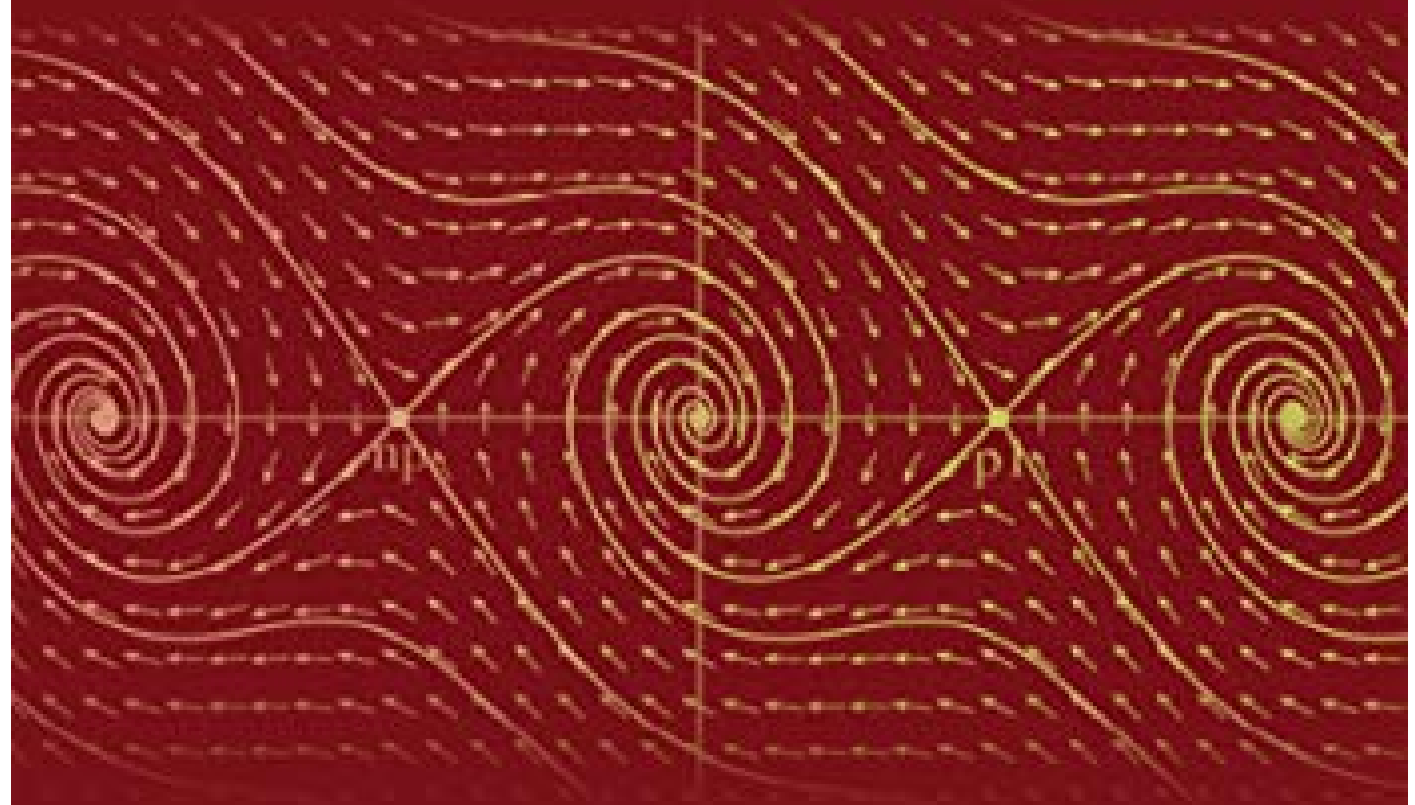


TEXTBOOKS IN MATHEMATICS

ORDINARY DIFFERENTIAL EQUATIONS

An Introduction to the Fundamentals

SECOND EDITION



Kenneth B. Howell



CRC Press
Taylor & Francis Group

A CHAPMAN & HALL BOOK

TEXTBOOKS IN MATHEMATICS

ORDINARY DIFFERENTIAL EQUATIONS

An Introduction to the Fundamentals

Second Edition

Kenneth B. Howell

University of Alabama in Huntsville, USA



CRC Press

Taylor & Francis Group

Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

Contents

Preface (With Important Information for the Reader)

I	The Basics	1
1	The Starting Point: Basic Concepts and Terminology	3
1.1	Differential Equations: Basic Definitions and Classifications	3
1.2	Why Care About Differential Equations? Some Illustrative Examples	8
1.3	More on Solutions	14
	Additional Exercises	17
2	Integration and Differential Equations	21
2.1	Directly-Integrable Equations	21
2.2	On Using Indefinite Integrals	23
2.3	On Using Definite Integrals	24
2.4	Integrals of Piecewise-Defined Functions	28
	Additional Exercises	32
II	First-Order Equations	35
3	Some Basics about First-Order Equations	37
3.1	Algebraically Solving for the Derivative	37
3.2	Constant (or Equilibrium) Solutions	39
3.3	On the Existence and Uniqueness of Solutions	42
3.4	Confirming the Existence of Solutions (Core Ideas)	44
3.5	Details in the Proof of Theorem 3.1	47
3.6	On Proving Theorem 3.2	58
3.7	Appendix: A Little Multivariable Calculus	59
	Additional Exercises	63
4	Separable First-Order Equations	65
4.1	Basic Notions	65
4.2	Constant Solutions	70
4.3	Explicit Versus Implicit Solutions	75
4.4	Full Procedure for Solving Separable Equations	77
4.5	Existence, Uniqueness, and False Solutions	78
4.6	On the Nature of Solutions to Differential Equations	81
4.7	Using and Graphing Implicit Solutions	83
4.8	On Using Definite Integrals with Separable Equations	88
	Additional Exercises	90

5	Linear First-Order Equations	93
5.1	Basic Notions	93
5.2	Solving First-Order Linear Equations	96
5.3	On Using Definite Integrals with Linear Equations	100
5.4	Integrability, Existence and Uniqueness	102
	Additional Exercises	103
6	Simplifying Through Substitution	105
6.1	Basic Notions	105
6.2	Linear Substitutions	107
6.3	Homogeneous Equations	110
6.4	Bernoulli Equations	113
	Additional Exercises	114
7	The Exact Form and General Integrating Factors	117
7.1	The Chain Rule	117
7.2	The Exact Form, Defined	119
7.3	Solving Equations in Exact Form	121
7.4	Testing for Exactness — Part I	127
7.5	“Exact Equations”: A Summary	129
7.6	Converting Equations to Exact Form	130
7.7	Testing for Exactness — Part II	137
	Additional Exercises	141
8	Review Exercises for Part of Part II	143
9	Slope Fields: Graphing Solutions Without the Solutions	145
9.1	Motivation and Basic Concepts	145
9.2	The Basic Procedure	147
9.3	Observing Long-Term Behavior in Slope Fields	152
9.4	Problem Points in Slope Fields, and Issues of Existence and Uniqueness	158
9.5	Tests for Stability	165
	Additional Exercises	172
10	Numerical Methods I: The Euler Method	177
10.1	Deriving the Steps of the Method	177
10.2	Computing via the Euler Method (Illustrated)	180
10.3	Using the Results of the Method	183
10.4	Reducing the Error	185
10.5	Error Analysis for the Euler Method	187
	Additional Exercises	193
11	The Art and Science of Modeling with First-Order Equations	197
11.1	Preliminaries	197
11.2	A Rabbit Ranch	198
11.3	Exponential Growth and Decay	201
11.4	The Rabbit Ranch, Again	204
11.5	Notes on the Art and Science of Modeling	207
11.6	Mixing Problems	211
11.7	Simple Thermodynamics	214
	Additional Exercises	215

12 Numerical Methods II: Beyond the Euler Method	221
12.1 Forward and Backward Euler Methods	221
12.2 The Improved Euler Method	223
12.3 A Few Other Methods Worth Brief Discussion	230
12.4 The Classic Runge-Kutta Method	232
12.5 Some Additional Comments	240
Additional Exercises	240
 III Second- and Higher-Order Equations	 243
13 Higher-Order Equations: Extending First-Order Concepts	245
13.1 Treating Some Second-Order Equations as First-Order	246
13.2 The Other Class of Second-Order Equations “Easily Reduced” to First-Order	250
13.3 Initial-Value Problems	253
13.4 On the Existence and Uniqueness of Solutions	256
Additional Exercises	259
 14 Higher-Order Linear Equations and the Reduction of Order Method	 263
14.1 Linear Differential Equations of All Orders	263
14.2 Introduction to the Reduction of Order Method	266
14.3 Reduction of Order for Homogeneous Linear Second-Order Equations	267
14.4 Reduction of Order for Nonhomogeneous Linear Second-Order Equations	272
14.5 Reduction of Order in General	275
Additional Exercises	277
 15 General Solutions to Homogeneous Linear Differential Equations	 279
15.1 Second-Order Equations (Mainly)	279
15.2 Homogeneous Linear Equations of Arbitrary Order	290
15.3 Linear Independence and Wronskians	291
Additional Exercises	294
 16 Verifying the Big Theorems and an Introduction to Differential Operators	 299
16.1 Verifying the Big Theorem on Second-Order, Homogeneous Equations	299
16.2 Proving the More General Theorems on General Solutions and Wronskians	306
16.3 Linear Differential Operators	307
Additional Exercises	314
 17 Second-Order Homogeneous Linear Equations with Constant Coefficients	 317
17.1 Deriving the Basic Approach	317
17.2 The Basic Approach, Summarized	320
17.3 Case 1: Two Distinct Real Roots	322
17.4 Case 2: Only One Root	323
17.5 Case 3: Complex Roots	327
17.6 Summary	333
Additional Exercises	334
 18 Springs: Part I	 337
18.1 Modeling the Action	337
18.2 The Mass/Spring Equation and Its Solutions	341
Additional Exercises	350

19 Arbitrary Homogeneous Linear Equations with Constant Coefficients	353
19.1 Some Algebra	353
19.2 Solving the Differential Equation	356
19.3 More Examples	360
19.4 On Verifying Theorem 19.2	362
19.5 On Verifying Theorem 19.3	368
Additional Exercises	369
20 Euler Equations	371
20.1 Second-Order Euler Equations	371
20.2 The Special Cases	374
20.3 Euler Equations of Any Order	378
20.4 The Relation Between Euler and Constant Coefficient Equations	381
Additional Exercises	382
21 Nonhomogeneous Equations in General	385
21.1 General Solutions to Nonhomogeneous Equations	385
21.2 Superposition for Nonhomogeneous Equations	389
21.3 Reduction of Order	391
Additional Exercises	391
22 Method of Undetermined Coefficients (aka: Method of Educated Guess)	395
22.1 Basic Ideas	395
22.2 Good First Guesses for Various Choices of g	398
22.3 When the First Guess Fails	402
22.4 Method of Guess in General	404
22.5 Common Mistakes	407
22.6 Using the Principle of Superposition	408
22.7 On Verifying Theorem 22.1	409
Additional Exercises	412
23 Springs: Part II (Forced Vibrations)	415
23.1 The Mass/Spring System	415
23.2 Constant Force	417
23.3 Resonance and Sinusoidal Forces	418
23.4 More on Undamped Motion under Nonresonant Sinusoidal Forces	424
Additional Exercises	426
24 Variation of Parameters (A Better Reduction of Order Method)	431
24.1 Second-Order Variation of Parameters	431
24.2 Variation of Parameters for Even Higher Order Equations	439
24.3 The Variation of Parameters Formula	442
Additional Exercises	444
25 Review Exercises for Part III	447
IV The Laplace Transform	449
26 The Laplace Transform (Intro)	451
26.1 Basic Definition and Examples	451
26.2 Linearity and Some More Basic Transforms	457
26.3 Tables and a Few More Transforms	459

26.4	The First Translation Identity (and More Transforms)	464
26.5	What Is “Laplace Transformable”? (and Some Standard Terminology)	466
26.6	Further Notes on Piecewise Continuity and Exponential Order	471
26.7	Proving Theorem 26.5	474
	Additional Exercises	477
27	Differentiation and the Laplace Transform	481
27.1	Transforms of Derivatives	481
27.2	Derivatives of Transforms	486
27.3	Transforms of Integrals and Integrals of Transforms	488
27.4	Appendix: Differentiating the Transform	493
	Additional Exercises	496
28	The Inverse Laplace Transform	499
28.1	Basic Notions	499
28.2	Linearity and Using Partial Fractions	501
28.3	Inverse Transforms of Shifted Functions	507
	Additional Exercises	509
29	Convolution	511
29.1	Convolution: The Basics	511
29.2	Convolution and Products of Transforms	515
29.3	Convolution and Differential Equations (Duhamel’s Principle)	519
	Additional Exercises	523
30	Piecewise-Defined Functions and Periodic Functions	525
30.1	Piecewise-Defined Functions	525
30.2	The “Translation Along the T -Axis” Identity	528
30.3	Rectangle Functions and Transforms of More Piecewise-Defined Functions	533
30.4	Convolution with Piecewise-Defined Functions	537
30.5	Periodic Functions	540
30.6	An Expanded Table of Identities	545
30.7	Duhamel’s Principle and Resonance	546
	Additional Exercises	553
31	Delta Functions	557
31.1	Visualizing Delta Functions	557
31.2	Delta Functions in Modeling	558
31.3	The Mathematics of Delta Functions	562
31.4	Delta Functions and Duhamel’s Principle	566
31.5	Some “Issues” with Delta Functions	568
	Additional Exercises	572
V	Power Series and Modified Power Series Solutions	575
32	Series Solutions: Preliminaries	577
32.1	Infinite Series	577
32.2	Power Series and Analytic Functions	582
32.3	Elementary Complex Analysis	591
32.4	Additional Basic Material That May Be Useful	594
	Additional Exercises	599

33 Power Series Solutions I: Basic Computational Methods	603
33.1 Basics	603
33.2 The Algebraic Method with First-Order Equations	605
33.3 Validity of the Algebraic Method for First-Order Equations	615
33.4 The Algebraic Method with Second-Order Equations	620
33.5 Validity of the Algebraic Method for Second-Order Equations	628
33.6 The Taylor Series Method	631
33.7 Appendix: Using Induction	636
Additional Exercises	641
34 Power Series Solutions II: Generalizations and Theory	647
34.1 Equations with Analytic Coefficients	647
34.2 Ordinary and Singular Points, the Radius of Analyticity, and the Reduced Form	648
34.3 The Reduced Forms	652
34.4 Existence of Power Series Solutions	653
34.5 Radius of Convergence for the Solution Series	659
34.6 Singular Points and the Radius of Convergence	662
34.7 Appendix: A Brief Overview of Complex Calculus	663
34.8 Appendix: The “Closest Singular Point”	667
34.9 Appendix: Singular Points and the Radius of Convergence for Solutions	671
Additional Exercises	678
35 Modified Power Series Solutions and the Basic Method of Frobenius	681
35.1 Euler Equations and Their Solutions	681
35.2 Regular and Irregular Singular Points (and the Frobenius Radius of Convergence)	685
35.3 The (Basic) Method of Frobenius	690
35.4 Basic Notes on Using the Frobenius Method	702
35.5 About the Indicial and Recursion Formulas	705
35.6 Dealing with Complex Exponents	712
35.7 Appendix: On Tests for Regular Singular Points	713
Additional Exercises	715
36 The Big Theorem on the Frobenius Method, with Applications	719
36.1 The Big Theorems	719
36.2 Local Behavior of Solutions: Issues	723
36.3 Local Behavior of Solutions: Limits at Regular Singular Points	724
36.4 Local Behavior: Analyticity and Singularities in Solutions	727
36.5 Case Study: The Legendre Equations	730
36.6 Finding Second Solutions Using Theorem 36.2	734
Additional Exercises	739
37 Validating the Method of Frobenius	743
37.1 Basic Assumptions and Symbology	743
37.2 The Indicial Equation and Basic Recursion Formula	744
37.3 The Easily Obtained Series Solutions	748
37.4 Second Solutions When $r_2 = r_1$	751
37.5 Second Solutions When $r_1 - r_2 = K$	754
37.6 Convergence of the Solution Series	761

VI Systems of Differential Equations (A Brief Introduction)	763
38 Systems of Differential Equations: A Starting Point	765
38.1 Basic Terminology and Notions	765
38.2 A Few Illustrative Applications	769
38.3 Converting Differential Equations to First-Order Systems	773
38.4 Using Laplace Transforms to Solve Systems	777
38.5 Existence, Uniqueness and General Solutions for Systems	779
38.6 Single N^{th} -Order Differential Equations	783
Additional Exercises	786
39 Critical Points, Direction Fields and Trajectories	789
39.1 The Systems of Interest and Some Basic Notation	789
39.2 Constant/Equilibrium Solutions	791
39.3 “Graphing” Standard Systems	793
39.4 Sketching Trajectories for Autonomous Systems	795
39.5 Critical Points, Stability and Long-Term Behavior	800
39.6 Applications	803
39.7 Existence and Uniqueness of Trajectories	809
39.8 Proving Theorem 39.2	811
Additional Exercises	815
40 Numerical Methods III: Systems and Higher-Order Equations	821
40.1 Brief Review of the Basic Euler Method	821
40.2 The Euler Method for First-Order Systems	822
40.3 Extending Euler’s Method to Second-Order Differential Equations	829
Additional Exercises	836
Appendix: Author’s Guide to Using This Text	839
A.1 Overview	839
A.2 Chapter-by-Chapter Guide	840
Answers to Selected Exercises	849
Index	887