improving the quality of textured images by reducing noise without altering fine textures.

• **Restoration of medical images:** Anisotropic diffusion is widely used in the restoration of medical images, such as magnetic resonance imaging (MRI) or tomographic images, to improve quality and facilitate diagnostic analysis.

Anisotropic diffusion provides a powerful alternative to isotropic diffusion in scenarios where the preservation of contours and directional structures is crucial. However, it is important to note that it can introduce certain artifacts, and the choice between anisotropic diffusion and isotropic diffusion will depend on the specificities of each application and the objectives of image restoration.

1.7 Weak convergence

1.7.1 Definition of weak convergence

Weak convergence is a concept used to describe the convergence of a sequence of functions or measures in a weaker sense compared to norm convergence [8]. It is also known as convergence in distribution or convergence in the weak-topology. Weak convergence focuses on the behavior of functions or measures when integrated against a set of test functions.

1.7.2 Relationship between weak convergence and norm convergence

In the context of function spaces, weak convergence is typically weaker than norm convergence. While norm convergence implies weak convergence, the converse is not always true. Weak convergence measures the convergence of functionals, such as integrals or function evaluations, rather than the pointwise convergence of functions. It provides a more flexible notion of convergence that allows for the convergence of a wider class of functions.

1.7.3 Importance of weak convergence in the analysis of Image restoration algorithms

Weak convergence plays a crucial role in the analysis of image restoration algorithms. It allows for the study of the convergence behavior of iterative algorithms used in image restoration, such as denoising or deblurring algorithms. By considering weak convergence, one can analyze the convergence properties of these algorithms in terms of functionals, which provides insights into their stability, accuracy, and robustness.

1.8 Functional spaces (Banach, Hilbert, Sobolev)

1.8.1 Definition of functional spaces

Complete metric space

Let (E, d) be a complete metric space, meaning a metric space in which every Cauchy sequence converges to an element of E.

Banach space

A Banach space is a complete normed vector space, which means it has an algebraic structure and a norm that satisfies certain mathematical properties [10]. Banach spaces are used to study the properties of functions and operators defined on these spaces.

Hilbert space

A Hilbert space is a Banach space that possesses an additional structure called an inner product, allowing the measurement of angles and lengths of vectors in the space. Hilbert spaces are used to study concepts such as convergence, orthogonality, and projections [11].

Sobolev space

The Sobolev space, denoted as $H^1(\Omega)$, is a functional space that encompasses functions whose partial derivatives up to order one are square integrable over a domain Ω. This space is used to model functions with properties of regularity and continuity [9].

Banach fixed-point theorem

Définition 1.1 *Let $\Phi : E \to E$ be a contractive mapping, which means that there exists a real constant k with $0 \leq k < 1$, satisfying:*

$$d(\Phi(x), \Phi(y)) \leq kd(x, y), \quad \forall\ x, y \in E$$

then, there exists a unique fixed point x^* of Φ such that $\Phi(x^*) = x^*$.

1.8.2 Role of functional spaces in the analysis and modeling of functions

Function spaces play a crucial role in the analysis and modeling of functions. They provide a framework for studying the properties and behavior of functions, such as continuity, differentiability, and integrability. Function spaces also facilitate the formulation and analysis of mathematical models and algorithms used in image restoration.

1.8.3 Examples off unctional spaces used in image restoration

In the context of image restoration, various function spaces are employed to represent images and model their properties. Some examples of function spaces commonly used in image restoration include:

L^2 **spaces:** These are Hilbert spaces consisting of square-integrable functions. They are often used to represent images in the frequency domain.

H^1 **spaces :** These are Sobolev spaces that contain functions with square-integrable derivatives up to the first order. They are utilized to model functions with certain regularity and smoothness properties.

1.9 Monotone and semi-continuous functions

.monotone and semi-continuous of $H^1(\Omega) to (H^1(\Omega))\prime$:

Monotone: This means that the operator satisfies a monotonicity condition. Specifically, for any pair of elements u and v in the space $H^1(\Omega)$, the operator satisfies the following inequality: $< A(u) - A(v), u - v > \geq 0$. This implies that the operator preserves or increases the order of the elements.

Semi-continuous: This means that the operator is not necessarily continuous but exhibits some regularity or partial continuity. Specifically, this means that the operator can have jumps or discontinuities at specific points, but overall remains regular and continuous in its behavior.

More specifically, in the context of the proof, this means that the considered operator, denoted A, satisfies both the monotonicity property (for all $u, v \in H^1(\Omega), < A(u) - A(v), u - v > \geq$

0) and the semi-continuity property (a certain partial regularity of the operator). These properties are important in the proof because they ensure certain inequalities or relationships necessary for establishing the results of the proof.

1.10 Partial differential equations (PDEs)

1.10.1 Introduction to partial differential equations (PDEs)

Partial differential equations (PDEs) are mathematical equations that describe the relationships between an unknown function and its partial derivatives with respect to multiple independent variables [12]. They are widely used to model complex mathematical and physical phenomena, such as heat flow, vibrations of a string, wave propagation, and more.

1.10.2 Types of PDEs (elliptic, parabolic, hyperbolic)

There are different types of PDEs, classified based on their mathematical and physical properties:

1. Elliptic PDEs:

- Characteristics: Elliptic PDEs are primarily used for stationary problems, meaning situations where the phenomena do not depend on time but only on space. They can also be used to model systems in equilibrium.

- Examples: The Laplace equation and the Poisson equation are examples of widely used elliptic PDEs in physics and engineering to solve problems related to electric potential, heat diffusion, and fluid flow.

2. Parabolic PDEs:

- Characteristics: Parabolic PDEs are used to model phenomena that evolve over time. They describe diffusion or propagation processes over time.

- Examples: The heat equation (or diffusion equation) is an example of a parabolic PDE commonly used to describe the propagation of heat in a material over time.

3. Hyperbolic PDEs:

- Characteristics: Hyperbolic PDEs describe phenomena that propagate in both space and time, such as acoustic waves, electromagnetic waves, or seismic waves.

- Examples: The wave equation and the advection equation are examples of hyperbolic

PDEs. The wave equation models the propagation of mechanical waves, such as sound waves, and the advection equation describes the transport of a conserved quantity in a fluid. Each type of PDE has specific mathematical and physical properties, making them suitable for different situations and problems. The choice of the appropriate PDE depends on the nature of the phenomenon to be modeled and the boundary conditions of the problem. By solving these PDEs, we can obtain solutions that provide a precise and detailed description of the studied phenomena, which is essential in many scientific and engineering fields.

1.10.3 Use of PDEs in image restoration

PDEs have found numerous applications in image restoration as they provide a powerful mathematical framework for modeling and solving restoration problems. They are often used to describe diffusion, regularization, and denoising processes in image restoration.

1.10.4 Examples of PDEs used in image restoration

Some examples of commonly used PDEs in image restoration are [12]:

1. The heat equation (or diffusion equation):

The heat equation is a parabolic PDE used to model the diffusion of heat in a medium. It is commonly employed for denoising and regularization of images by attenuating variations in brightness over time. The equation is given by:

$$\frac{\partial u}{\partial t}(x, y, t) = \triangle u(x, y, t)$$

$$u(x, y, 0) = u_0(x, y)$$

where $u$represents the image, ∇u is the gradient of the image, and $u_0(x, y)$ is the initial state of the image.

2. The Perona-Malik Equation: The Perona-Malik equation is an anisotropic diffusion PDE that is often used for image restoration. It is designed to preserve image contours while reducing noise. The equation is written as:

$$\frac{\partial u}{\partial t} = div(c|\nabla u|)\nabla u)$$

where u represents the image,∇u is the gradient of the image, div denotes the divergence operator, and $c(|\nabla u|)$ is a function that controls the diffusion process based on the gradient

magnitude.
3. The Monge-Ampère equation: The Monge-Ampère equation is an elliptic PDE used in image restoration to solve optimization problems, such as surface reconstruction or texture restoration. It is given by:

$$det(\nabla^2 u) = f(x, y)$$

where u represents the image, $\nabla^2 u$ is the Hessian matrix of u, and $f(x, y)$ is a given function representing the desired image properties.
4. The total variation (TV) equation: The TV equation is a partial differential equation used for image denoising and restoration. It is a popular method for preserving edges and reducing noise in images. The TV equation can be written as:

$$\frac{\partial u}{\partial t} = div(|\nabla u|)\nabla u)$$

The TV equation aims to minimize the total variation of the image, which measures the overall changes in pixel intensities. By promoting sparsity in the gradient of the image, the TV equation effectively smooths the image while preserving sharp edges.
Numerical methods, such as finite difference or finite element methods, are commonly used to solve these PDEs. Iterative algorithms, like the Split Bregman method or the Chambolle-Pock algorithm, are often employed to find the solution efficiently.
The TV equation has been widely applied in various image processing tasks, including denoising, deblurring, inpainting, and image segmentation. It has shown to be effective in removing noise while preserving important details, making it a popular choice in the field of image restoration.

Chapter 2

Theoretical study of the problem of denoising and image restoration

2.1 Introduction to the nonlinear anisotropic diffusion model

We present an introduction to the nonlinear anisotropic diffusion model and its application in image restoration. We begin by discussing the motivation underlying the use of anisotropic diffusion for image restoration tasks.

2.1.1 Motivation for anisotropic diffusion

We highlight the motivation behind the use of anisotropic diffusion for image restoration. Traditional linear filtering techniques, such as averaging and Gaussian blur, have limitations when it comes to preserving edges and details while removing noise. These techniques apply a uniform blur to the entire image, resulting in the loss of important details and a reduction in image quality.

Anisotropic diffusion offers a more advanced and adaptable approach to image restoration. Instead of applying a uniform blur, anisotropic diffusion takes into account the local characteristics of the image, such as edges and textures, to adjust the diffusion process. This allows for the preservation of important edges and details while reducing noise.

One of the main challenges in image restoration is the preservation of edges, which can be damaged or blurred during the filtering process. Anisotropic diffusion overcomes this challenge by adapting the diffusion rate based on intensity variations and the image gradient. As a result, edges are better preserved, and details are more faithfully represented.

Furthermore, anisotropic diffusion allows for the treatment of complex textures and structures present in images. By using adaptive diffusion coefficients, it is possible to better preserve the specific characteristics of each region in the image.

2.1.2 Nonlinear models for Image restoration

1. Advantages of nonlinear models:

• Nonlinear models offer a more flexible and adaptive approach to image restoration compared to linear filtering techniques. Unlike linear filters that apply a uniform transformation to the entire image, nonlinear models allow for selective modification of pixels based on their local characteristics.

• Nonlinear models are capable of better handling the complex structures of the image, such as edges, textures, and objects of interest, while preserving their integrity and original appearance. These models are particularly effective for restoring images with significant intensity variations or regions containing fine details and sharp transitions.

2. Handling complex structures:

• Nonlinear models employ sophisticated mechanisms to handle the complex structures of the image. For example, anisotropic diffusion uses an adaptive diffusion coefficient that takes into account local intensity and gradient variations to control the diffusion process. This helps preserve edges and details while reducing noise.

• Nonlinear models can also use regularization terms to promote spatial coherence in homogeneous regions while preserving complex structures. This avoids excessive suppression of fine details and maintains a natural appearance in the restored image.

3. Examples of nonlinear models

• In addition to anisotropic diffusion, there are many other nonlinear models used in image restoration, such as total variation (TV)-based restoration, sparsity-based denoising models, texture decomposition models, etc. Each model has its own advantages and is suitable for specific image restoration scenarios.

2.1.3 Anisotropic diffusion

1. Definition of anisotropic diffusion:

• Anisotropic diffusion is a nonlinear model that allows for different diffusion rates in different regions of the image. Unlike isotropic diffusion, which applies a uniform diffusion rate in all directions, anisotropic diffusion adapts to the local characteristics of the image.

• Anisotropic diffusion is particularly effective in preserving edges and details while reducing noise. It allows for the preservation of important image structures while smoothing homogeneous regions.

2. Principle of edge and detail preservation:

• One of the key advantages of anisotropic diffusion is its ability to preserve edges and fine details in the image. During the diffusion process, regions containing edges or sharp transitions are less smoothed compared to homogeneous regions.

• This preservation of edges is achieved by using a diffusivity function that is sensitive to local intensity and gradient variations. The diffusivity function controls the amount of diffusion applied in each region of the image, prioritizing regions with significant variations.

3. Use of the diffusivity function:

• The diffusivity function is a crucial element of anisotropic diffusion. It determines how the diffusion rate varies based on the local characteristics of the image. Adaptive diffusivity functions are used to adapt to the image's edges and details.

• The diffusivity function is typically based on the properties of the image's intensity gradient. It can be defined in various ways, such as exponential function, threshold function, or function based on statistical models.

2.2 Problem treated

The non linear anisotropic diffusion method is an image processing technique that preserves the contours and complex structures of an image while reducing noise. This method is based on a nonlinear anisotropic diffusion equation that adaptively regulates local image variations based on local characteristics [14, 15][?]. The diffusion conductivity depends on image characteristics such as texture, contour direction, and regularity.

In the field of image processing, it is common to encounter problems of degradation and noise that affect the quality of images. One of the commonly used approaches to restore such images is by applying anisotropic diffusion techniques. This method aims to reduce noise while preserving important features of the original image.

The original image (a) represents the raw and unaltered visual information that we intend to analyze or process. However, due to various factors such as transmission, compression, or sensor imperfections, the original image can be degraded and contain noise.

The noisy image(b) is a version of the original image in which noise has been introduced. Noise can originate from various sources such as the capture environment, electromagnetic interference, or compression artifacts. Noise makes the image more challenging to interpret and can compromise the accuracy of subsequent analysis.

To restore the noisy image and recover as much as possible the details and clarity of the original image, anisotropic diffusion is applied (c). This image processing method is based on diffusion processes that gradually reduce noise while preserving the important contours and structures of the image. Anisotropic diffusion uses adaptive diffusion coefficients to better adapt to different regions of the image, allowing for more efficient and precise restoration.

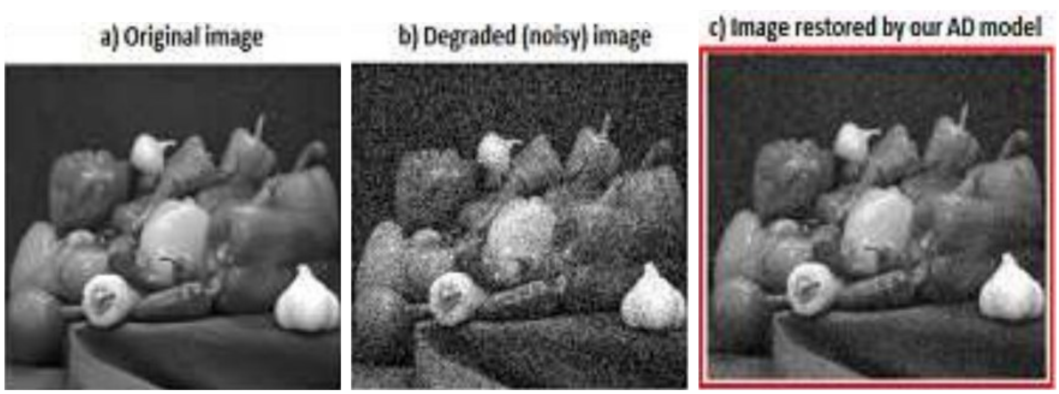


Figure 2.1: : image processed with AD

2.3 The model of anisotropic diffusion

The proposed image noise reduction algorithm uses nonlinear anisotropic diffusion. This method is mathematically described by a parabolic equation given by:

$$\begin{cases} \dfrac{\partial u}{\partial t} = div(\psi_{K(u)}(|\nabla u|^2)\nabla u), \\ u(0,x,y) = u_0, \quad (x,y) \in \Omega, \\ \nabla u \cdot v = 0 \quad , \quad on\ (0,T) \times \partial\Omega. \end{cases} \tag{2.1}$$

where u_0 is the initial noisy image, its domain is $\Omega \subset R^2$ and v is the normal to $\partial\Omega$. This equation is based on nonlinear anisotropic diffusion, which is a method used to remove noise while preserving image edges.The diffusivity (or edge-stopping) function $\psi_k(u)$ is defined as follows:

$$\psi_K(u)(s^2) = \begin{cases} \alpha\sqrt{\dfrac{K(u)}{\beta.s^2+\eta}}, & s > 0, \\ 1, \quad if & s = 0, \end{cases} \tag{2.2}$$

where α, $\beta \in [0.5, 0.8]$, and $\eta \in [0.5, 1]$.The conductance diffusivity in this model depends on the state of the image u at time t. When the gradient magnitude exceeds a certain threshold, the corresponding edge is enhanced. Various approaches exist for determining the conductance parameter, such as using a fixed value or making it a function of time. In this model, we propose an automatic computation of the conductance parameter based on image noise estimation at each iteration.

The conductance parameter $k(u)$ is computed using the following formula :

$$K(u) = ||u||_F \frac{median(u)}{\varepsilon \cdot n(u)}, \tag{2.3}$$

where $\varepsilon \in (0, 1]$, $||u||_F$ is the Frobenius norm of image u, median(u) represents its median value and $n(u)$ is the number of its pixels in the image. This parameter estimation method utilizes statistics of the image to determine the conductance diffusivity.

2.4 Existence and uniqueness of the solution

Théorème 2.1 *Suppose that (2.2) is satisfied. Then, for any initial condition* $u_0 \in L^2(\Omega)$*, there exists a unique weak solution* u *to problem (2.1). Furthermore, if* $u_0 \geq 0$*, then* $u \geq 0$*.*

Proof.

We define the set χ as the set of functions u that satisfy the following conditions:

$$\chi = \left\{ u \in C([0,T]; L^2(\Omega)) \cap L^2(]0,T[; H^1(\Omega)) \;\middle|\; \| u \|_{L^2(]0,T[;H^1(\Omega))} \leq R \right\},$$

where R is a fixed constant.

The set χ is a collection of functions that satisfy two requirements. Firstly, these functions must be continuous over the interval $[0,T]$ and belong to the space $L^2(\Omega)$ for all t within this interval. This ensures that the functions u are well-behaved throughout the spatial domain Ω and at any given time t.

Secondly, the functions u must also belong to the space $L^2(]0,T[; H^1(\Omega))$, which means their spatial derivatives must also be integrable over $]0,T[$. This is important to ensure that weak solutions possess spatial derivatives that are regular enough to be considered within the framework of the given equatio Finally, the $L^2(]0,T[; H^1(\Omega))$ norm of the functions u is bounded by a constant R. This allows for controlling the size of solutions and ensures that the set χ is well-defined and bounded.

In summary, the set χ contains all functions u that are both sufficiently regular in time and space to be considered as potential candidates for the weak solution of the given problem.

Weak formulation

To demonstrate the existence of a weak solution u to problem (2.1), we use the method of variations. Let $v \in \chi$, where χ is the set defined previously.

We multiply equation (2.1) by a test function $\varphi \in H^1(\Omega)$ such that $\varphi = 0$ on $\partial\Omega$. Next, we integrate the equation over the domain Ω.

$$\int_\Omega \frac{\partial u}{\partial t} \varphi \, dxdy = \int_\Omega \operatorname{div}\left(\psi K(v) |\nabla u|^2 \nabla u\right) \varphi \, dxdy. \tag{2.4}$$

By using the divergence theorem, we can rewrite the right-hand side term as :

$$\int_\Omega \frac{\partial u}{\partial t} \varphi \, dxdy = \int_\Omega \nabla \cdot \left(\psi K(v) |\nabla u|^2 \nabla u\right) \varphi \, dxdy - \int_\Omega \nabla \left(\psi K(v) |\nabla u|^2 \nabla u\right) \cdot \nabla \varphi \, dxdy. \tag{2.5}$$

By using the divergence theorem again, we can rewrite the first term on the right-hand side:

$$\int_\Omega \nabla \cdot \left(\psi K(v)|\nabla u|^2 \nabla u\right) \varphi \, dxdy = \int_{\partial\Omega} \left(\psi K(v)|\nabla u|^2 \nabla u\right) \cdot \mathbf{n}\varphi, ds \tag{2.6}$$

where $\mathbf{n}$ is the outward normal vector of the domain Ω.
Now, we can rewrite equation (2.5) as follows :

$$\int_\Omega \frac{\partial u}{\partial t}\varphi \, dxdy = -\int_{\partial\Omega} \left(\psi K(v)|\nabla u|^2 \nabla u\right) \cdot \mathbf{n}\varphi \, ds - \int_\Omega \nabla \left(\psi K(v)|\nabla u|^2 \nabla u\right) \cdot \nabla\varphi \, dxdy. \tag{2.7}$$

Since we have assumed that u satisfies homogeneous Dirichlet conditions on $\partial\Omega$ (i.e., $u \cdot \mathbf{n} = 0$ on $\partial\Omega$), the first term on the right-hand side becomes zero. We thus obtain the weak formulation:

$$\int_\Omega \frac{\partial u}{\partial t}\varphi \, dxdy + \int_\Omega \nabla \left(\psi K(v)|\nabla u|^2 \nabla u\right) \cdot \nabla\varphi \, dxdy = 0 \quad \forall \varphi \in H^1(\Omega). \tag{2.8}$$

In summary, the weak formulation (or variational formulation) consists of finding a function $u \in \chi$ such that for all $\varphi \in H^1(\Omega)$ (with $\varphi = 0$ on $\partial\Omega$), equation (6) is satisfied.

We can also show that there exists a constant $\alpha > 0$ such that:
${}_{H^1(\Omega)} < A_\nu(t)u, u >_{H^1(\Omega)} \geq \alpha \parallel u \parallel^2_{H^1(\Omega)} \qquad \forall u \in H^1(\Omega)$
By using a well-known result due to Lions, we conclude that for each $v \in \chi$, problem (2.6) has a unique weak solution $u = \Phi(v)$.
then to prove the invariance of the set χ, we aim to show that Φ is a contraction on χ and preserves this set. By performing calculations using equation (2.9) and the monotonicity of the function $r \mapsto \psi_{K(u)}(|r|^2)r)$, we can establish the existence of a positive constant α_0 such that the following inequality holds :

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t} \parallel u(t) - \overline{u} \parallel^2_{L^2} + \alpha_0 \int_\Omega |\nabla(u(t,x,y) - \overline{u}(t,x,y)|^2 dxdy \\ &\leq \int_\Omega |\nabla(u(t,x,y)) - u(t,x,y)||K(v(t,x,y) - K(\overline{v}(t,x,y))|dxdy, \end{aligned} \tag{2.9}$$

for all $t \in (0, T)$.
This inequality implies the following inequality:

$$\parallel u(t) - \overline{u}(t) \parallel^2_{L^2} + \frac{\alpha_0}{2} \int_0^t \int_\Omega |\nabla(u(t,x,y) - \overline{u}(t,x,y)|^2 dxdydt,$$

$$\leq C \int_0^t \int_\Omega |\nabla(v(t,x,y) - \overline{v}(t,x,y)|^2 dxdydt, \tag{2.10}$$

where $u = \Phi(v)$ and $\overline{u} = \Phi(\overline{v})$, and C is a constant.

Definition of operator $A_\nu(t)$:

The operator $A_\nu(t)$ is a linear operator that acts on a function $u \in H^1(\Omega)$ and maps it to an element of $(H^1(\Omega))'$, which is the dual space of $H^1(\Omega)$. It is defined as follows:

$$\langle A_\nu(t)u, \varphi\rangle = \int_\Omega \psi K(v)|\nabla u|^2 \nabla u \cdot \nabla\varphi \, dxdy \quad \forall \varphi \in H^1(\Omega) \tag{2.11}$$

where $\langle\cdot,\cdot\rangle$ represents the inner product in $(H^1(\Omega))'$, which is the dual space of $H^1(\Omega)$. Note that to define $A_\nu(t)u$ in $(H^1(\Omega))'$, we use the inner product $\langle\cdot,\cdot\rangle$.

Properties of operator $A_\nu(t)$:

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$$\langle A_\nu(t)u, \varphi\rangle = \int_\Omega \psi K(v)|\nabla u|^2 \nabla u \cdot \nabla\varphi \, dxdy \quad \forall \varphi \in H^1(\Omega), \tag{2.12}$$

where $\langle\cdot,\cdot\rangle$ represents the inner product in $(H^1(\Omega))'$, which is the dual space of $H^1(\Omega)$. Note that to define $A_\nu(t)u$ in $(H^1(\Omega))'$, we use the inner product $\langle\cdot,\cdot\rangle$.

Properties of operator $A_\nu(t)$

1. **Linearity:** The operator $A_\nu(t)$ is linear, which means that for any $u, \overline{u} \in H^1(\Omega)$ and any scalar α,

$$A_\nu(t)(u + \alpha\overline{u}) = A_\nu(t)u + \alpha A_\nu(t)\overline{u}. \tag{2.13}$$

2. **Monotonicity:** The operator $A_\nu(t)$ is monotone, meaning that for any $u, \overline{u} \in H^1(\Omega)$,

$$\langle A_\nu(t)u - A_\nu(t)\overline{u}, u - \overline{u}\rangle \geq 0. \tag{2.14}$$

This monotonicity property is essential for demonstrating the contraction of the operator Φ in the proof.

3. **Continuity:** The operator $A_\nu(t)$ is continuously linear from $H^1(\Omega)$ into $(H^1(\Omega))'$. This

means that there exists a positive constant C such that for any $u \in H^1(\Omega)$,

$$\|A_\nu(t)u\|_{(H^1(\Omega))'} \leq C\|u\|_{H^1(\Omega)}, \tag{2.15}$$

where $\|\cdot\|_{(H^1(\Omega))'}$ is the norm in the space $(H^1(\Omega))'$.

Théorème 2.2 ***(Banach fixed-point theorem)*** *Let (X, d) be a complete metric space and $\Phi : X \to X$ be a contraction mapping on X with a contraction factor $0 < \rho < 1$, that is, for all $x, y \in X$,*

$$d(\Phi(x), \Phi(y)) \leq \rho \cdot d(x, y). \tag{2.16}$$

Then, Φ has a unique fixed point in X, which means there exists a unique element $x^* \in X$ such that $\Phi(x^*) = x^*$.

Application of the Banach fixed-point theorem:
In our proof, we have defined the operator $\Phi : \chi \to \chi$, which maps each $v \in \chi$ to the solution u of problem (2.6). To show that Φ has a fixed point, i.e., a weak solution $u \in \chi$ such that $\Phi(u) = u$, we need to demonstrate two important properties :
1. **Invariance of χ:** We must show that the operator Φ preserves the set χ. In other words, if $v \in \chi$, then the weak solution $u = \Phi(v)$ also belongs to χ.
2. **Contraction of Φ:** We must show that the operator Φ is contracting on χ, meaning that there exists a contraction factor $0 < \rho < 1$ such that for any $v, \overline{v} \in \chi$,

$$\|\Phi(v) - \Phi(\overline{v})\|_{L^2(0,T;H^1(\Omega))} \leq \rho \cdot \|v - \overline{v}\|_{L^2(0,T;H^1(\Omega))}, \tag{2.17}$$

where $\|\cdot\|_{L^2(0,T;H^1(\Omega))}$ is an appropriate norm for the space χ.
Once we establish the invariance of χ and the contraction of Φ, we can apply the Banach Fixed-Point Theorem to conclude that there exists a unique weak solution $u \in \chi$ such that $\Phi(u) = u$.
we also have serval Remarks:
1.**Model modifications:** We made modifications to the initial model by replacing the function (2.1) with the function (2.8). This modification is justified because, in certain image denoising or restoration applications, the gradient amplitude typically does not exceed

a certain value, even for sharp edges. Thus, the choice of $\psi_{k(u)}$ is appropriate in such situations and leads to results that are more tailored to the specificities of the problem at hand.
2. **Existence of a weak solution:** Thanks to Proposition 3.1 and its proof, we can assert that the solution u of equation (2.1) can be obtained iteratively as the limit as n approaches infinity of u_n, where u_n represents the weak solution of problem (2.1). This convergence allows us to conclude that the solution u of (2.1) can also be obtained as the limit of the finite difference scheme mentioned in the previous section:

$$u(t+1) = u(t) + \operatorname{div}(\psi K(u)(|\nabla u|^2)\nabla u) \quad \text{in } \Omega, \quad u(t)\cdot v = 0 \quad \text{on } \partial\Omega. \tag{2.18}$$

This observation is crucial in demonstrating the convergence of the numerical scheme to the weak solution of equation (2.1).
3. **Reformulation of equation (2.1):** Equation (2.1) can be reformulated in an alternative form :

$$\frac{\partial}{\partial t}\sqrt{k(u)} = \frac{1}{2}\operatorname{div}(g_0(|\nabla u|^2)\nabla u) + \frac{1}{4}\frac{k'(u)}{\sqrt{k(u)}}g_0(|\nabla u|^2)(|\nabla u|^2) \quad \text{in } (0,T)\times\Omega, \tag{2.19}$$

subject to the initial condition $u(0) = u_0$ in Ω and the Neumann boundary condition $\nabla u \cdot v = 0$ on $(0,T)\times\partial\Omega$, where $g_0(s) = \dfrac{\alpha}{\sqrt{\beta s + \eta}}$ for $s > 0$. Neglecting the lower-order term, we obtain the simplified equation:

$$\frac{\partial}{\partial t}\sqrt{k(u)} = \frac{1}{2}\operatorname{div}(g_0(|\nabla u|^2)\nabla u). \tag{2.20}$$

with the Neumann boundary condition. This reformulation of equation (2.1) in an alternative form allows us to better understand its behavior and analyze it using methods specific to nonlinear parabolic equations. These methods can be applied to establish the existence of solutions for equation (2.1) under more general conditions on $K(u)$ (e.g., $K(u) > 0$). It is crucial to note that when $K(u)$ is constant, problem (2.1) reduces to the model of bounded variation flow, which is well-posed in the space of functions of bounded variation.

$\square$

Chapter 3

Numerical solution and implementation

3.1 Numerical solution

A robust numerical approximation scheme is used to discretize the continuous mathematical model described by equation (2.1). The scheme involves a 4-nearest-neighbors discretization of the Laplacian operator, denoted by Δu.By applying this discretization, equation (2.1) can be approximated as:

$$\frac{\partial u}{\partial t} = div(\psi_K(u)(|\nabla u|^2)\nabla u) \Rightarrow u(x,y,t+1) - u(x,y,t) \cong div(\psi K(u)(|\nabla u|^2)\Delta u)$$

,

which leads to the following approximating scheme:

$$u_{i,j}^{t+1} = u_{i,j}^{t} + \lambda \sum_{q \in N(i,j)} \psi_{K(u)} \left(\left| \nabla u_{i,j}^{q}(t) \right| \right)^2 \left| \nabla u_{i,j}^{q}(t) \right| \tag{3.1}$$

.

In this scheme u represents the image at time t and $\lambda \in (0,1)$,N(p) represents the set of pixels representing the 4-neighborhood of pixel $(i,j) = (x,y)$,to illustrate the concept of a 4-neighborhood, let's consider a 2D image represented by a matrix. Each element of the matrix represents a pixel in the image.

Suppose we have a 5x5 matrix representing our image. Each element in the matrix can be identified by its coordinates (i, j), where i represents the row and j represents the column of the element.

For a pixel located at position (i, j), the four 4-neighborhood neighbors are the pixels located at positions:
(i-1, j): neighboring pixel located in the upper row
(i+1, j): neighboring pixel located in the lower row
(i, j-1): neighboring pixel located in the previous column
(i, j+1): neighboring pixel located in the next column

These four neighbors represent the pixels directly adjacent to the central pixel (i, j) in the vertical and horizontal directions. and the gradient magnitude of the image in a particular direction at iteration t is computed as follows:

$$\nabla u^{p,q}(t) = u(q,t) - u(p,t) \tag{3.2}$$

. The restoration algorithm given by (2.4) is applied to the current image for $t = 0, 1, ..., N$, where N is the maximum number of iterations. Our iterative noise removal approach converges quite rapidly to the desired solution. More about the convergence of this finite difference scheme is discussed in the next section. It produces the smoothed image u^N from the degraded image $u(0) = u_0$ in a relatively low number of steps, so the N value should be kept relatively low.

3.2 Numerical solving algorithms

The numerical solving algorithm corresponding to the anisotropic diffusion equation you presented can be described as follows:

1. Initialization:

Define the algorithm parameters such as the number of iterations, the coefficient λ, and load the input image.

2. Preprocessing:

Convert the input image to grayscale if necessary.

3. Iteration loop:

a. For each time iteration t:

i. Traverse each pixel (i, j) of the image.

ii. Calculate the image gradient $\nabla u_{i,j}^{q}(t)$ for each neighboring pixel q of (i, j).

iii. Calculate the anisotropic diffusion term $\psi_{K(u)}\left(\left|\nabla u_{i,j}^{q}(t)\right|\right)^{2}\left|\nabla u_{i,j}^{q}(t)\right|$ for each neighboring pixel (q) of (i, j).

iv. Calculate the new pixel value$u_{i,j}^{t+1}$ using the update equation:

$$u_{i,j}^{t+1} = u_{i,j}^{t} + \lambda \sum_{q \in N(i,j)} \psi_{K(u)}(((|\nabla u_{i,j}^{q}(t)))^{2}(|\nabla u_{i,j}^{q}(t))$$

b. Update the image with the new pixel values.

c. Repeat steps a and b for the specified number of iterations.

This algorithm iterates over each pixel of the image at each time iteration, calculating the anisotropic diffusion term for neighboring pixels and updating the pixel value using the approximation equation. The algorithm repeats for a defined number of iterations until the solution converges to a desired state.

It is worth noting that this algorithm is a general representation of the numerical solving method for the anisotropic diffusion equation, and variations may exist depending on the specifics of the implementation and choices of parameters.

3.3 Computer implementation

The implementation of the anisotropic diffusion model using Python as the programming language can be done by following the following steps :

1. Importing libraries: The appropriate Python libraries need to be imported to facilitate the implementation of the anisotropic diffusion model. Some commonly used libraries for image processing include NumPy, OpenCV, scikit-image, matplotlib, etc. These libraries provide functionalities such as image manipulation, matrix calculations, result visualization, etc.

2. Loading the input image: Use a function from the appropriate library to load the image from a file. For example, using OpenCV, one can use the ‘cv2.imread()‘ function to load the image.

3. Preprocessing the image: If necessary, apply preprocessing operations on the input image, such as converting to grayscale, resizing, normalization, etc. These operations can be performed using the appropriate functions from the library used.

4. Implementing the anisotropic diffusion algorithm: Use an iteration loop to implement the numerical solving algorithm for the anisotropic diffusion equation. At each iteration, traverse each pixel of the image, calculate the anisotropic diffusion terms for neighboring pixels, and update the pixel value using the approximation equation.

5. Displaying the results: At the end of the iterations, display the input image and the resulting image after anisotropic diffusion. Use the appropriate visualization functions from the chosen library, such as 'imshow()' from matplotlib or 'cv2.imshow()' from OpenCV.

Regarding the software architecture, it depends on the complexity of your image processing system. For a simple implementation of the anisotropic diffusion model, a linear software architecture may suffice, where the different steps of the process (loading the image, preprocessing, anisotropic diffusion, displaying the results) are executed sequentially. However, for more complex systems, a modular architecture based on functions, classes, or modules can be used to organize and reuse code effectively.

3.4 Results

In this section, we present a collection of figures that visually depict the results obtained from applying our model to image restoration tasks. Each figure provides a side-by-side comparison between the original degraded image and the restored image using our approach.

Figure 3.1: Representation of an image "apple" processed by AD model

Figure 3.2: Representation of an image "animal" processed by AD model

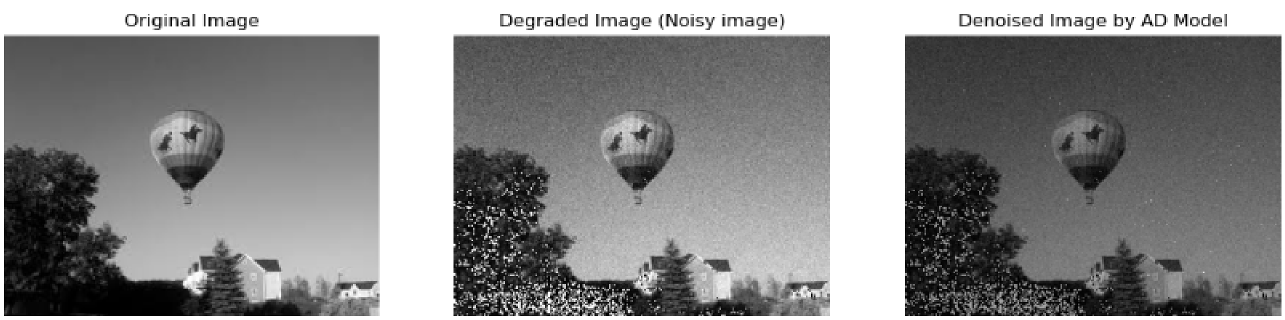

Figure 3.3: Representation of an image "blimp" processed by AD model

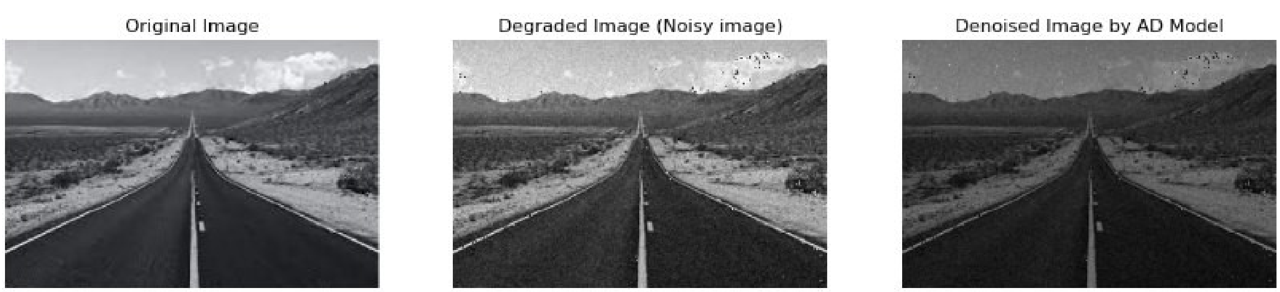

Figure 3.4: Representation of an image "road" processed by AD model

Conclusion

Our work is primarily focused on developing mathematical methods for image restoration based on differential equations. Specifically, we have developed a non-linear anisotropic diffusion-based model for image restoration. This model aims to enhance the quality of degraded images by reducing noise and preserving important contours and details.

To achieve this, we conducted a thorough study of partial differential equations (PDEs) and variational models, with a specific focus on anisotropic diffusion, for image restoration tasks. We gained a deep understanding of the theoretical principles behind these models and the underlying mathematical concepts. Subsequently, we designed our customized non-linear anisotropic diffusion model tailored to our image restoration application.

To analyze our model, we conducted a comprehensive mathematical study of its properties, including well-posedness, stability, and convergence. We ensured the existence and uniqueness of solutions to the model's equations, studied how the model responded to perturbations in initial conditions or parameters, and analyzed whether the model's solution converged to the desired result as computations progressed.

Following that, we performed numerical experiments to evaluate the effectiveness of our image restoration model. We applied the model to artificially corrupted images with different types of noise and compared the restored results with the original, uncorrupted images.

The results of our numerical experiments demonstrated that our non-linear anisotropic diffusion model efficiently reduced noise and blur in corrupted images while preserving essential contours and details.

In conclusion, our work successfully generated advanced mathematical methods for image restoration, based on differential equations. We developed a non-linear anisotropic diffusion model, rigorously analyzed its mathematical properties, and demonstrated its effectiveness through numerical experiments. These results confirm the relevance and utility of our model in the field of image restoration. This research provides a solid foundation for future research aimed at improving our model, exploring its application in various image restoration scenarios, and utilizing it in practical image processing applications.

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