

TEXTBOOKS IN MATHEMATICS

# Principles of Fourier Analysis

SECOND EDITION

Kenneth B. Howell



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A CHAPMAN & HALL BOOK

TEXTBOOKS in MATHEMATICS

# Principles of Fourier Analysis

## SECOND EDITION

Kenneth B. Howell

The University of Alabama in Huntsville  
USA



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**Principles of Fourier Analysis** develops the four core theories of Fourier analysis—the classical theory for Fourier series, the classical theory for Fourier transforms, the generalized theory uniting and extending the classical theories, and the theory for discrete Fourier transforms and FFTs.

The text is written in an engaging style, with the reader motivated and enlightened through the use of easily understood arguments and examples. Any non-rigorous development is later backed up by solid, mathematically rigorous arguments. As the author says, “Good proofs keep us honest.”

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