

TEXTBOOKS IN MATHEMATICS

Principles of Fourier Analysis

SECOND EDITION

Kenneth B. Howell

 CRC Press
Taylor & Francis Group
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Kenneth B. Howell

The University of Alabama in Huntsville

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Contents

Preface	xiii
Sample Courses	xv
I Preliminaries	1
1 The Starting Point	3
1.1 Fourier's Bold Conjecture	3
1.2 Mathematical Preliminaries and the Following Chapters	5
Additional Exercises	6
2 Basic Terminology, Notation and Conventions	7
2.1 Numbers	7
2.2 Functions, Formulas and Variables	8
2.3 Operators and Transforms	12
3 Basic Analysis I: Continuity and Smoothness	15
3.1 (Dis)Continuity	15
3.2 Differentiation	22
3.3 Basic Manipulations and Smoothness	25
3.4 Addenda	27
Additional Exercises	34
4 Basic Analysis II: Integration and Infinite Series	37
4.1 Integration	37
4.2 Infinite Series (Summations)	41
Additional Exercises	48
5 Symmetry and Periodicity	49
5.1 Even and Odd Functions	49
5.2 Periodic Functions	51
5.3 Sines and Cosines	53
Additional Exercises	56
6 Elementary Complex Analysis	57
6.1 Complex Numbers	57
6.2 Complex-Valued Functions	59
6.3 The Complex Exponential	61
6.4 Functions of a Complex Variable	66
Additional Exercises	71

7	Functions of Several Variables	73
7.1	Basic Extensions	73
7.2	Single Integrals of Functions with Two Variables	78
7.3	Double Integrals	82
7.4	Addenda: Proving Theorems 7.7 and 7.9	84
	Additional Exercises	90
II	Fourier Series	93
8	Heuristic Derivation of the Fourier Series Formulas	95
8.1	The Frequencies	95
8.2	The Coefficients	96
8.3	Summary	99
	Additional Exercises	100
9	The Trigonometric Fourier Series	101
9.1	Defining the Trigonometric Fourier Series	101
9.2	Computing the Fourier Coefficients	107
9.3	Partial Sums and Graphing	115
	Additional Exercises	117
10	Fourier Series over Finite Intervals (Sine and Cosine Series)	121
10.1	The Basic Fourier Series	121
10.2	The Fourier Sine Series	123
10.3	The Fourier Cosine Series	125
10.4	Using These Series	126
	Additional Exercises	127
11	Inner Products, Norms and Orthogonality	129
11.1	Inner Products	129
11.2	The Norm of a Function	131
11.3	Orthogonal Sets of Functions	132
11.4	Orthogonal Function Expansions	134
11.5	The Schwarz Inequality for Inner Products	135
11.6	Bessel's Inequality	137
	Additional Exercises	141
12	The Complex Exponential Fourier Series	145
12.1	Derivation	145
12.2	Notation and Terminology	147
12.3	Computing the Coefficients	149
12.4	Partial Sums	150
	Additional Exercises	151
13	Convergence and Fourier's Conjecture	155
13.1	Pointwise Convergence	155
13.2	Uniform and Nonuniform Approximations	161
13.3	Convergence in Norm	169
13.4	The Sine and Cosine Series	173
	Additional Exercises	175

14 Convergence and Fourier's Conjecture: The Proofs	179
14.1 Basic Theorem on Pointwise Convergence	179
14.2 Convergence for a Particular Saw Function	186
14.3 Convergence for Arbitrary Saw Functions	195
14.4 A Divergent Fourier Series	196
15 Derivatives and Integrals of Fourier Series	201
15.1 Differentiation of Fourier Series	201
15.2 Differentiability and Convergence	206
15.3 Integrating Periodic Functions and Fourier Series	210
15.4 Sine and Cosine Series	214
Additional Exercises	216
16 Applications	219
16.1 The Heat Flow Problem	219
16.2 The Vibrating String Problem	226
16.3 Functions Defined by Infinite Series	234
16.4 Verifying the Heat Flow Problem Solution	243
Additional Exercises	247
III Classical Fourier Transforms	249
17 Heuristic Derivation of the Classical Fourier Transform	251
17.1 Riemann Sums over the Entire Real Line	251
17.2 The Derivation	253
17.3 Summary	255
18 Integrals on Infinite Intervals	257
18.1 Absolutely Integrable Functions	257
18.2 The Set of Absolutely Integrable Functions	261
18.3 Many Useful Facts	261
18.4 Functions with Two Variables	268
Additional Exercises	276
19 The Fourier Integral Transforms	279
19.1 Definitions, Notation and Terminology	279
19.2 Near-Equivalence	281
19.3 Linearity	283
19.4 Invertibility	284
19.5 Other Integral Formulas (A Warning)	286
19.6 Some Properties of the Transformed Functions	287
Additional Exercises	294
20 Classical Fourier Transforms and Classically Transformable Functions	297
20.1 The First Extension	298
20.2 The Set of Classically Transformable Functions	302
20.3 The Complete Classical Fourier Transforms	304
20.4 What Is and Is Not Classically Transformable?	308
20.5 Finite Duration and Finite Bandwidth Functions	310
20.6 More on Terminology, Notation and Conventions	313
Additional Exercises	314

21	Some Elementary Identities: Translation, Scaling and Conjugation	319
21.1	Translation	319
21.2	Scaling	327
21.3	Practical Transform Computing	328
21.4	Complex Conjugation and Related Symmetries	332
	Additional Exercises	335
22	Differentiation and Fourier Transforms	339
22.1	The Differentiation Identities	339
22.2	Rigorous Derivation of the Differential Identities	346
22.3	Higher Order Differential Identities	349
22.4	Anti-Differentiation and Integral Identities	351
	Additional Exercises	356
23	Gaussians and Gaussian-Like Functions	359
23.1	Basic Gaussians	359
23.2	General Gaussians	364
23.3	Gaussian-Like Functions	368
	Additional Exercises	373
24	Convolution and Transforms of Products	375
24.1	Derivation of the Convolution Formula	375
24.2	Basic Formulas and Properties of Convolution	377
24.3	Algebraic Properties	379
24.4	Computing Convolutions	382
24.5	Existence, Smoothness and Derivatives of Convolutions	388
24.6	Convolution and Fourier Analysis	392
	Additional Exercises	395
25	Correlation, Square-Integrable Functions and the Fundamental Identity	399
25.1	Correlation	399
25.2	Square-Integrable/Finite Energy Functions	403
25.3	The Fundamental Identity	412
	Additional Exercises	416
26	Generalizing the Classical Theory: A Naive Approach	419
26.1	Delta Functions	419
26.2	Transforms of Periodic Functions	426
26.3	Arrays of Delta Functions	429
26.4	The Generalized Derivative	432
	Additional Exercises	444
27	Fourier Analysis in the Analysis of Systems	447
27.1	Linear, Shift-Invariant Systems	447
27.2	Computing Outputs for LSI Systems	454
	Additional Exercises	461
28	Multi-Dimensional Fourier Transforms	463
28.1	Basic Definitions	463
28.2	Computing Multi-Dimensional Transforms	466
	Additional Exercises	470

29 Identity Sequences	471
29.1 An Elementary Identity Sequence	471
29.2 General Identity Sequences	473
29.3 Gaussian Identity Sequences	477
29.4 Verifying Identity Sequences	481
29.5 An Application (with Exercises)	485
29.6 Laplace Transforms as Fourier Transforms	487
Additional Exercises	489
30 Gaussians as Test Functions and Proofs of Important Theorems	491
30.1 Testing for Equality with Gaussians	491
30.2 The Fundamental Theorem on Invertibility	492
30.3 The Fourier Differential Identities	495
30.4 The Fundamental and Convolution Identities of Fourier Analysis	501
IV Generalized Functions and Fourier Transforms	509
31 A Starting Point for the Generalized Theory	511
31.1 Starting Points	511
Additional Exercises	514
32 Gaussian Test Functions	515
32.1 The Space of Gaussian Test Functions	515
32.2 On Using the Space of Gaussian Test Functions	519
32.3 Other Test Function Spaces and a Confession	521
32.4 More on Gaussian Test Functions	522
32.5 Norms and Operational Continuity	529
Additional Exercises	535
33 Generalized Functions	537
33.1 Functionals	537
33.2 Generalized Functions	540
33.3 Basic Algebra of Generalized Functions	547
33.4 Generalized Functions Based on Other Test Function Spaces	553
33.5 Some Consequences of Functional Continuity	553
33.6 The Details of Functional Continuity	559
Additional Exercises	564
34 Sequences and Series of Generalized Functions	567
34.1 Sequences and Limits	567
34.2 Infinite Series (Summations)	574
34.3 A Little More on Delta Functions	577
34.4 Arrays of Delta Functions	579
Additional Exercises	583
35 Basic Transforms of Generalized Fourier Analysis	587
35.1 Fourier Transforms	587
35.2 Generalized Scaling of the Variable	592
35.3 Generalized Translation/Shifting	597
35.4 The Generalized Derivative	605
35.5 Transforms of Limits and Series	613

35.6 Adjoint-Defined Transforms in General	614
35.7 Generalized Complex Conjugation	621
Additional Exercises	623
36 Generalized Products, Convolutions and Definite Integrals	629
36.1 Multiplication and Convolution	630
36.2 Definite Integrals of Generalized Functions	639
36.3 Appendix: On Defining Generalized Products and Convolutions	643
Additional Exercises	646
37 Periodic Functions and Regular Arrays	649
37.1 Periodic Generalized Functions	649
37.2 Fourier Series for Periodic Generalized Functions	655
37.3 On Proving Theorem 37.5	663
Additional Exercises	671
38 Pole Functions and General Solutions to Simple Equations	673
38.1 Basics on Solving Simple Algebraic Equations	674
38.2 Homogeneous Equations with Polynomial Factors	677
38.3 Nonhomogeneous Equations with Polynomial Factors	689
38.4 The Pole Functions	693
38.5 Pole Functions in Transforms, Products and Solutions	700
Additional Exercises	705
V The Discrete Theory	707
39 Periodic, Regular Arrays	709
39.1 The Index Period and Other Basic Notions	709
39.2 Fourier Series and Transforms of Periodic, Regular Arrays	711
Additional Exercises	720
40 Sampling, Discrete Fourier Transforms and FFTs	721
40.1 Some General Conventions and Terminology	721
40.2 Sampling and the Discrete Approximation	722
40.3 The Discrete Approximation and Its Transforms	725
40.4 The Discrete Fourier Transforms	737
40.5 Discrete Transform Identities	741
40.6 Fast Fourier Transforms	747
Additional Exercises	756
Tables, References and Answers	761
Table A.1: Fourier Transforms of Some Common Functions	763
Table A.2: Identities for the Fourier Transforms	767
References	769
Answers to Selected Exercises	771
Index	783

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Principles of Fourier Analysis develops the four core theories of Fourier analysis—the classical theory for Fourier series, the classical theory for Fourier transforms, the generalized theory uniting and extending the classical theories, and the theory for discrete Fourier transforms and FFTs.

The text is written in an engaging style, with the reader motivated and enlightened through the use of easily understood arguments and examples. Any non-rigorous development is later backed up by solid, mathematically rigorous arguments. As the author says, “Good proofs keep us honest.”

This book serves as a text and reference for everyone who uses or may use Fourier analysis, including the beginning student first discovering Fourier analysis and the more advanced student desiring a deeper understanding. This book may also be used in a general applied analysis course. Parts should be of interest to professionals who are already experts in Fourier analysis since the generalized theory presented here substantially extends the theory presented elsewhere.

New to this Edition

- A brief discussion of the Haar wavelets
- The construction of a continuous, periodic function with a divergent Fourier series
- A brief discussion of the classic sampling theorem for band-limited functions
- A short chapter on multi-dimensional Fourier transforms
- A discussion relating the Fourier and Laplace transforms

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