

Geometric Inverse Problems

With Emphasis on Two Dimensions

**GABRIEL P. PATERNAIN
MIKKO SALO
GUNTHER UHLMANN**



Geometric Inverse Problems

With Emphasis on Two Dimensions

GABRIEL P. PATERNAIN

University of Cambridge

MIKKO SALO

University of Jyväskylä, Finland

GUNTHER UHLMANN

University of Washington



CAMBRIDGE
UNIVERSITY PRESS

Contents

<i>Foreword by András Vasy</i>	<i>page xi</i>
<i>Preface</i>	xv
<i>Acknowledgements</i>	xxiii
1 The Radon Transform in the Plane	1
1.1 Uniqueness and Stability	1
1.2 Range and Support Theorems	5
1.3 The Normal Operator and Singularities	8
1.4 The Funk Transform	19
2 Radial Sound Speeds	25
2.1 Geodesics of a Radial Sound Speed	25
2.2 Travel Time Tomography	31
2.3 Geodesics of a Rotationally Symmetric Metric	36
2.4 Geodesic X-ray Transform	39
2.5 Examples and Counterexamples	45
3 Geometric Preliminaries	52
3.1 Non-trapping and Strict Convexity	52
3.2 Regularity of the Exit Time	58
3.3 The Geodesic Flow and the Scattering Relation	63
3.4 Complex Structure	65
3.5 The Unit Circle Bundle of a Surface	75
3.6 The Unit Sphere Bundle in Higher Dimensions	81
3.7 Conjugate Points and Morse Theory	87
3.8 Simple Manifolds	99
4 The Geodesic X-ray Transform	107
4.1 The Geodesic X-ray Transform	107
4.2 Transport Equations	110

4.3	Pestov Identity	112
4.4	Injectivity of the Geodesic X-ray Transform	114
4.5	Stability Estimate in Non-positive Curvature	117
4.6	Stability Estimate in the Simple Case	123
4.7	The Higher Dimensional Case	126
5	Regularity Results for the Transport Equation	130
5.1	Smooth First Integrals	130
5.2	Folds and the Scattering Relation	132
5.3	A General Regularity Result	135
5.4	The Adjoint $I_{\mathcal{A}}^*$	138
6	Vertical Fourier Analysis	142
6.1	Vertical Fourier Expansions	142
6.2	The Fibrewise Hilbert Transform	149
6.3	Symmetric Tensors as Functions on SM	151
6.4	The X-ray Transform on Tensors	157
6.5	Guillemin–Kazhdan Identity	162
6.6	The Higher Dimensional Case	166
7	The X-ray Transform in Non-positive Curvature	171
7.1	Tensor Tomography	171
7.2	Stability for Functions	173
7.3	Stability for Tensors	179
7.4	Carleman Estimates	182
7.5	The Higher Dimensional Case	186
8	Microlocal Aspects, Surjectivity of I_0^*	189
8.1	The Normal Operator	189
8.2	Surjectivity of I_0^*	196
8.3	Stability Estimates Based on the Normal Operator	201
8.4	The Normal Operator with a Matrix Weight	203
9	Inversion Formulas and Range	208
9.1	Motivation	208
9.2	Properties of Solutions of the Jacobi Equation	211
9.3	The Smoothing Operator W	213
9.4	Fredholm Inversion Formulas	218
9.5	Revisiting the Euclidean Case	224
9.6	Range	227
9.7	Numerical Implementation	229

10	Tensor Tomography	233
10.1	Holomorphic Integrating Factors	233
10.2	Tensor Tomography	236
10.3	Range for Tensors	238
11	Boundary Rigidity	241
11.1	The Boundary Rigidity Problem	241
11.2	Boundary Determination	244
11.3	Determining the Lens Data and Volume	253
11.4	Rigidity in a Given Conformal Class	256
11.5	Determining the Dirichlet-to-Neumann Map	257
11.6	Calderón Problem	260
11.7	Boundary Rigidity for Simple Surfaces	267
12	The Attenuated Geodesic X-ray Transform	269
12.1	The Attenuated X-ray Transform in the Plane	269
12.2	Injectivity Results for Scalar Attenuations	271
12.3	Surjectivity of I_{\perp}^*	275
12.4	Discussion on General Weights	276
13	Non-Abelian X-ray Transforms	277
13.1	Scattering Data	277
13.2	Pseudo-linearization Identity	280
13.3	Elementary Background on Connections	281
13.4	Structure Equations Including a Connection	283
13.5	Scattering Rigidity and Injectivity for Connections	286
13.6	An Alternative Proof of Tensor Tomography	290
13.7	General Skew-Hermitian Attenuations	293
13.8	Injectivity for Connections and Higgs Fields	295
13.9	Scattering Rigidity for Connections and Higgs Fields	298
13.10	Matrix Holomorphic Integrating Factors	299
13.11	Stability Estimate	302
14	Non-Abelian X-ray Transforms II	304
14.1	Scattering Rigidity and Injectivity Results for $\mathfrak{gl}(n, \mathbb{C})$	304
14.2	A Factorization Theorem from Loop Groups	308
14.3	Proof of Theorems 14.1.1 and 14.1.2	310
14.4	General Lie Groups	314
14.5	Range of $I_{A,0}$ and $I_{A,\perp}$	317
14.6	Surjectivity of $I_{A,0}^*$ and $I_{A,\perp}^*$	321
14.7	Adding a Matrix Field	324

15	Open Problems and Related Topics	326
15.1	Open Problems	326
15.2	Related Topics	327
	<i>References</i>	332
	<i>Index</i>	342