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وزارة التعليم العالي و البحث العلمي
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Faculty of Technology

Department of Automatic and Electrotechnic



جامعة البليدة 1

كلية التكنولوجيا

قسم الالية و الإلكتروني تقني

2nd year in Electrical engineering

Course title :

Waves and Vibrations

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Course structuring and planning

Course information

University : Blida 1

Faculty : Technology

Department : Automatic and Electrotechnic

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Course title : waves and Vibrations

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Preface

The Wave and the vibration is an art that requires mastery of several disciplines such as electricity, mechanics and physics.

The vibration has an extremely broad field of application, it concerns the study of mechanical and electrical systems.

The waves, there are familiar kinds of waves in ropes and springs.

The present course is divided into two main parts, the first one deal with Vibrations in four detailed chapters.

The second part of the course covers fours chapters in waves.

Through educational sequences allowing the assimilation of the expected concepts, the present course in confined.

The content of the present support corresponds in accordance with the framework taught in electrical engineering, it is attended for second year students in electrical engineering.

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Part A : Vibrations

Chapter 1 : generality on vibrations

1.1. Objectives

At the end on the present chapter, the student will be able to develop the Lagrange equations and to identify the type of oscillation.

1.2. Introduction

The vibrations are oscillations of an object near its equilibrium point. The oscillation can be regular like the movement of a spring or pendulum or random like the movement of a tire on a gravel path.

1.3. Necessity

Among other things, the student need to know:

- Complex numbers.
- Total and partial derivatives (optional but useful for finding your way).

1.4. Definitions of vibrational movements

Before defining the vibrations in depth, it is useful to learn about some physical concepts such as the stable and the unstable equilibrium point [1].

1.4.1. Stable equilibrium point

If we move away a particle placed at his equilibrium point, its movement will allow it to return to his initial position (Fig.1)

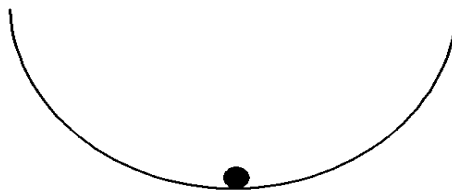


Fig.1

1.4.2. unstable equilibrium point

If we move away a particle placed at his equilibrium point, its movement will not allow it to return to his initial position (Fig.2)

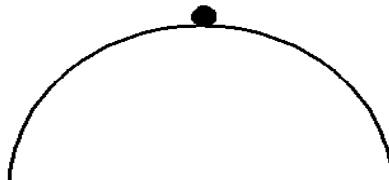


Fig.2

1.4.3. Examples on vibratory movements

1.4.3.1. Mass attached to spring

The system below is composed of a mass (m) connected to a spring (k) vertically [1] [2] [3].

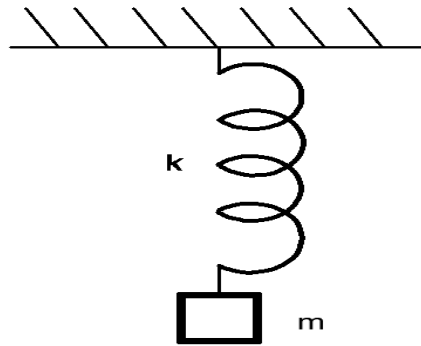


Fig.3

The mass is attached to one end of the spring while the other end is held fixed. At rest, the mass is pulled down and then released. The present system will go back and forth around the equilibrium position. This displacement is called vibrational movement.

1.4.3.2. Simple pendulum

The mass is pulled from its equilibrium position with an angle (θ) and then released as can be shown in the figure bellow.

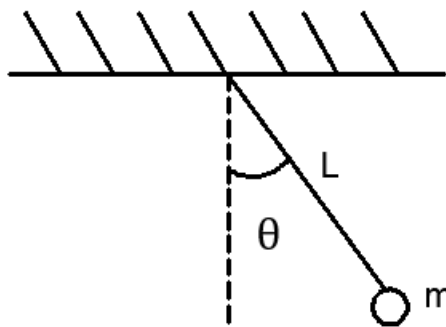


Fig.4

The present system will go back and forth around the equilibrium position. This displacement is called vibrational movement.

1.4.3.3. Disk attached to an inextensible wire

The system below consists of a disc suspended in the center with an inextensible wire. When the disc is rotated with an angle (θ) and then released, the system will go back and forth around the equilibrium position. This displacement is called vibrational movement [1].

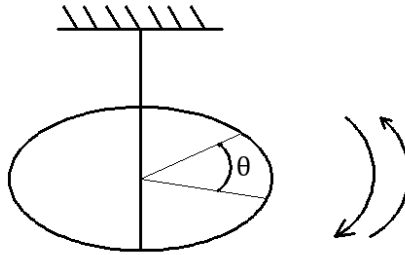


Fig.5

1.4.3.4. Electrical circuit

The system below consists of a resistance, inductance and capacitance [1].

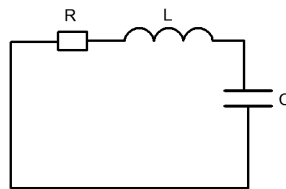


Fig.6

1.5. Different types of vibrations

1.5.1 Undamped free oscillation

The system in this case ensures vibrations without any external force. The movement will never attenuate, it vibrates indefinitely.

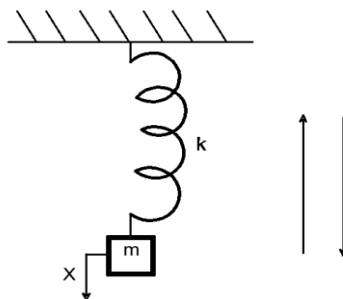


Fig.7

1.5.2. Damped free oscillation

The energy in the present system will gradually dissipate during a period of time.

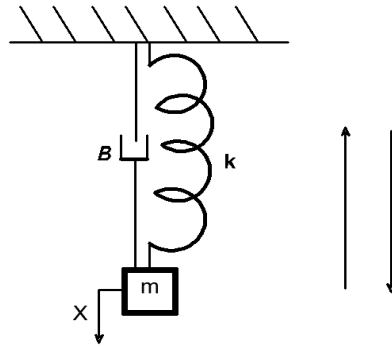


Fig.8

1.5.3. Undamped forced oscillation

The system in this case ensures vibrations through an external force which create the movement. The energy in the present system is conserved. The movement will not attenuate.

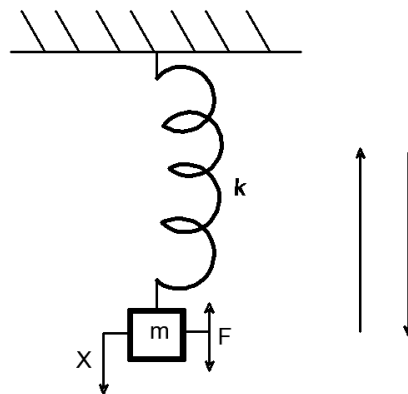


Fig.9

1.5.4. Damped forced oscillation

The system in this case ensures vibrations through an external force which create the movement. The energy in the present system will gradually dissipate during a period of time [4].

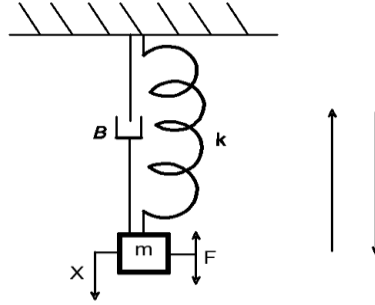


Fig.10

1.6. Methods of resolution

The study of vibrational movement is carried out through the determination of the differential equation movement. The present methods aim to identify the oscillator system viz; position, speed, amplitude, frequency, etc. Generally there exist three methods for the resolution including:

- The theorem of conservation of energy
- The fundamental relationship of dynamics
- The Lagrange formalism

1.6.1 Kinetic energy

The kinetic energy is an energy that related to a movement (it can be rotation or translation direction).

Mathematically we can write the following equations:

$$T = \frac{1}{2} m \dot{x}^2 \quad (1)$$

$$T = \frac{1}{2} J \dot{\theta}^2 \quad (2)$$

Where, $\dot{x} = \frac{dx}{dt}$, J : is the inertia moment, $\dot{\theta} = \frac{d\theta}{dt}$ is the angular velocity, x : is the direction, θ : is the angular velocity.

a. Inertia Moment

The moment of inertia characterizes the distribution of mass around an axis of rotation. The most frequently encountered moments are the following [1]:

- A bar with mass (m) and a length (l) rotates around an axis (Δ) (the bar rotates around its center):

$$J = \frac{1}{12} ml^2$$
- A bar with mass (m) and a length (l) rotates around an axis (Δ) (the bar rotates on its extremity): $J = \frac{1}{3} ml^2$
- Punctual mass attached to a wire: $J = ml^2$
- A disk: $J = \frac{1}{2} mR^2$
- A cercle: $J = mR^2$

1.6.2. Potential energy

The potential energy can take the following forms:

- Gravitation: $U = mgh$

Note : $U = 0$ in the reference position

- Elastic deformation: $U = \frac{1}{2} kx^2$
- Suspended mass: $U = mgl(1 - \cos\theta) \approx \mp mgl(\cos\theta)$

1.7. Lagrange method

The Lagrangian L represents a difference between the kinetic and the potential energies. Mathematically, we can write the following equation:

$$L = T - U \quad (3)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = F \quad (4)$$

Where,

D : is the dissipation energy , F : is the external force

Chapter 2: Free oscillation with one degree of freedom

2.1. Objectives

At the end on the present chapter, the student will be able to develop the Lagrange equations for a free oscillation for one degree of freedom.

2.2. Necessity

Generalities on vibration movements

2.3. Introduction

The free undamped oscillation is a form in which the energy is conserved, and therefore, there is no external force which opposes the movement [1].

2.4. Free undamped oscillation

a. Mass attached to a spring

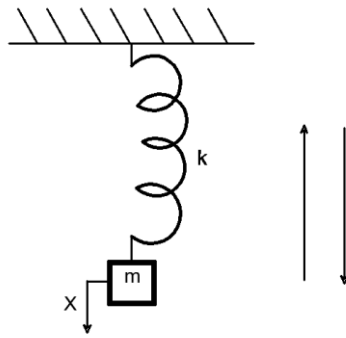


Fig.1

Considering a mass attached to a spring as can be shown in the figure.1, the proposed mechanical system could be resolved using the Lagrange equation as follows:

$$T = \frac{1}{2} m \dot{x}^2 \quad (1)$$

$$U = \frac{1}{2} k x^2 \quad (2)$$

$$L = T - U \quad (3)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = F \quad (4)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad (5)$$

Such as :

$$\begin{cases} \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = m\ddot{x} \\ \frac{\partial L}{\partial x} = kx \end{cases}$$

$$m\ddot{x} + kx = 0 \quad (6)$$

Where, $\omega_0^2 = \frac{k}{m}$ (clean pulsation or proper pulsation)

$$\ddot{x} + \omega_0^2 x = 0 \quad (7)$$

The above equation is simply a differential equation in second order, the final solution could be written as follows:

$$x(t) = A \cos(\omega_0 t + \varphi) \quad (8)$$

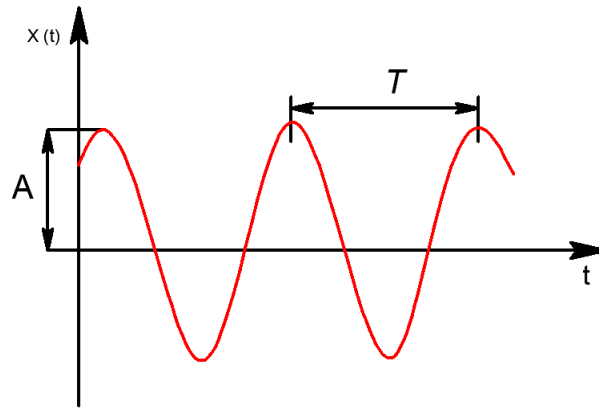


Fig.2

Where: **A**: is the amplitude of oscillation, **φ**: is the initial phase,

Finally, we have to use the initial condition in order to determine the initial pulsation and phase;

IC: $t = 0, x_0 = A, \dot{x}_0 = 0$

$$\dot{x} = \frac{dx}{dt} = -A \sin(\omega_0 t + \varphi)$$

$\varphi = 2\pi n, , \text{for } n = 0 \rightarrow \varphi = 0$

$$x_0 = A \cos(w_0 t_0 + \varphi) = A \cos(w_0 \times 0 + 0) = A$$

$w_0 = 2\pi f = \frac{2\pi}{T}$, T : is the period of oscillation.

b. Simple pendulum

The system below contains a mass m attached to an extensible wire of a length L (the mass of the wire is negligible). The oscillation is ensured by moving away the object from its equilibrium position with an angle θ and then released. The system in this case will oscillates around its equilibrium position.

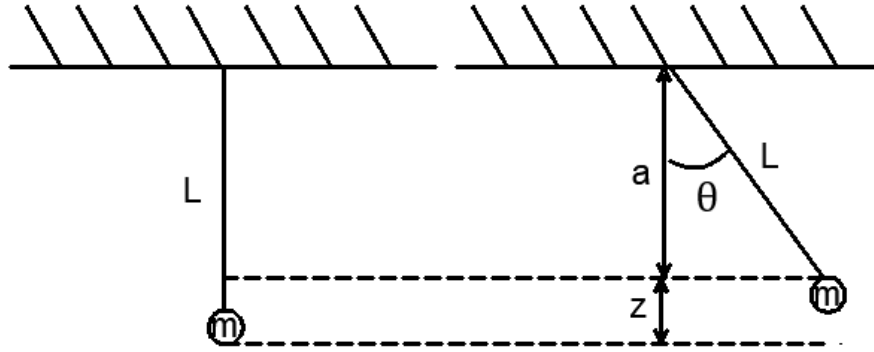


fig.3

Where, $a = L \cos \theta$, $z = L - a = L - L \cos \theta$

The Lagrangian equations could be written as follows:

$$T = \frac{1}{2} J \dot{\theta}^2 = \frac{1}{2} m l^2 \dot{\theta}^2 \quad (9)$$

$$U = mgh = mgl(1 - \cos \theta) \quad (10)$$

$$L = T - U \quad (11)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} + \frac{\partial D}{\partial \dot{\theta}} = F \quad (12)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \quad (13)$$

Such as :

$$\begin{cases} \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = ml^2 \ddot{\theta} \\ \frac{\partial L}{\partial \theta} = -mgl \sin \theta \end{cases}$$

$$ml^2 \ddot{\theta} + mgl \sin \theta = 0 \rightarrow l \ddot{\theta} + g \sin \theta = 0 \quad (14)$$

Where, $w_0^2 = \frac{g}{l}$ (clean pulsation or proper pulsation)

$$\ddot{\theta} + w_0^2 \theta = 0 \quad (15)$$

The above equation is simply a differential equation in second order, the final solution could be written as follows:

$$\theta(t) = A \cos(w_0 t + \varphi) \quad (16)$$

Where: A : is the amplitude of oscillation, φ : is the initial phase.

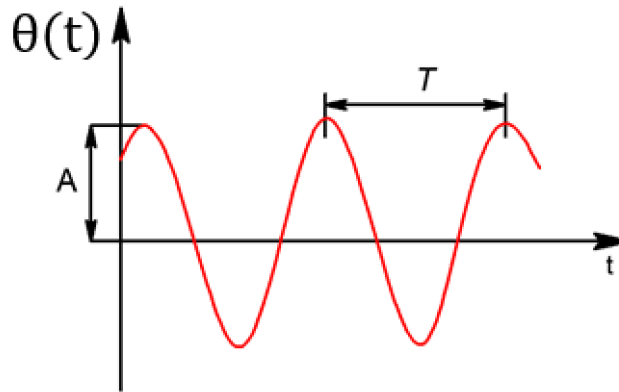


Fig.4

Finally, we have to use the initial condition in order to determine the initial pulsation and phase;

$$\text{IC: } t = 0, \theta_0 = A, \dot{\theta}_0 = 0$$

$$\dot{\theta} = \frac{d\theta}{dt} = -A \sin(w_0 t + \varphi)$$

$$\varphi = 2\pi n, \text{ for } n = 0 \rightarrow \varphi = 0$$

$$\theta_0 = A \cos(w_0 t_0 + \varphi) = A \cos(w_0 \times 0 + 0) = A$$

Where, $w_0 = 2\pi f = \frac{2\pi}{T}$, T : is the period of oscillation.

c. Electrical circuit (LC)

The free undamped oscillation could be found in electrical circuit composed of inductance L and capacitor C as can be shown the figure below.

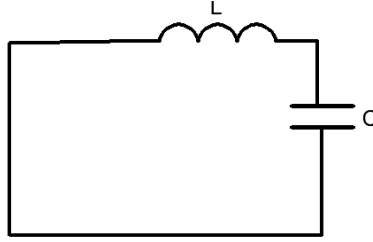


Fig.5

Using the mesh equation we can found the followings:

$$u_l + u_c = 0 \quad (17)$$

Where: $u_l = L \frac{di}{dt}$, $u_c = \frac{1}{C} \int i dt$, $i = \dot{q} = \frac{dq}{dt}$, $\frac{di}{dt} = \frac{d^2q}{dt^2} = \ddot{q}$

$$L\ddot{q} + \frac{1}{C}q = 0 \quad (18)$$

$$\ddot{q} + \frac{1}{LC}q = 0 \quad (19)$$

The above equation is simply a differential equation in second order, the final solution could be written as follows:

$$q(t) = A \cos(w_0 t + \varphi) \quad (20)$$

Finally, we have to use the initial condition in order to determine the initial pulsation and phase;

IC: $t = 0$, $q_0 = A$, $\dot{q} = 0$

$$\dot{q} = \frac{dq}{dt} = -A \sin(w_0 t + \varphi)$$

$$\varphi = 2\pi n, \text{ for } n = 0 \rightarrow \varphi = 0$$

$$q_0 = A \cos(w_0 t_0 + \varphi) = A \cos(w_0 \times 0 + 0) = A$$

2.5 Free damped oscillation

In this type of oscillation, the vibrational movement is gradually attenuates, this is due to the presence of an external force. For a mechanical system it is called “ a friction force” and for an electrical system we add a resistance.

2.5.1. Mass attached to a spring and damper (shock absorber)

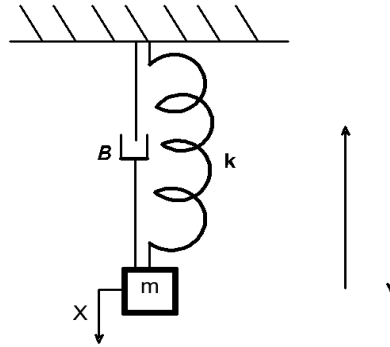


Fig.6

Considering a mass attached to a spring and a damper as can be shown in the figure.6 the Lagrangian equations could be written as follows:

$$T = \frac{1}{2} m \dot{x}^2 \quad (21)$$

$$U = \frac{1}{2} k x^2 \quad (22)$$

$$D = \frac{1}{2} \beta \dot{x}^2 \quad (23)$$

$$L = T - U \quad (24)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = F \quad (25)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = 0 \quad (26)$$

Such as :

$$\begin{cases} \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = m\ddot{x} \\ \frac{\partial L}{\partial x} = kx \\ \frac{\partial D}{\partial \dot{x}} = \beta\dot{x} \end{cases}$$

$$m\ddot{x} + \beta\dot{x} + kx = 0 \quad (27)$$

Where, $w_0^2 = \frac{k}{m}$ and $2\lambda = \frac{\beta}{m}$ (damping coefficient)

$$\ddot{x} + 2\lambda\dot{x} + w_0^2 x = 0 \quad (28)$$

Here, we suppose that: $x = Ae^{rt}$, $\dot{x} = Are^{rt}$, $\ddot{x} = Ar^2e^{rt}$

The above equation is simply a differential equation in second order with a constant coefficient. The solution is based on the discriminant Δ' of the following characteristic equation:

$$r^2 + 2\lambda r + w_0^2 = 0 \quad (29)$$

The discriminant could be written as follows: $\Delta' = \lambda^2 - w_0^2$

Here, there exist three possible solution:

- $\Delta' > 0 \Rightarrow \lambda^2 > w_0^2$: strong damping (hypercritical)

Here, there are two real roots: $r_{1,2} = -\lambda \pm B$

Where: $B = \sqrt{\Delta'} = \sqrt{\lambda^2 - w_0^2}$

In this situation, the damping is strong (heavy), the oscillation stops rapidly after a short period of time.

Finally, we have the following solution:

$$x(t) = e^{-\lambda t}(A_1 e^{r_1 t} + A_2 e^{-r_2 t}) \quad (30)$$

- $\Delta' = 0 \Rightarrow \lambda^2 = w_0^2$: critical damping (critical)

$$r = -\lambda$$

Finally, we have the following solution:

$$x(t) = e^{-\lambda t}(a + bt) \quad (31)$$

Critical damping; quickest return to equilibrium position without oscillation

- $\Delta' < 0 \Rightarrow \lambda^2 < w_0^2$: low damping (pseudo-periodic)

Here, there are two complex roots: $r = -\lambda \pm jw$

Where: $w = \sqrt{w_0^2 - \lambda^2}$

In this situation, the damping is low (light), the oscillation weakens after a good period of time.

Finally, we have the following solution:

$$x(t) = Ae^{-\lambda t}\cos(wt - \varphi) \quad (32)$$

The representation of three cases of study is illustrated in the figure.8.

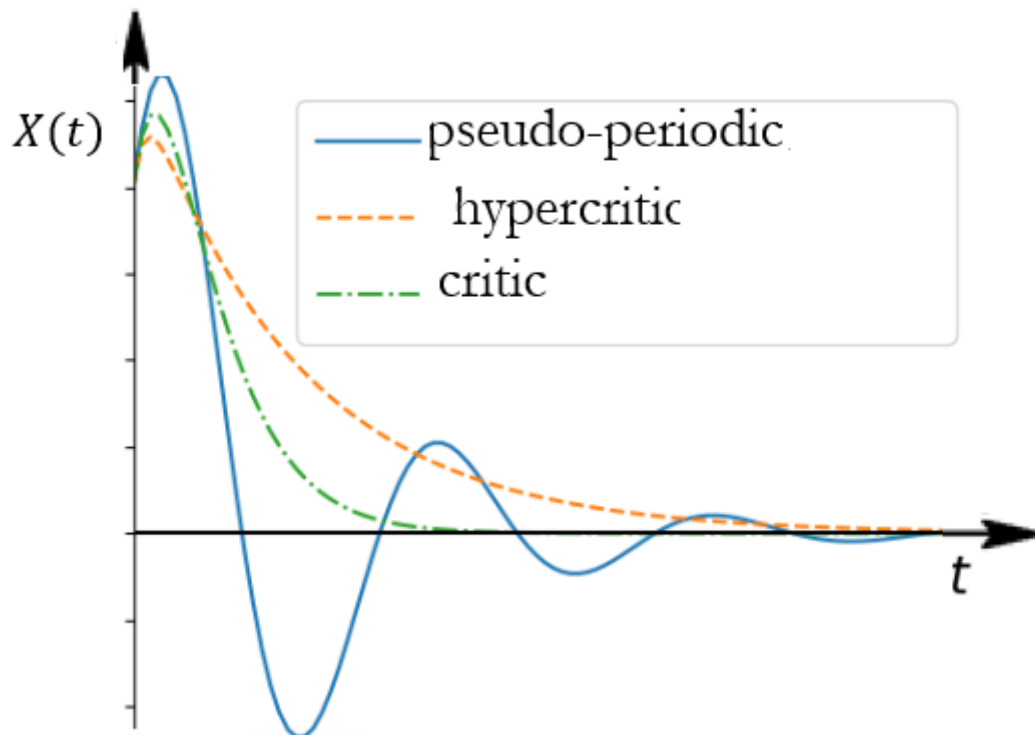


Fig.7

2.5.1. Logarithmic decrement

The logarithmic decrement δ is parameter which indicates the rate of decrease in amplitude of two successive oscillations (A_n, A_{n+1}) [4].

$$\frac{A_n}{A_{n+1}} = \frac{Ae^{-\lambda t_n} \cos(\omega t_n - \varphi)}{Ae^{-\lambda t_{n+1}} \cos(\omega t_{n+1} - \varphi)} = \frac{e^{-\lambda t_n}}{e^{-\lambda t_{n+1}}} = e^{\lambda(t_{n+1}-t)} = e^{\lambda(\Delta t)}$$

$$\ln \frac{A_n}{A_{n+1}} = \ln e^{\lambda(\Delta t)}$$

$$\ln \frac{A_n}{A_{n+1}} = \lambda(\Delta t) \Rightarrow \ln \frac{A_n}{A_{n+1}} = \lambda T$$

Note :

Mathematically, we can inverse the operation as follows :

$$\frac{A_{n+1}}{A_n} = \frac{Ae^{-\lambda t_{n+1}} \cos(\omega t_{n+1} - \varphi)}{Ae^{-\lambda t_n} \cos(\omega t_n - \varphi)} = \frac{e^{-\lambda t_{n+1}}}{e^{-\lambda t_n}} = e^{-\lambda t_{n+1} + \lambda t} = e^{-\lambda(t_{n+1}-t)} = e^{-\lambda(\Delta t)}$$

$$\ln \frac{A_{n+1}}{A_n} = \ln e^{-\lambda(\Delta t)}$$

$$\ln \frac{A_{n+1}}{A_n} = -\lambda(\Delta t) \Rightarrow \ln \frac{A_{n+1}}{A_n} = -\lambda T$$

Where, T is the period between two consecutive amplitude

Finally, we can write the logarithmic decrement expression as follows:

$$\ln \frac{A_n}{A_{n+1}} = \lambda T = \delta \quad (33)$$

We can generalise the above expression for two amplitudes (A_t, A_{t+nT}) by the following

$$\delta = \frac{1}{n} \ln \frac{A_t}{A_{t+nT}} = \lambda T \quad (34)$$

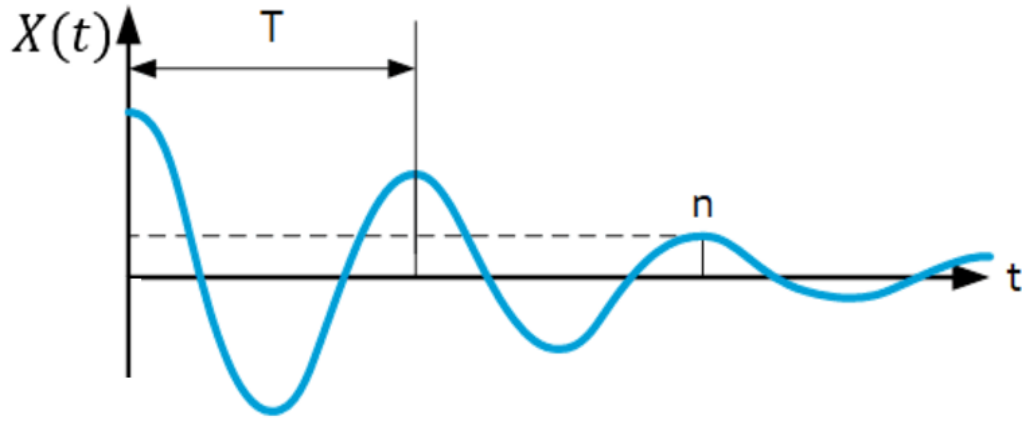


Fig.8

b. Electrical circuit (RLC)

The free damped oscillation could be found in electrical circuit composed of inductance L , resistor R and capacitor C as illustrated in the figure below [4].

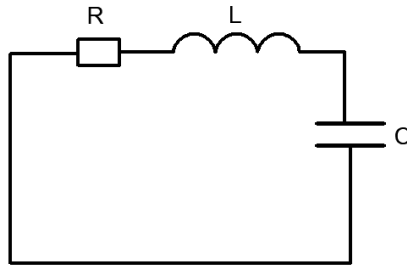


Fig.9

Here, we can use the mesh equation we found the followings:

$$u_l + u_R + u_c = 0 \quad (35)$$

Where: $u_l = L \frac{di}{dt}$, $u_c = \frac{1}{C} \int i dt$, $u_R = Ri$, $i = \dot{q} = \frac{dq}{dt}$, $\frac{di}{dt} = \frac{d^2q}{dt^2} = \ddot{q}$

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0 \quad (36)$$

Where, $w_0^2 = \frac{1}{LC}$ (clean pulsation or proper pulsation), $2\lambda = \frac{R}{L}$ (λ is called stock absorber coefficient)

$$\ddot{q} + 2\lambda\dot{q} + w_0^2q = 0 \quad (37)$$

2.6. Analogy of mechanical-electrical systems

The table.1 shows the main parameters for a free oscillation systems

Mechanical system	Electrical system
x	q
$w_0 = \sqrt{\frac{k}{m}}$	$w_0 = \frac{1}{\sqrt{LC}}$
K	C
B	R
M	L
F	e

Table.1 analogy of mechanical-electrical systems

Chapter 3: Forced oscillation with one degree of freedom

3.1 Objectives

At the end on the present chapter, the students will be able to develop the Lagrange equations for forced oscillation with one degree of freedom.

3.2. Necessity

Resolve the equations of mechanical-electrical systems for the second chapter.

3.3. Introduction

The damped oscillation is a form in which an external force (friction force) is applied to the system at hand.

3.4. Forced dumped oscillation

a. Mass attached to a spring and shock absorber with an external force

In this type of vibration, a frictional force is applied to the system as can be shown the figure.1.

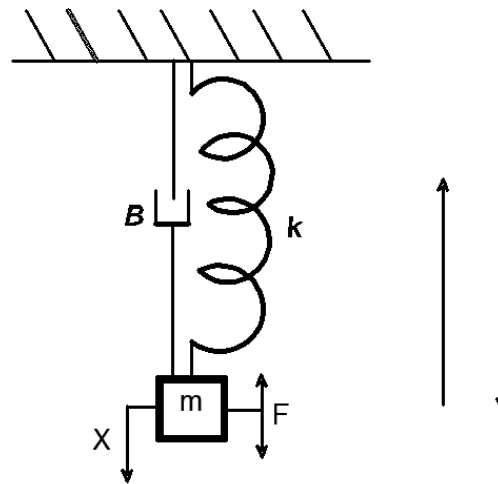


fig.1

The Lagrangian equations could be written as follows:

$$T = \frac{1}{2} m \dot{x}^2 \quad (1)$$

$$U = \frac{1}{2} k x^2 \quad (2)$$

$$D = \frac{1}{2} \beta \dot{x}^2 \quad (3)$$

$$w_{\text{ext}} = Fx \quad (4)$$

$$L = T - U \quad (5)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = \frac{w_{ext}}{\partial x} \quad (6)$$

Such as :

$$\begin{cases} \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = m\ddot{x} \\ \frac{\partial L}{\partial x} = kx \\ \frac{\partial D}{\partial \dot{x}} = \beta\dot{x} \\ \frac{w_{ext}}{\partial x} = F \end{cases}$$

$$m\ddot{x} + \beta\dot{x} + kx = F \quad (7)$$

$$\ddot{x} + 2\lambda\dot{x} + w_0^2 x = \frac{F}{m} \quad (8)$$

The equation.8 is a differential equation in second order with a second member. The solution $x(t)$ is a sum of a general solution without a second member $x_g(t)$, and a particular solution $x_p(t)$ [2].

Here, we make the following suppositions:

$$x = x_0 e^{j(wt+\varphi)}, \dot{x} = jwx_0 e^{j(wt+\varphi)}, \ddot{x} = -w^2 x_0 e^{j(wt+\varphi)}, F = f_0 e^{j\omega t}$$

The above variables are placed in the equation.8 as follows:

$$-w^2 x_0 e^{j(wt+\varphi)} + 2\lambda jwx_0 e^{j(wt+\varphi)} + w_0^2 x_0 e^{j(wt+\varphi)} = f_0 e^{j\omega t}$$

$$(-w^2 x_0 + 2\lambda jwx_0 + w_0^2 x_0) e^{j(wt+\varphi)} = f_0 e^{j\omega t}$$

$$(-w^2 + w_0^2 + 2\lambda jw) x_0 e^{j(wt+\varphi)} = f_0 e^{j\omega t}$$

$$(\sqrt{(-w^2 + w_0^2)^2 + 4\lambda^2 w^2}) e^{j(\arctan(\frac{2\lambda w}{-w^2 + w_0^2}))} x_0 e^{j(wt+\varphi)} = f_0 e^{j\omega t}$$

$$(\sqrt{(-w^2 + w_0^2)^2 + 4\lambda^2 w^2}) x_0 e^{j(\arctan(\frac{2\lambda w}{-w^2 + w_0^2}) + wt + \varphi)} = f_0 e^{j\omega t}$$

Where:

$$\begin{cases} x_0 = \frac{f_0}{\sqrt{(-w^2 + w_0^2)^2 + 4\lambda^2 w^2}} \\ \varphi = -\arctang \frac{2\lambda w}{-w^2 + w_0^2} \end{cases} \quad (9)$$

Note:

- For a complex number: $z = a + jb = |z|e^{j(\arctg(\frac{b}{a}))}$ $\begin{cases} |z| = \sqrt{a^2 + b^2} \\ \varphi = \arctg(\frac{b}{a}) \end{cases}$
- $e^a e^b = e^{a+b}$
- **Variation of x_0 :**

$$x_0(w) = \frac{f_0}{\sqrt{(-w^2 + w_0^2)^2 + 4\lambda^2 w^2}}$$

$$\frac{dx_0}{dw} = 0 \Rightarrow \frac{d}{dw} ((-w^2 + w_0^2)^2 + 4\lambda^2 w^2) = 0 \Rightarrow \frac{d}{dw} (w^4 - 2w^2 w_0^2 + w_0^4 + 4\lambda^2 w^2) = 0$$

$$4w^3 - 4w w_0^2 + 8\lambda^2 w = 0 \Rightarrow w^2 - w_0^2 + 2\lambda^2 = 0 \Rightarrow w = \sqrt{w_0^2 - 2\lambda^2}$$

Where, w is the resonance pulsation (pulse) and we can write the following formulas:

$$w = w_r = \sqrt{w_0^2 - 2\lambda^2}$$

By replacing the w_r in the equation. 9 we found:

$$x_0(w) = \frac{f_0}{\sqrt{(-w^2 + w_0^2)^2 + 4\lambda^2 w^2}}$$

$$x_0(w) = \frac{f_0}{\sqrt{(-w_0^2 + 2\lambda^2 + w_0^2)^2 + 4\lambda^2 (w_0^2 - 2\lambda^2)}}$$

$$x_0(w) = \frac{f_0}{\sqrt{(-w_0^2 + 2\lambda^2 + w_0^2)^2 + 4\lambda^2 w_0^2 - 8\lambda^4}}$$

$$x_0(w) = \frac{f_0}{\sqrt{4\lambda^4 + 4\lambda^2 w_0^2 - 8\lambda^4}} = \frac{f_0}{\sqrt{-4\lambda^4 + 4\lambda^2 w_0^2}}$$

Finally:

$$x_0(w) = \frac{f_0}{2\lambda \sqrt{w_0^2 - \lambda^2}}$$

Where, $w' = \sqrt{w_0^2 - \lambda^2}$ and we can write the following formulas:

$$x_{0\max} = \frac{f_0}{2\lambda w'}$$

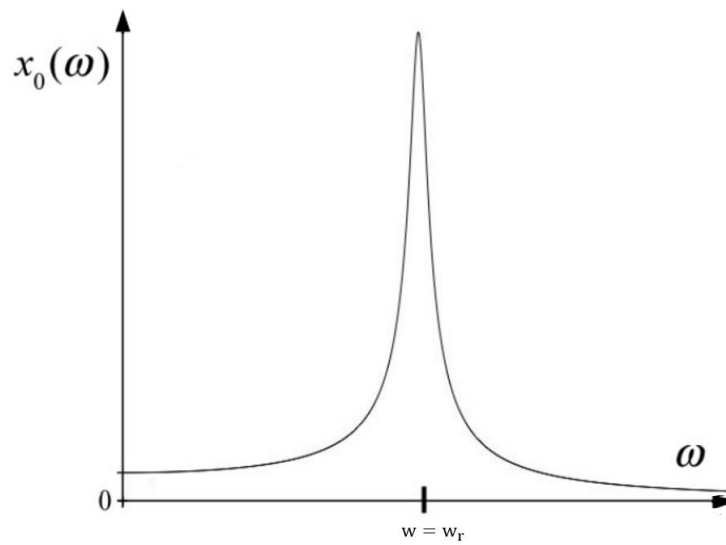


Fig.2

Characteristics:

- Bandwidth: the bandwidth is defined as $\Delta w = w_2 - w_1$.
- Quality factor : the quality factor Q is inversely proportional to the friction coefficient. Note that more the friction coefficient increases, the value of the resonance pulsation w_r moves away from the proper pulsation w_0 .

b. Mechanical impedance

Mechanical impedances are known for mechanical systems (subjected to a force that varies sinusoidally) [1].

$$m\ddot{x} + \beta\dot{x} + kx = F$$

Here, we make the followings:

$$\left\{ \begin{array}{l} \ddot{x} = \frac{dv}{dt} = \dot{v}, \dot{x} = v, x = \int v dt \\ v = v_0 e^{j(\omega t + \varphi)} \Rightarrow v = \bar{v}_0 e^{j\omega t} \text{ where, } \bar{v}_0 = v_0 e^{j\varphi} \\ \dot{v} = \frac{dv}{dt} = j\omega \bar{v}_0 e^{j\omega t} \\ \int v dt = \frac{\bar{v}_0}{j\omega} e^{j\omega t} \\ F = f_0 e^{j\omega t} \end{array} \right. \quad (10)$$

these variables will be placed in the differential equation as follows:

$$\begin{aligned} m \frac{dv}{dt} + \beta v + k \int v dt &= f_0 \sin(\omega t) \\ m j\omega \bar{v}_0 e^{j\omega t} + \beta \bar{v}_0 e^{j\omega t} + k \frac{\bar{v}_0}{j\omega} e^{j\omega t} &= f_0 e^{j\omega t} \\ m j\omega \bar{v}_0 + \beta \bar{v}_0 - j \frac{k}{\omega} \bar{v}_0 &= f_0 \\ \left[\beta + j \left(m\omega - \frac{k}{\omega} \right) \right] \bar{v}_0 &= f_0 \end{aligned}$$

Finally,

$$f_0 = \bar{v}_0 Z \quad (11)$$

Where, Z : is the mechanical impedance (ohm), the module and the phase are given as follows:

$$\left\{ \begin{array}{l} |Z| = \sqrt{\beta^2 + \left(m\omega - \frac{k}{\omega} \right)^2} \\ \varphi = \arctg\left(\frac{m\omega - \frac{k}{\omega}}{\beta}\right) \end{array} \right.$$

c. Electrical impedance

For an electrical system carried by a sinusoidal current we distinguish an electrical impedance.

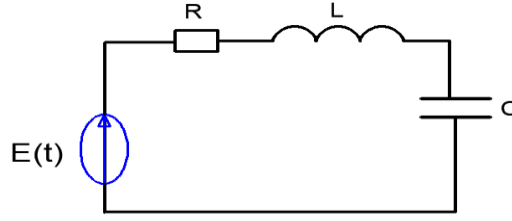


Fig.3

The RLC circuit with a sinusoidal generator source, the Kirchhoff Voltage Law is written as follows:

$$Ri + L \frac{di}{dt} + \frac{1}{C} \int i dt = E_0 \cos(\omega t) \quad (12)$$

where:

$$\left\{ \begin{array}{l} i = I_0 e^{j(\omega t + \varphi)} \Rightarrow i = \bar{I}_0 e^{j\omega t} \text{ where, } \bar{I}_0 = I_0 e^{j\varphi} \\ \frac{di}{dt} = j\omega \bar{I}_0 e^{j\omega t} \\ \int i dt = \frac{\bar{I}_0}{j\omega} e^{j\omega t} \\ e(t) = e_0 e^{j\omega t} \end{array} \right. \quad (13)$$

$$R\bar{I}_0 e^{j\omega t} + jL\omega \bar{I}_0 e^{j\omega t} + \frac{1}{jC\omega} \bar{I}_0 e^{j\omega t} = e_0 e^{j\omega t} \quad (14)$$

$$R\bar{I}_0 e^{j\omega t} + jL\omega \bar{I}_0 e^{j\omega t} - \frac{j}{C\omega} \bar{I}_0 e^{j\omega t} = e_0 e^{j\omega t} \quad (15)$$

$$\left[R + j \left(L\omega - \frac{1}{C\omega} \right) \right] \bar{I}_0 = e_0 \quad (16)$$

Finally,

$$e_0 = \bar{I}_0 Z \quad (17)$$

Where, Z : is the electrical impedance (ohm), the module and the phase are given as follows:

$$\left\{ \begin{array}{l} |Z| = \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega} \right)^2} \\ \varphi = \arctg\left(\frac{L\omega - \frac{1}{C\omega}}{R}\right) \end{array} \right. \quad (18)$$

Chapter 04 : Oscillation with two degrees of freedom

4.1 Objectives

At the end on the present chapter, the students will be able to develop the Lagrange equations for systems with two degrees of freedom.

4.2. Necessity

Able to resolve the equations of mechanical-electrical systems

4.3. Introduction

In the present chapter, the degree of freedom in the systems is two. This means that there exist two independent variables in the system.

In this type of systems, the coupling can take generally three forms namely, inertia, elasticity or capacity.

4.4. Forced undamped oscillation

a. Mass attached to a spring with an external force

considering the mechanical system as shown in the figure.1, the system contains two masses (m_1, m_2) with a translation on (x, y) respectively. Three springs (k_1, k_2, k_3) and an external force applied to the mass m_1 [4]

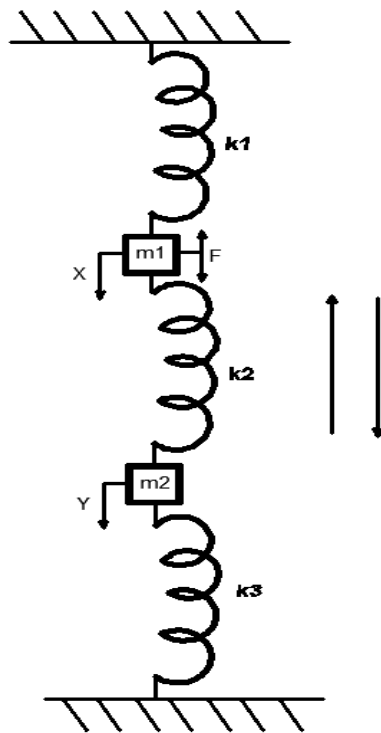


Fig.1

The Lagrangian equation could be written as follows:

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 \quad (1)$$

$$U = \frac{1}{2}k_1(0 - x)^2 + \frac{1}{2}k_2(x - y)^2 + \frac{1}{2}k_3(y - 0)^2 \quad (2)$$

$$D = 0 \quad (3)$$

$$w_{\text{ext}} = Fx \quad (4)$$

$$L = T - U \quad (5)$$

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 - \frac{1}{2}k_1(0 - x)^2 - \frac{1}{2}k_2(x - y)^2 - \frac{1}{2}k_3(y - 0)^2$$

$$\begin{bmatrix} \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = \frac{w_{\text{ext}}}{\partial x} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \frac{\partial D}{\partial \dot{y}} = \frac{w_{\text{ext}}}{\partial y} \end{bmatrix}$$

$$\begin{bmatrix} m_1\ddot{x} + (k_1 + k_2)x - k_2y = F \\ m_2\ddot{y} - k_2x + (k_2 + k_3)y = 0 \end{bmatrix} \quad (6)$$

Such as :

$$\left\{ \begin{array}{l} \frac{\partial L}{\partial \dot{x}} = m_1\dot{x} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = m_1\ddot{x} \\ \frac{\partial L}{\partial x} = -k_1x - k_2x - k_2y \\ \frac{\partial D}{\partial \dot{x}} = 0 \\ \frac{w_{\text{ext}}}{\partial x} = F \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{\partial L}{\partial \dot{y}} = m_2\dot{y} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = m_2\ddot{y} \\ \frac{\partial L}{\partial y} = k_2x - k_2y - k_3y \\ \frac{\partial D}{\partial \dot{y}} = 0 \\ \frac{w_{\text{ext}}}{\partial y} = 0 \end{array} \right.$$

Here, we make the following simplifications :

$$\begin{cases} k_1 = k_2 = k_3 = k \\ m_1 = m_2 = m \\ x = x_0 e^{j\omega t}, \dot{x} = j\omega x_0 e^{j\omega t}, \ddot{x} = -\omega^2 x_0 e^{j\omega t} \\ y = y_0 e^{j\omega t}, \dot{y} = j\omega y_0 e^{j\omega t}, \ddot{y} = -\omega^2 y_0 e^{j\omega t} \\ F = f_0 e^{j\omega t} \end{cases}$$

Then, we replace the above simplification to the equation.6 as follows:

$$\begin{bmatrix} -m\omega^2 x_0 e^{j\omega t} + (k_1 + k_2)x_0 e^{j\omega t} - k_2 y_0 e^{j\omega t} = f_0 e^{j\omega t} \\ -m\omega^2 y_0 e^{j\omega t} - k_2 x_0 e^{j\omega t} + (k_2 + k_3)y_0 e^{j\omega t} = 0 \end{bmatrix} \quad (7)$$

$$\begin{bmatrix} (-\omega^2 + 2\frac{k}{m})x_0 - \frac{k}{m}y_0 = f_0 \\ -\frac{k}{m}x_0 + (-\omega^2 + 2\frac{k}{m})y_0 = 0 \end{bmatrix}$$

$$x_0 = \frac{\begin{vmatrix} f_0 & -\frac{k}{m} \\ 0 & (-\omega^2 + 2\frac{k}{m}) \end{vmatrix}}{\begin{vmatrix} (-\omega^2 + 2\frac{k}{m}) & -\frac{k}{m} \\ -\frac{k}{m} & (-\omega^2 + 2\frac{k}{m}) \end{vmatrix}} = \frac{f_0 \left(-\omega^2 + 2\frac{k}{m}\right)}{(-\omega^2 + 2\frac{k}{m})^2 - (\frac{k}{m})^2}$$

$$y_0 = \frac{\begin{vmatrix} (-\omega^2 + 2\frac{k}{m}) & f_0 \\ -\frac{k}{m} & 0 \end{vmatrix}}{\begin{vmatrix} (-\omega^2 + 2\frac{k}{m}) & -\frac{k}{m} \\ -\frac{k}{m} & (-\omega^2 + 2\frac{k}{m}) \end{vmatrix}} = \frac{f_0 \frac{k}{m}}{(-\omega^2 + 2\frac{k}{m})^2 - (\frac{k}{m})^2}$$

b. Electrical circuit

Considering an electrical circuit (see the figure.3). Each cell in the proposed electrical circuit contains an inductance and a capacity. Between the first and the second cells a capacity is placed. The generator delivers energy in the first cell.

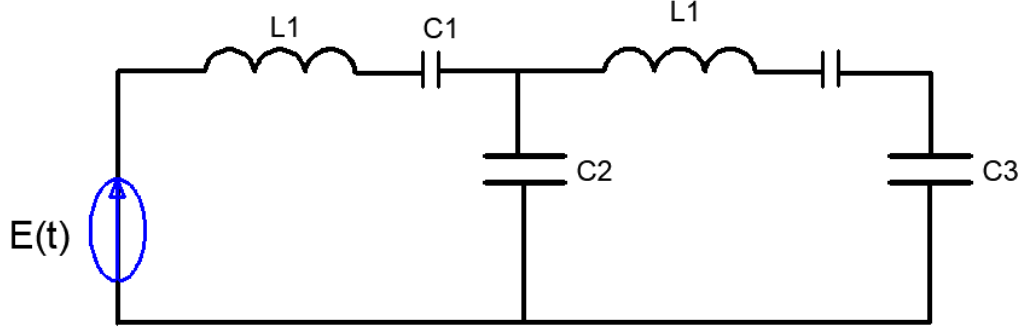


Fig.2

Here, we can use the mesh equation we found the followings:

For the first cell:

$$u_{L1} + u_{C1} + u_{C2} = E(t) \quad (8)$$

$$L_1 \ddot{q}_1 + \frac{1}{C_1} q_1 + \frac{q_1 - q_2}{C_1} = u_0 \cos(\omega t) \quad (9)$$

For the second cell:

$$u_{L2} + u_{C3} + u_{C2} = 0 \quad (10)$$

$$L_2 \ddot{q}_2 + \frac{1}{C_3} q_2 + \frac{q_2 - q_1}{C_2} = 0 \quad (11)$$

Where: $u_L = L \frac{di}{dt}$, $u_C = \frac{1}{C} \int i dt$, $u_R = Ri$, $i = \dot{q} = \frac{dq}{dt}$, $\frac{di}{dt} = \frac{d^2q}{dt^2} = \ddot{q}$

By making the following simplifications:

$$\begin{cases} L_1 = L_2 = L \\ C_1 = C_2 = C_3 = C \\ q_1 = Ae^{j\omega t}, \dot{q}_1 = j\omega A e^{j\omega t}, \ddot{q}_1 = -\omega^2 A e^{j\omega t} \\ q_2 = Ae^{j\omega t}, \dot{q}_2 = j\omega A e^{j\omega t}, \ddot{q}_2 = -\omega^2 A e^{j\omega t} \\ E = e_0 e^{j\omega t} \end{cases}$$

$$\begin{bmatrix} \ddot{q}_1 + \frac{1}{LC}q_1 + \frac{q_1 - q_2}{LC} = u_0 \cos(\omega t) \\ \ddot{q}_2 + \frac{1}{LC}q_2 + \frac{q_2 - q_1}{LC} = 0 \end{bmatrix}$$

$$\begin{bmatrix} (-\omega^2 + \frac{2}{LC})A - \frac{1}{LC}B = e_0 \\ -\frac{1}{LC}A + (-\omega^2 + \frac{2}{LC})B = 0 \end{bmatrix}$$

$$x_0 = \frac{\begin{vmatrix} e_0 & -\frac{1}{LC} \\ 0 & (-\omega^2 + \frac{2}{LC}) \end{vmatrix}}{\begin{vmatrix} (-\omega^2 + \frac{2}{LC}) & -\frac{1}{LC} \\ -\frac{1}{LC} & (-\omega^2 + \frac{2}{LC}) \end{vmatrix}} = \frac{E_0 \left(-\omega^2 + \frac{2}{LC}\right)}{(-\omega^2 + \frac{2}{LC})^2 - (\frac{1}{LC})^2}$$

$$y_0 = \frac{\begin{vmatrix} (-\omega^2 + \frac{2}{LC}) & e_0 \\ -\frac{1}{LC} & 0 \end{vmatrix}}{\begin{vmatrix} (-\omega^2 + \frac{2}{LC}) & -\frac{1}{LC} \\ -\frac{1}{LC} & (-\omega^2 + \frac{2}{LC}) \end{vmatrix}} = \frac{e_0 \frac{1}{LC}}{(-\omega^2 + \frac{2}{LC})^2 - (\frac{1}{LC})^2}$$

Part B: Waves

Chapter 01: Propagation phenomena

1.1 Introduction

A mechanical wave is a phenomenon of propagation of a disturbance in an environment without transporting matter [5] [6] [7] [1].

1.2 Mechanical wave characteristics

- The wave propagates from a source and in all directions.
- The propagation allows the transport of energy.
- A wave can be longitudinal or transversal

- **The amplitude**

The amplitude " A " is the maximum value of an oscillation around its equilibrium position. The unit of the amplitude is a meter [m].

- **The wavelength**

The wavelength " λ " is the distance between two successive points on the wave over a period. The unit of the wavelength is a meter [m].

- **The number of wave**

The number of wave is defined as " $k = \frac{2\pi}{\lambda}$ ", its unit is radian per meter [rad/m]

- **The period**

The period " T " is the time required to complete one cycle. Its unit is a second [s]

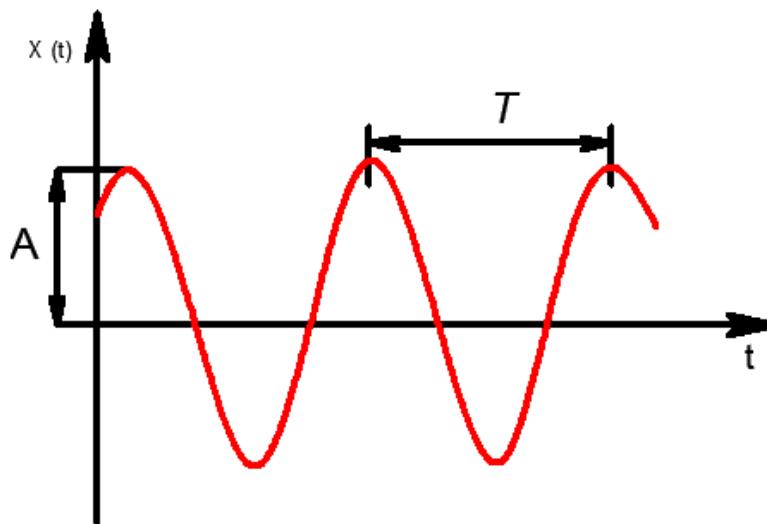


fig.1

- **The phase**

The phase " φ " represents the instantaneous position of the periodic wave in the period. Therefore, the phase shift " $\Delta\varphi$ " appears in the case where several waves are plotted in the same curve.

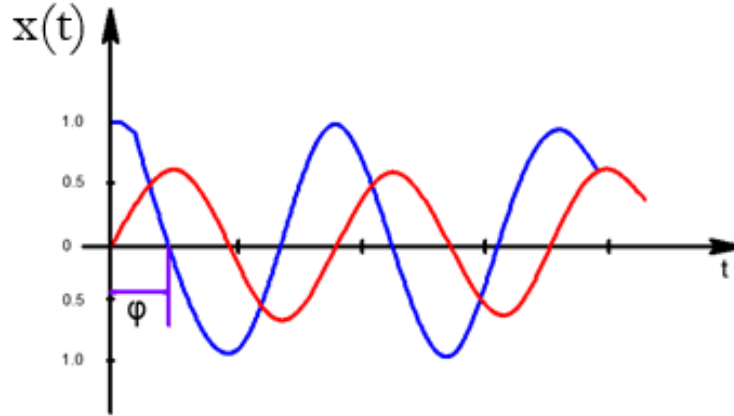


fig.2

- **The frequency**

The frequency " f " is related to the period by the following equation: $f = \frac{1}{T}$ [Hz]

- **The pulsation**

The pulsation ω is related to the frequency by the following equation: $\omega = 2\pi f$ [rad/s]

1.3 Mathematical wave expression

A sinusoidal wave may spread in two direction ways namely, the positive and the negative one.

When the propagation takes the positive direction, the wave is called progressive, and we can write the following equation [4]:

$$y = A \sin(\omega t - kx + \varphi) \quad (1)$$

When the propagation takes the negative direction, the wave is called regressive, and we can write the following equation:

$$y = A \sin(\omega t + kx + \varphi) \quad (2)$$

1.4 Phase speed and group speed

The phase speed v_ϕ is the speed displacement of a point in the wave. The mathematical expression of the phase speed relate the pulsation and the number of wave as follows: $v_\phi = \frac{\omega}{k}$.

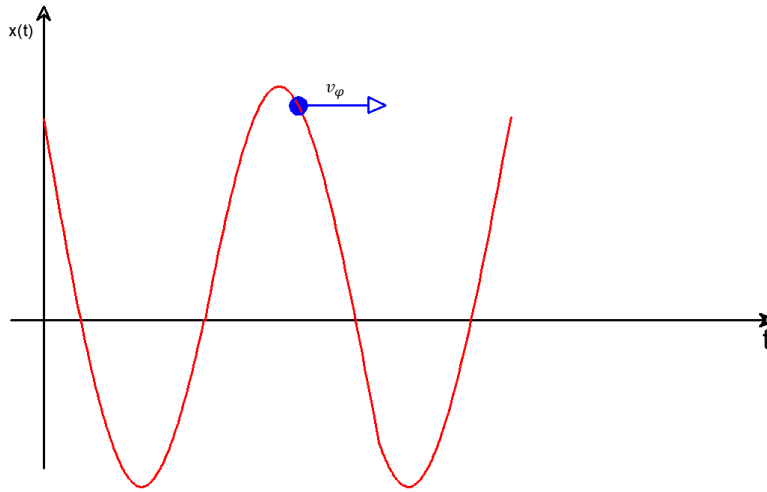


fig.3

The group speed v_g is the propagation speed of the group packet, the group speed exist for the case of two or more waves. we can write the following equation: $v_g = \frac{d\omega}{dk}$.

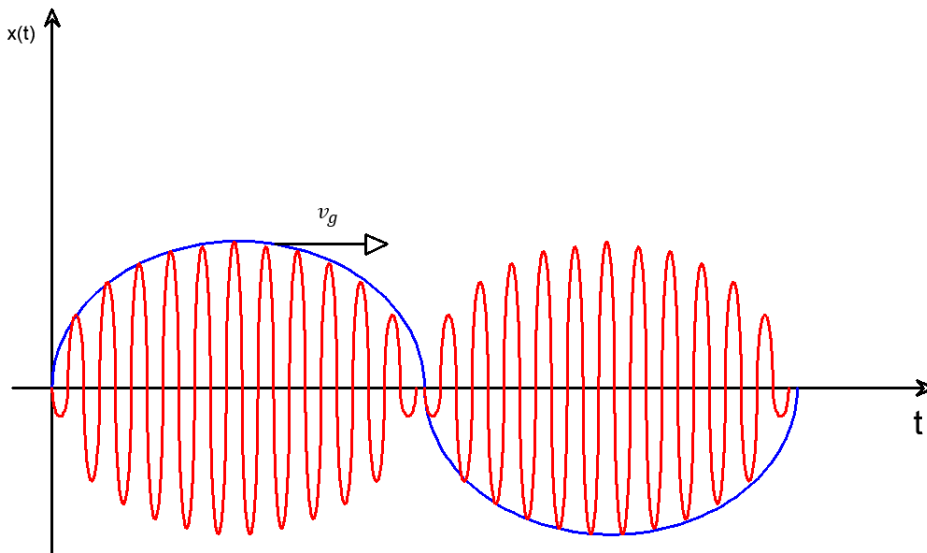


fig.4

1.5 Propagation equation

In several branches of physics, the evolution in time and space is described by a partial differential equation. Mathematically, we can write the following equation [3] [8]:

$$\frac{\partial^2 s}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 s}{\partial t^2} \quad (3)$$

Where:

$s(t, x)$: is a function of time and space variables. It can be a displacement, electric or magnetic field component,.. etc.

c : is the cerility or the speed constant.

The above equation is called “Alembert” equation and it is a one-dimensional wave equation.

1.5.1 Alembert equation resolution

In order to solve the Alembert equation, we need to change some variables. The main objective of this change is to pass from $s(t, x)$ to $s(\alpha, \beta)$. Mathematically we can write the following equation:

$$\begin{cases} \alpha = t - \frac{x}{c} \\ \beta = t + \frac{x}{c} \end{cases} \quad (4)$$

- To begin with, we can write the followings:

$$\begin{cases} \frac{\partial s}{\partial x} = \frac{\partial s}{\partial \alpha} \left(\frac{\partial \alpha}{\partial x} \right) + \frac{\partial s}{\partial \beta} \left(\frac{\partial \beta}{\partial x} \right) \\ \frac{\partial s}{\partial t} = \frac{\partial s}{\partial \alpha} \left(\frac{\partial \alpha}{\partial t} \right) + \frac{\partial s}{\partial \beta} \left(\frac{\partial \beta}{\partial t} \right) \\ \frac{\partial s}{\partial x} = \frac{\partial s}{\partial \alpha} \left(-\frac{1}{c} \right) + \frac{\partial s}{\partial \beta} \left(\frac{1}{c} \right) \\ \frac{\partial s}{\partial t} = \frac{1}{c} \left(-\frac{\partial s}{\partial \alpha} + \frac{\partial s}{\partial \beta} \right) \end{cases} \quad (5)$$

Note:

Using the equation.4: $\frac{\partial \alpha}{\partial x} = -\frac{1}{c}$ and $\frac{\partial \beta}{\partial x} = \frac{1}{c}$ and replace it in the third step of the equation.5.

$$\left\{ \begin{array}{l} \frac{\partial^2 s}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial s}{\partial x} \right) \\ \frac{\partial^2 s}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial s}{\partial x} \right) \frac{\partial \alpha}{\partial \alpha} + \frac{\partial}{\partial x} \left(\frac{\partial s}{\partial x} \right) \frac{\partial \beta}{\partial \beta} \\ \frac{\partial^2 s}{\partial x^2} = \frac{\partial}{\partial \alpha} \left(\frac{\partial s}{\partial x} \right) \frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta} \left(\frac{\partial s}{\partial x} \right) \frac{\partial \beta}{\partial x} \\ \frac{\partial^2 s}{\partial x^2} = \frac{\partial}{\partial \alpha} \left(\frac{\partial s}{\partial x} \right) \left(-\frac{1}{c} \right) + \frac{\partial}{\partial \beta} \left(\frac{\partial s}{\partial x} \right) \left(\frac{1}{c} \right) \\ \frac{\partial^2 s}{\partial x^2} = \frac{\partial}{\partial \alpha} \left(\frac{1}{c} \left(-\frac{\partial s}{\partial \alpha} + \frac{\partial s}{\partial \beta} \right) \right) \left(-\frac{1}{c} \right) + \frac{\partial}{\partial \beta} \left(\frac{1}{c} \left(-\frac{\partial s}{\partial \alpha} + \frac{\partial s}{\partial \beta} \right) \right) \left(\frac{1}{c} \right) \\ \frac{\partial^2 s}{\partial x^2} = -\frac{1}{c^2} \frac{\partial}{\partial \alpha} \left(-\frac{\partial s}{\partial \alpha} + \frac{\partial s}{\partial \beta} \right) + \frac{1}{c^2} \frac{\partial}{\partial \beta} \left(-\frac{\partial s}{\partial \alpha} + \frac{\partial s}{\partial \beta} \right) \\ \frac{\partial^2 s}{\partial x^2} = \frac{1}{c^2} \left(\frac{\partial^2 s}{\partial \alpha^2} + \frac{\partial^2 s}{\partial \beta^2} - 2 \frac{\partial^2 s}{\partial \alpha \partial \beta} \right) \end{array} \right. \quad (6)$$

Note:

Using the equation.5: $\frac{\partial s}{\partial x} = \frac{1}{c} \left(-\frac{\partial s}{\partial \alpha} + \frac{\partial s}{\partial \beta} \right)$ and replace it in the fifth step of the equation.6.

- Likewise,

$$\left\{ \begin{array}{l} \frac{\partial s}{\partial t} = \frac{\partial s}{\partial t} \left(\frac{\partial \alpha}{\partial \alpha} \right) + \frac{\partial s}{\partial t} \left(\frac{\partial \beta}{\partial \beta} \right) \\ \frac{\partial s}{\partial t} = \frac{\partial s}{\partial \alpha} \left(\frac{\partial \alpha}{\partial t} \right) + \frac{\partial s}{\partial \beta} \left(\frac{\partial \beta}{\partial t} \right) \\ \frac{\partial s}{\partial t} = \frac{\partial s}{\partial \alpha} + \frac{\partial s}{\partial \beta} \end{array} \right. \quad (7)$$

Note:

Using the equation.2: $\frac{\partial \alpha}{\partial t} = 1$ and $\frac{\partial \beta}{\partial t} = 1$ and replace it in the second step of the equation.5.

$$\left\{ \begin{array}{l} \frac{\partial^2 s}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial s}{\partial t} \right) \\ \frac{\partial^2 s}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial s}{\partial t} \right) \frac{\partial \alpha}{\partial \alpha} + \frac{\partial}{\partial t} \left(\frac{\partial s}{\partial t} \right) \frac{\partial \beta}{\partial \beta} \\ \frac{\partial^2 s}{\partial t^2} = \frac{\partial}{\partial \alpha} \left(\frac{\partial s}{\partial t} \right) \frac{\partial \alpha}{\partial t} + \frac{\partial}{\partial \beta} \left(\frac{\partial s}{\partial t} \right) \frac{\partial \beta}{\partial t} \\ \frac{\partial^2 s}{\partial t^2} = \frac{\partial}{\partial \alpha} \left(\frac{\partial s}{\partial t} \right) + \frac{\partial}{\partial \beta} \left(\frac{\partial s}{\partial t} \right) \\ \frac{\partial^2 s}{\partial t^2} = \frac{\partial}{\partial \alpha} \left(\frac{\partial s}{\partial \alpha} + \frac{\partial s}{\partial \beta} \right) + \frac{\partial}{\partial \beta} \left(\frac{\partial s}{\partial \alpha} + \frac{\partial s}{\partial \beta} \right) \\ \frac{\partial^2 s}{\partial t^2} = \left(\frac{\partial^2 s}{\partial \alpha^2} + \frac{\partial^2 s}{\partial \beta^2} + 2 \frac{\partial^2 s}{\partial \alpha \partial \beta} \right) \end{array} \right. \quad (8)$$

Note:

Using the equation.4: $\frac{\partial \alpha}{\partial t} = 1$ and $\frac{\partial \beta}{\partial t} = 1$ and replace it in the fourth step of the equation.8.

Using the third line of the equation.7: $\frac{\partial s}{\partial t} = \frac{\partial s}{\partial \alpha} + \frac{\partial s}{\partial \beta}$ and replace it in the fifth step of the equation.8.

• **Discussion:**

- The main results are the followings:

$$\frac{\partial^2 s}{\partial x^2} = \frac{1}{c^2} \left(\frac{\partial^2 s}{\partial \alpha^2} + \frac{\partial^2 s}{\partial \beta^2} - 2 \frac{\partial^2 s}{\partial \alpha \partial \beta} \right)$$

$$\frac{\partial^2 s}{\partial t^2} = \left(\frac{\partial^2 s}{\partial \alpha^2} + \frac{\partial^2 s}{\partial \beta^2} + 2 \frac{\partial^2 s}{\partial \alpha \partial \beta} \right)$$

By replacing the above equation in the equation.1 $\left(\frac{\partial^2 s}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 s}{\partial t^2} \right)$, we obtain the following equation:

$$-\frac{\partial^2 s}{\partial \alpha \partial \beta} = \frac{\partial^2 s}{\partial \alpha \partial \beta} \rightarrow 2 \frac{\partial^2 s}{\partial \alpha \partial \beta} = 0 \rightarrow \frac{\partial^2 s}{\partial \alpha \partial \beta} = 0 \rightarrow \frac{\partial}{\partial \alpha} \left(\frac{\partial s}{\partial \beta} \right) = 0$$

- The quantity $\frac{\partial s}{\partial \beta}$ is a function of β only since its derivative with respect to α is zero, and we can write the following equation: $\left(\frac{\partial s}{\partial \beta} \right) = k(\beta)$
- The general solution of the Alembert equation is written as follow:

$$s(\alpha, \beta) = f\left(t - \frac{x}{c}\right) + g\left(t + \frac{x}{c}\right)$$

Where:

$f\left(t - \frac{x}{c}\right)$ represents the shape of the shaking at t time, with a progressive propagation in the positive direction as can be shown in the figure.5

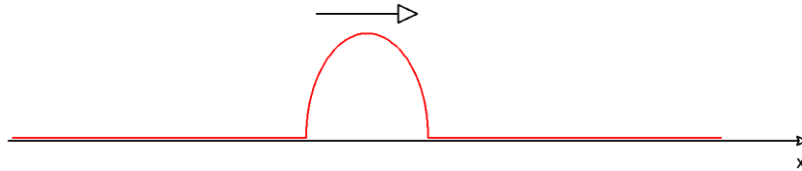


fig.5

$f\left(t + \frac{x}{c}\right)$ represents the shape of the shaking at t time, with a regressive propagation in the negative direction as can be shown in the figure.6

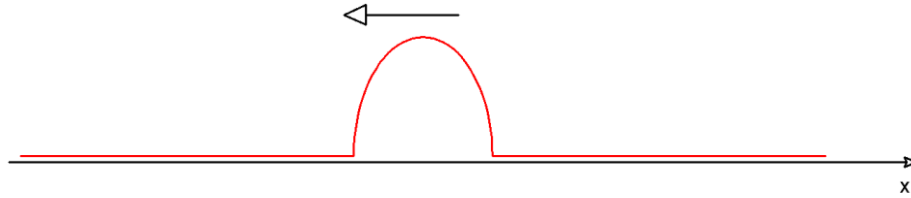


fig.6

1.5.2 Physical interpretation of the Alembert equation solutions

We note that at the moment t and the abscise x , we can express the function as $f\left(t - \frac{x}{c}\right)$ as shown in the Fig.5. The next values at $t + \Delta t$ and $x + \Delta x$, the new function could be expressed as $f\left((t + \Delta t) - \frac{(x + \Delta x)}{c}\right)$.

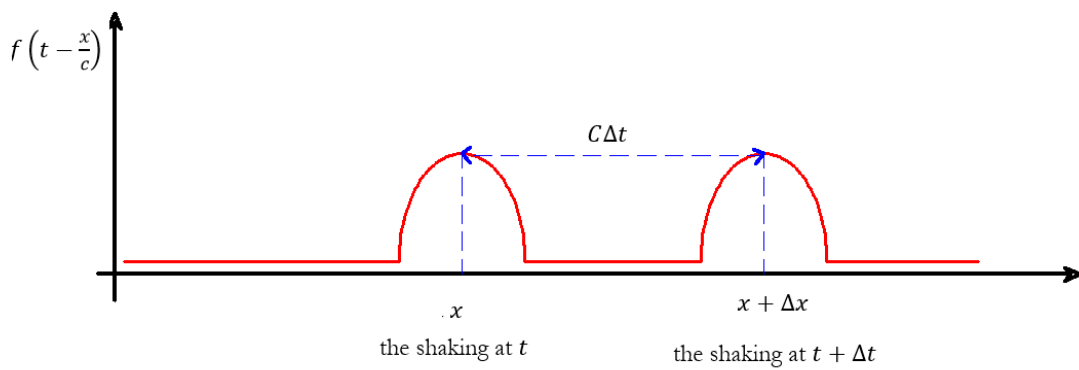


Fig.7

- **Discussion:**

- $f\left(t - \frac{x}{c}\right)$ is a progressive function with one dimension
- The figure.3 allows to see the propagation phenomena of the function $(f\left(t - \frac{x}{c}\right))$ without deformation and with a celerity through an increasing way.
- The propagation phenomena without deformation and with a celerity through a decreasing way is expressed by the following function $(f\left(t + \frac{x}{c}\right))$.

Chapter 02: Vibrating Strings

2.1. Introduction

This chapter aims to establish the equations of the vibrating string basing on the Alembert theorem.

2.2. Vibrating rope (string)

Considering an extensible rope with a mass μ (a mass per unit length). The rope is subjected to a force T (where T represents the constant intensity, T is greater than the gravity force) Fig.1.

When we move the rope from its horizontal equilibrium position, the PQ element finds itself subjected to three forces, namely [4]:

- The weight: $dp = \mu g dx$ ($p = mgh$)
- The tangential force: $-\vec{T}(t, x)$ at the P point
- The tangential force: $\vec{T}(t, x + dx)$ at the Q point

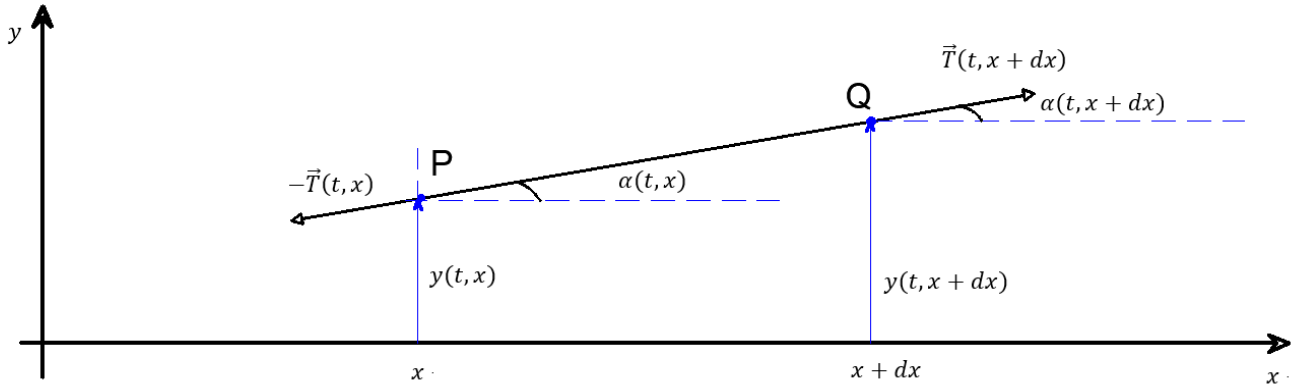


Fig.1

By using the fundamental dynamic relationship, we can write the following equation:

$$\vec{T}(t, x + dx) - \vec{T}(t, x) = \mu dx \vec{\gamma} \quad (7)$$

The projection of the vectors of the equation.7 along the axis OX and OY gives the following equations:

$$|\vec{T}(t, x + dx)| \cos \alpha(t, x + dx) - |\vec{T}(t, x)| \cos \alpha(t, x) = 0 \quad (8)$$

$$|\vec{T}(t, x + dx)| \sin \alpha(t, x + dx) - |\vec{T}(t, x)| \sin \alpha(t, x) = \mu dx \frac{\partial^2 y}{\partial t^2} \quad (9)$$

The equation.8 makes the $\cos \alpha(t, x + dx)$ and $\cos \alpha(t, x)$ equal to 1 since the angle α is very weak (very small) and the displacement of the rope element is vertical. This allows to write from equation.8 the following:

$$|\vec{T}(t, x + dx)| - |\vec{T}(t, x)| = 0 \rightarrow \vec{T}(t, x + dx) = \vec{T}(t, x) = cte = T \quad (10)$$

By replacing the equation.10 into the equation.9 we found the following:

$$|T \sin \alpha(t, x + dx)| - |T \sin \alpha(t, x)| = \mu dx \frac{\partial^2 y}{\partial t^2} \quad (11)$$

$$T \left(\frac{\sin \alpha(t, x + dx) - \sin \alpha(t, x)}{dx} \right) = \mu \frac{\partial^2 y}{\partial t^2} \quad (12)$$

Finally,

$$\begin{aligned} T \left(\frac{\sin \alpha(t, x + dx) - \sin \alpha(t, x)}{\partial x} \right) &= \mu \frac{\partial^2 y}{\partial t^2} \\ \left(\frac{\sin \alpha(t, x + dx) - \sin \alpha(t, x)}{\partial x} \right) &= \frac{\mu}{T} \frac{\partial^2 y}{\partial t^2} \end{aligned} \quad (13)$$

According to the figure.2,

$$tg \alpha = \frac{y(t, x + dx) - y(t, x)}{dx} = \frac{\Delta y}{\Delta x} = \frac{\partial y}{\partial x} \quad (14)$$

Since the angle α is very small, we can make $\sin \alpha \approx tg \alpha$ and by derivation we found the following:

$$\frac{\partial \sin \alpha}{\partial x} = \frac{\partial}{\partial x} tg \alpha \quad (15)$$

And by using the result of the equation.14 into the equation.15 we found:

$$\frac{\partial \sin \alpha}{\partial x} = \frac{\partial}{\partial x} tg \alpha = \frac{\partial^2 y}{\partial x^2} \quad (16)$$

Finally, from the equation.13 and 16 we obtain:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T} \frac{\partial^2 y}{\partial t^2} \rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2} \quad (17)$$

Comparing the equation.17 and the equation.1, we found that the celerity is equation to $c = \frac{T}{\mu}$ for this example.

$$\frac{\partial^2 s}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 s}{\partial t^2} \quad (1) \quad \Rightarrow \quad \frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T} \frac{\partial^2 y}{\partial t^2} \quad (17)$$

Discussion:

- The equation.17 represents a wave propagation along a rope.
- The function $y = f\left(t - \frac{x}{c}\right)$ represents the propagation in a positive direction (fig.2)

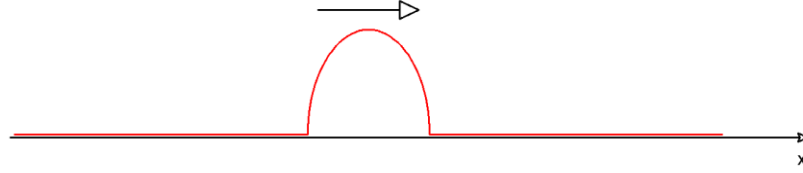


fig.2

- The function $y = f\left(t + \frac{x}{c}\right)$ represents the propagation in a negative direction (fig.3).

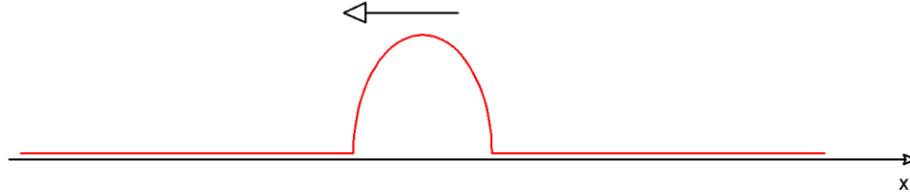


fig.3

- For a general case, where the wave propagates in all direction with a function wave is $s(x, y, z)$, the equation of wave propagation is written as follows:

$$\Delta s - \frac{1}{c^2} \frac{\partial^2 s}{\partial t^2} = 0 \quad (18)$$

Where: the Laplacian Δ is equal to $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

1.5.3 Impedance characteristic

The impedance characteristic of a mechanical wave is related to a force T and the celerity c . We can write the following expression: $Z = \frac{T}{c}$

For the example studied in the above section (rope, from where, $c = \sqrt{\frac{T}{\mu}}$), the impedance could be expressed

as follows: $Z = \frac{T}{c} = \frac{T\sqrt{\mu}}{\sqrt{T}} = \sqrt{T\mu}$

Where, μ is a linear mass (mass per length unit)

1.5.4 Progressive wave energy

During the moving shaking, the kinetic and the potential energies can contribute. The kinetic energy is due to the movement and the potential energy is due to the force of the rope. Mathematically, the system could be written as follows [1]:

$$\begin{cases} E_c = \frac{1}{2}\mu \int \left(\frac{\partial y}{\partial t}\right)^2 dx \\ E_p = \frac{1}{2}T \int \left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{1}{2}\mu c^2 \int \left(\frac{\partial y}{\partial x}\right)^2 dx \end{cases} \quad (19)$$

The total energy could be expressed as follows:

$$E_t = E_c + E_p = \frac{1}{2}\mu \left[\int \left(\frac{\partial y}{\partial t}\right)^2 dx + c^2 \int \left(\frac{\partial y}{\partial x}\right)^2 dx \right] \quad (20)$$

1.5.5 Energy transport

Considering a sinusoidal wave in a string of the following form:

$$y = A \sin(\omega t - kx) \quad (21)$$

The kinetic and the potential energies are expressed using the equation (19), where:

$$\left(\frac{\partial y}{\partial t}\right)^2 = A^2 \omega^2 \cos^2(\omega t - kx), \quad \left(\frac{\partial y}{\partial x}\right)^2 = A^2 k^2 \cos^2(\omega t - kx)$$

By replacing the above considerations in the kinetic and the potential energies, we found the followings:

$$\begin{cases} E_c = \frac{1}{2}\mu \int \left(\frac{\partial y}{\partial t}\right)^2 dx = \frac{1}{2}\mu \int A^2 \omega^2 \cos^2(\omega t - kx) dx \\ E_p = \frac{1}{2}T \int \left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{1}{2}\mu c^2 \int A^2 k^2 \cos^2(\omega t - kx) dx \end{cases} \quad (22)$$

$$\begin{cases} E_c = \frac{1}{2}\mu A^2 \omega^2 \int \cos^2(\omega t - kx) dx \\ E_p = \frac{1}{2}\mu c^2 A^2 k^2 \int \cos^2(\omega t - kx) dx \end{cases} \quad (23)$$

Note:

$$\begin{cases} \cos^2(X) + \sin^2(X) = 1 \Rightarrow \sin^2(X) = 1 - \cos^2(X) \\ \cos(2X) = \cos^2(X) - \sin^2(X) \end{cases}$$

$$\Rightarrow \cos(2X) = \cos^2(X) - 1 + \cos^2(X) \Rightarrow \cos(2X) = 2\cos^2(X) - 1 \Rightarrow \cos^2(X) = \frac{\cos(2X)}{2} + \frac{1}{2}$$

Using the final expression of $\cos^2(X)$, and by replacing the variable (X) by $(wt - kx)$. The new expression of the kinetic and the potential energy is expressed as follows:

$$\begin{cases} E_c = \frac{1}{2} \mu A^2 w^2 \int \left(\frac{\cos(2(wt - kx))}{2} + \frac{1}{2} \right) dx \\ E_p = \frac{1}{2} \mu c^2 A^2 k^2 \int \left(\frac{\cos(2(wt - kx))}{2} + \frac{1}{2} \right) dx \end{cases} \quad (24)$$

Here, we can use the following simplification:

- $(v_\phi = c = \frac{w}{k} \Rightarrow w = c \cdot k)$
- *the integral of $\cos X$ in a period of time is equal to zero*

Finally,

$$\begin{cases} E_c = \frac{1}{2} \mu A^2 w^2 \int_0^\lambda \left(\frac{\cos(2(wt - kx))}{2} + \frac{1}{2} \right) dx = \frac{1}{4} \mu A^2 w^2 \lambda \\ E_p = \frac{1}{2} \mu c^2 A^2 k^2 \int_0^\lambda \left(\frac{\cos(2(wt - kx))}{2} + \frac{1}{2} \right) dx = \frac{1}{4} \mu A^2 w^2 \lambda \end{cases} \quad (25)$$

Basing on the results of the equation.25, the total energy is finally expressed by the following expression:

$$E_t = E_c + E_p = \frac{1}{2} \mu A^2 w^2 \lambda \quad (26)$$

Chapter 03: Longitudinal waves in fluids

1.3. Characteristics of longitudinal wave

- In this type of wave, the direction of disturbance is parallel to the direction of the movement of the wave. For instance: we grip manually a few turns of a spring and release them quickly (fig.1).
- In fluids, only longitudinal vibrations can propagate, the existence of viscosity which dissipates the energy does not allow the transversal wave to be maintained.

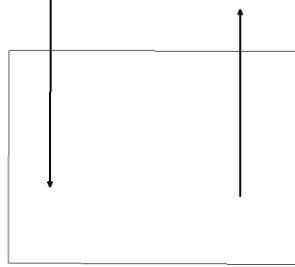


fig.1

1.3 Propagation of waves in a fluid

Considering a perfect and homogenous fluid. The wave surfaces are planes perpendicular to the direction of propagation as can be shown in (Fig.2) [1].

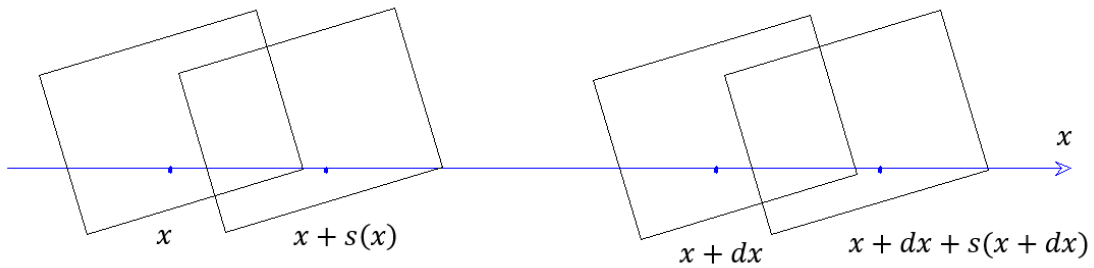


fig.2

Mathematically we can write the following equation:

$$\vartheta_0 = S((x + dx) - x) = sdx \quad (1)$$

Where:

- ϑ_0 is the volume limited between P_1 and P_2
- S is the plan section

The wave propagates by pressure variation. The propagation of the wave causes along the ox axis a displacement $s(x)$ at point x , and a displacement $s(x + dx)$ at point $x + dx$. The new expression of the volume could be written as follows:

$$\vartheta = S([x + dx + s(x + dx)] - [x + s(x)]) = S(dx + s(x + dx) - s(x)) \quad (2)$$

Knowing that:

$$s(x + dx) - s(x) = \frac{\partial s}{\partial x} dx \quad (3)$$

By replacing the equation.3 in the equation.2 we found the following expression:

$$\vartheta = S(dx + \frac{\partial s}{\partial x} dx) \quad (4)$$

During its passage, the wave causes a pressure variation given as follows:

$$P(x, t) = p - p_0 = -\frac{\theta}{\aleph} \quad (5)$$

Where, $\aleph$ and θ are respectively the compressibility and the expansion volume.

$$\theta = \frac{\Delta\vartheta}{\vartheta} = \frac{\Delta\vartheta}{\vartheta_0} \quad (6)$$

Where,

$$\Delta\vartheta = \vartheta - \vartheta_0 \quad (7)$$

According to the equation.4 and equation.1 we can write the following equation:

$$\Delta\vartheta = S(dx + \frac{\partial s}{\partial x} dx) - s dx \quad (8)$$

Finally:

$$\Delta\vartheta = S \frac{\partial s}{\partial x} dx \quad (9)$$

According to the equation.6, the expansion volume could be modified (basing on the results of the equations 8 and 1) as follows:

$$\theta = \frac{\Delta\vartheta}{\vartheta_0} = \frac{S \frac{\partial s}{\partial x} dx}{s dx} \quad (10)$$

Finally:

$$\theta = \frac{\partial s}{\partial x} \quad (11)$$

Using the above results, the pressure variation of equation.5 is modified as follows:

$$P(x, t) = -\frac{1}{\aleph} \frac{\partial s}{\partial x} \quad (12)$$

Basing on the fundamental relationship of dynamic (where: $m = \rho_0 v = \rho_0 S dx$, ρ_0 : is the density of fluid) we can write the following expression:

$$S[P(x) - P(x + dx)] = \rho_0 S dx \frac{\partial^2 s}{\partial t^2} \quad (13)$$

Knowing that:

$$P(x) - P(x + dx) = -\frac{\partial P}{\partial x} dx \quad (14)$$

By replacing the equation.14 in the equation.13 we found the following expression:

$$S[-\frac{\partial P}{\partial x} dx] = \rho_0 S dx \frac{\partial^2 s}{\partial t^2} \quad (15)$$

$$-\frac{\partial P}{\partial x} = \rho_0 \frac{\partial^2 s}{\partial t^2} \quad (16)$$

$$\rho_0 \frac{\partial^2 s}{\partial t^2} = -\frac{\partial P}{\partial x} \quad (17)$$

According to the equation.12, the first derivative gives the followings:

$$\frac{\partial P(x, t)}{\partial x} = \frac{1}{\aleph} \frac{\partial^2 s}{\partial x^2} \quad (18)$$

By replacing the equation.18 into the equation.17, we found:

$$\rho_0 \frac{\partial^2 s}{\partial t^2} = \frac{1}{\aleph} \frac{\partial^2 s}{\partial x^2} \quad (19)$$

Finally,

$$\frac{\partial^2 s}{\partial x^2} = \rho_0 \aleph \frac{\partial^2 s}{\partial t^2} \quad (20)$$

$$\frac{\partial^2 s}{\partial x^2} - \rho_0 \aleph \frac{\partial^2 s}{\partial t^2} \quad (21)$$

The present equation represents a propagation of longitudinal wave in fluid.

1.3.1. Characteristics

- The speed of the propagation wave in fluid is given as: $c = \sqrt{\frac{1}{\rho_0 \kappa}}$
- The solution of the equation of the propagation is given as:

$$S = f\left(t - \frac{x}{c}\right) + g\left(t + \frac{x}{c}\right) \quad (22)$$

1.4 Wave propagation in a cylindrical pipe

Considering a cylindrical pipe (with length L and section S) containing a fluid as can be shown in the figure.3.

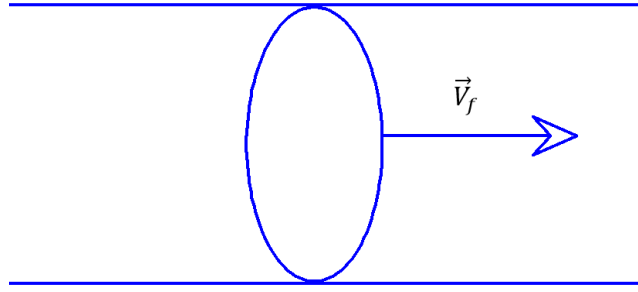


Fig.3

The propagation of the mechanical wave results a pressure variation. This leads to a displacement $S(x, t)$ with a speed $V_f(x, t)$. Mathematically we can write the following equation:

$$V_f(x, t) = \frac{\partial S}{\partial t} \quad (23)$$

1.4.1 characteristics

- The fluid is characterized by its flow:

$$D = \int \vec{V}_f dS \quad (24)$$

- The acoustic impedance is defined as follows: $\bar{Z}_a(x) = \frac{\bar{P}(x)}{S\bar{V}_f(x)}$

Where: $\bar{P}(x) = P e^{-j\frac{x}{V}}$, $\bar{V}_f(x) = V_f e^{-j\frac{x}{V}}$

- The impedance characteristic is defined as follows: $Z_c = \frac{\rho_0 V}{S}$

1.5 Doppler effect

When a fixed source emits a sound wave of a speed V in a homogenous fluid at rest, a fixed observer receives a frequency f_0 of that wave. However, if the source and/or the observer are not fixed, we distinguish a new frequency called apparent frequency f_a . This change in frequency amounts to the movement of the source or the observer or both of them.

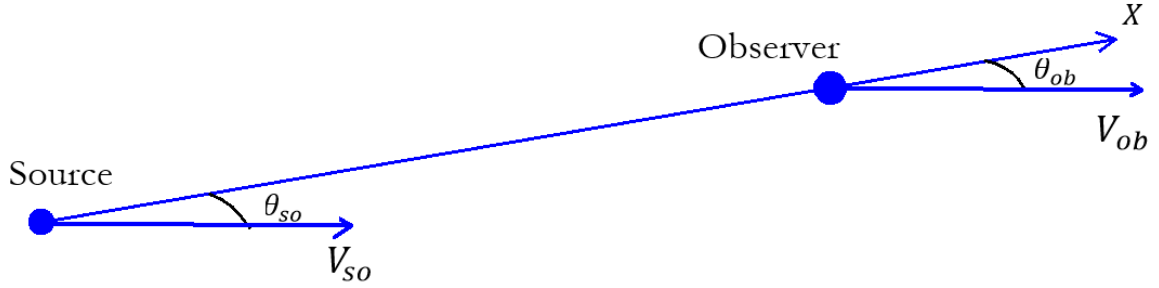


Fig.4

Where:

V_{so} : is the speed of the source

V_{ob} : is the speed of the observer

θ_{so} : is the angle between the source and the displacement vector X

θ_{ob} : is the angle between the observer and the displacement vector X

Mathematically, the apparent frequency is expressed as follows:

$$f_a = f_0 \frac{1 + \frac{V_{ob}}{V} \cos(\theta_{ob})}{1 + \frac{V_{so}}{V} \cos(\theta_{so})} \quad (25)$$

Chapter 04: Electromagnetic waves

4.1 Mathematical operators

4.1.1 Gradient

The gradient of a function S is expressed as follows:

$$\overrightarrow{\text{grad}}f = \vec{\nabla}(f) \quad (1)$$

Where $\vec{\nabla}$ is called “nabla”.

$$\overrightarrow{\text{grad}}f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} \quad \text{and} \quad \vec{\nabla}(f) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} \quad \rightarrow \quad \vec{\nabla} \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

4.1.2 Divergence

The divergence is the scalar product application of $\vec{\nabla}$ to a vector.

$$\text{div}(\vec{V}) = \vec{\nabla} \cdot \vec{V} \quad (2)$$

$$\text{div}(\vec{V}) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

4.1.3 Rotational

The rotational operator is applied only to vectors.

$$\overrightarrow{\text{rot}}(\vec{V}) = \vec{\nabla} \wedge \vec{V} \quad (3)$$

$$\overrightarrow{\text{rot}}(\vec{V}) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{pmatrix}$$

4.1.4 Laplacian

The Laplacian is used in electromagnetic field propagation equation. It could be applied to a vector ($\Delta \vec{V}$) or function (Δf).

For a vector we can write the following equation:

$$\Delta \vec{V} = \begin{pmatrix} \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \\ \frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2} \\ \frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2} \end{pmatrix} \quad (4)$$

For a function we can write the following equation:

$$\Delta f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} \\ \frac{\partial^2 f}{\partial y^2} \\ \frac{\partial^2 f}{\partial z^2} \end{pmatrix} \quad (5)$$

4.2 Propagation equation of an electromagnetic wave

The propagation equation of an electromagnetic wave could be expressed as follows:

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \quad (6)$$

Such as : $\frac{1}{c^2} = \mu_0 \epsilon_0$

Where,

$\vec{E}$: is the electromagnetic field

μ_0 : is the magnetic permeability

ϵ_0 : is the permittivity

Using the above operators, the propagation equation of an electromagnetic wave is modified as follows:

$$\Delta \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \quad (7)$$

Applications

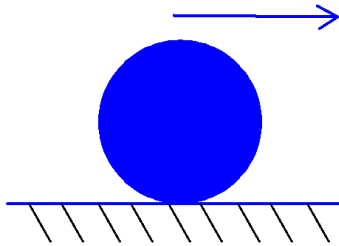
part A

Chapter 01: Generality on vibrations

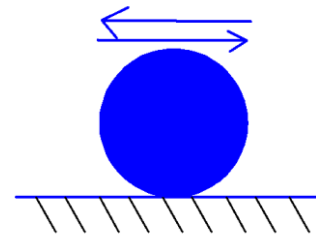
Exercise 01

Considering the below systems. Which of them represent an oscillatory movement?

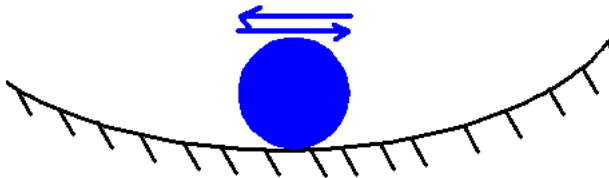
1. A Disc moving in one direction



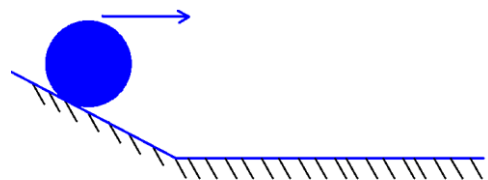
2. A Disc moving in two directions



3. A disc moving in a curved way



4. A disc moving in an inclined plane



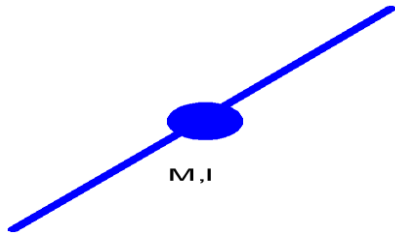
Correction:

2 and 3 represent oscillatory movement

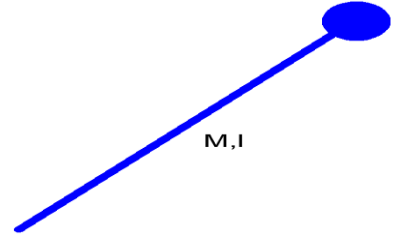
Exercise 02

Find the inertia moment of the following systems:

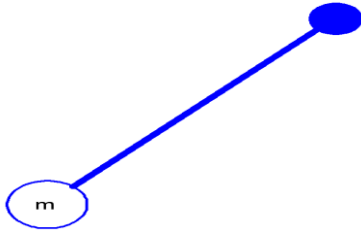
1. A bar (M, l) fixed to its center



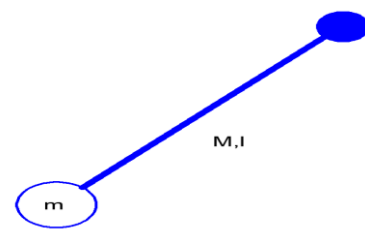
2. A bar (M, l) fixed to its extremity



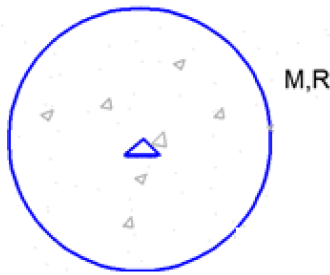
3. A point mass attached to a simple wire



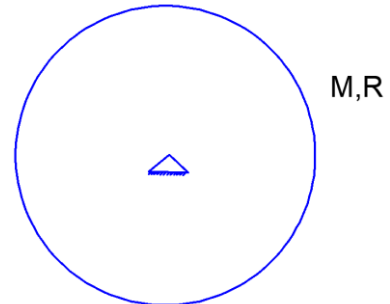
4. A point mass attached to a bar (M, l)



5. A disc (M, R) fixed at his center



6. A circle (M, R) fixed at his center



Correction;

1. A bar (M, l) fixed to its center: $J_c = \int x^2 dm = \lambda \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 dx = \frac{\lambda l^3}{12} = \frac{ml^2}{12}$

Using the Huygens: $J_{bar_center} = J_c + md^2 = \frac{ml^2}{12} + m \cdot 0^2 = \frac{ml^2}{12}$

2. A bar (M, l) fixed to its extremity: $J_e = J_c + md^2 = \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{3}$

3. A point mass attached to a simple wire: $J_m = ml^2$

4. A point mass attached to a bar:

$$J_{mass_bar} = J_m + J_{bar_center} = J_m + (J_c + m_b d^2)$$

$$J_{mass_bar} = ml^2 + \left(\frac{m_b l^2}{12} + m_b \frac{l^2}{4} \right)$$

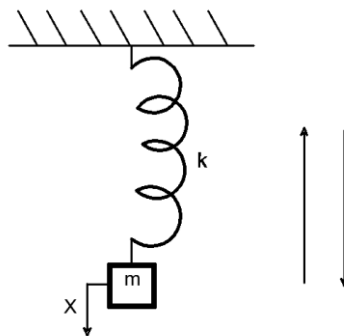
$$J_{mass_bar} = ml^2 + \frac{m_b l^2}{3}$$

5. $J_{disc} = \frac{MR^2}{2}$

6. $J_{circle} = MR^2$

Exercise 03

Considering the following oscillatory system:



Find the differential equation of the movement basing on:

1. The fundamental relationship of dynamic
2. The Lagrangian method

Correction:

1. $\sum F_i = m\ddot{x} \Rightarrow -kx = m\ddot{x} \Rightarrow m\ddot{x} + kx = 0$
2. $L = T - U$

Where:

$$T = \frac{1}{2}m\dot{x}^2, U = \frac{1}{2}kx^2$$

$$L = T - U$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

Such as :

$$\begin{cases} \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = m\ddot{x} \\ \frac{\partial L}{\partial x} = kx \end{cases}$$

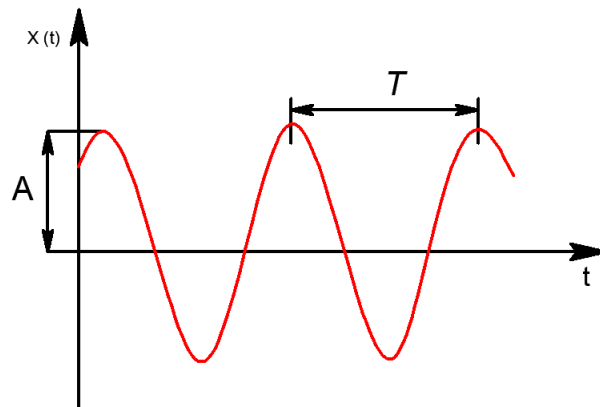
$$m\ddot{x} + kx = 0$$

Where, $\omega_0^2 = \frac{k}{m}$ (clean pulsation or proper pulsation)

$$\ddot{x} + \omega_0^2 x = 0$$

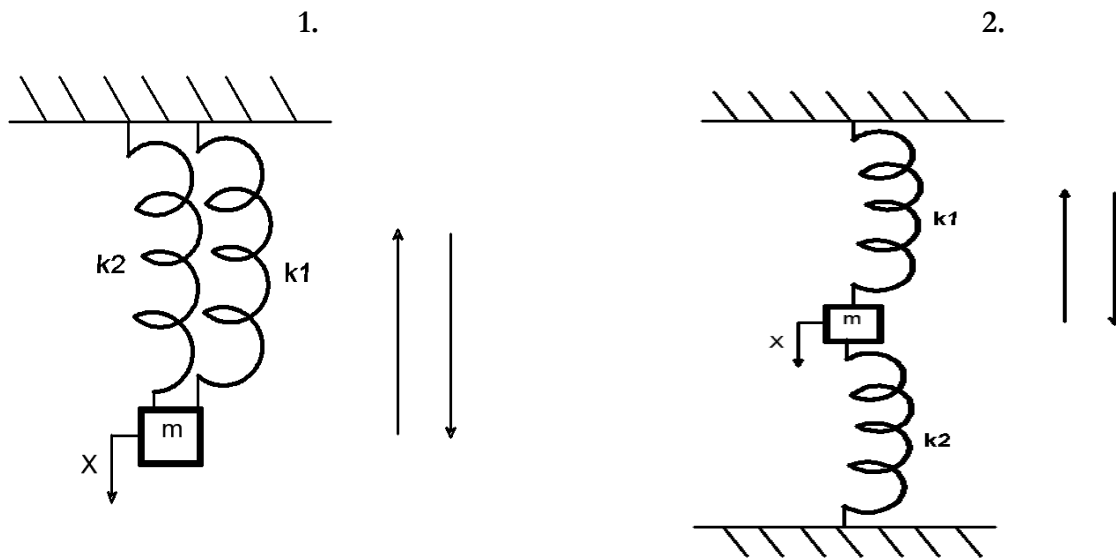
The above equation is simply a differential equation in second order, the final solution could be written as follows:

$$x(t) = A \cos(\omega_0 t + \varphi)$$



Exercise 04

Considering the following oscillatory systems:



Find the differential equation of the movement basing on the Lagrangian method

Correction:

First case:

$$T = \frac{1}{2} m \dot{x}^2$$

$$U = \frac{1}{2} k_1 (0 - x)^2 + \frac{1}{2} k_2 (0 - x)^2$$

$$L = T - U$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

$$\left\{ \begin{array}{l} \frac{\partial L}{\partial \dot{x}} = m \dot{x} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = m \ddot{x} \\ \frac{\partial L}{\partial x} = k_1 x + k_2 x \end{array} \right.$$

$$m\ddot{x} + (k_1 + k_2)x = 0$$

Where, $\omega_0^2 = \frac{k_1+k_2}{m}$ (clean pulsation or proper pulsation)

$$\ddot{x} + \omega_0^2 x = 0$$

Second case:

$$T = \frac{1}{2} m \dot{x}^2$$

$$U = \frac{1}{2} k_1 (0 - x)^2 + \frac{1}{2} k_2 (x - 0)^2$$

$$L = T - U$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

$$\left\{ \begin{array}{l} \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = m\ddot{x} \\ \frac{\partial L}{\partial x} = k_1 x + k_2 x \end{array} \right.$$

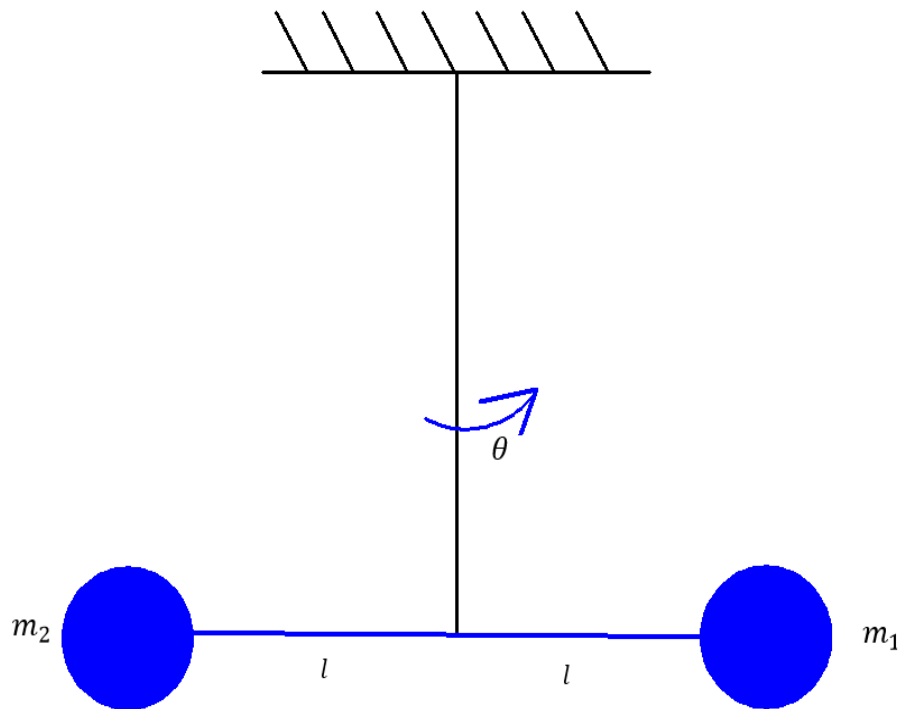
$$m\ddot{x} + (k_1 + k_2)x = 0$$

Where, $\omega_0^2 = \frac{k_1+k_2}{m}$ (clean pulsation or proper pulsation)

$$\ddot{x} + \omega_0^2 x = 0$$

Exercise 05

Considering the following torsion pendulum:



1. Find the differential equation of the movement basing on the Lagrangian method
2. find the solution of the differential equation ($\theta(0) = \frac{\pi}{3}$, $\dot{\theta}(0) = 0$)

Correction:

$$T = \frac{1}{2}J_{m1}\dot{\theta}^2 + \frac{1}{2}J_{m2}\dot{\theta}^2 = J\dot{\theta}^2 = ma^2\dot{\theta}^2$$

$$U = \frac{1}{2} C\theta^2$$

$$L = T - U$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\begin{cases} \frac{\partial L}{\partial \dot{\theta}} = 2ma^2\dot{\theta} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = 2ma^2\ddot{\theta} \\ \frac{\partial L}{\partial \theta} = C\theta \end{cases}$$

$$2ma^2\ddot{\theta} + C\theta = 0$$

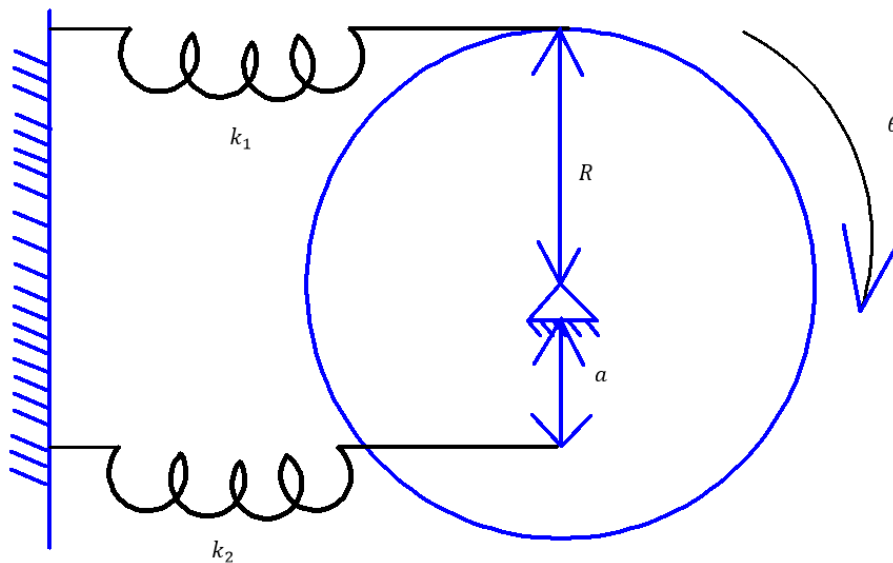
Where, $\omega_0^2 = \frac{C}{2ma^2}$ (clean pulsation or proper pulsation)

$$\ddot{\theta} + \omega_0^2\theta = 0$$

$$\theta(t) = \frac{\pi}{3} \cos\left(\sqrt{\frac{C}{2ma^2}} t\right)$$

Exercise 06

Considering the following mechanical system, where it is formed by a pulley fixed at her center and free to rotate around its fixation point. Two springs (k_1, k_2) are placed to the system with distances (R and a).



1. Extract the energies of the proposed system
2. Establish the differential equation using the Lagrangian method
3. Deduce the solution

Correction:

$$T = \frac{1}{2} J_p \dot{\theta}^2 = \frac{1}{2} M R^2 \dot{\theta}^2$$

$$U = \frac{1}{2} k_1 R^2 \theta^2 + \frac{1}{2} k_2 a^2 \theta^2$$

$$L = T - U$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\begin{cases} \frac{\partial L}{\partial \dot{\theta}} = M R^2 \dot{\theta} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = M R^2 \ddot{\theta} \\ \frac{\partial L}{\partial \theta} = (k_1 R^2 + k_2 a^2) \theta \end{cases}$$

$$M R^2 \ddot{\theta} + (k_1 R^2 + k_2 a^2) \theta = 0$$

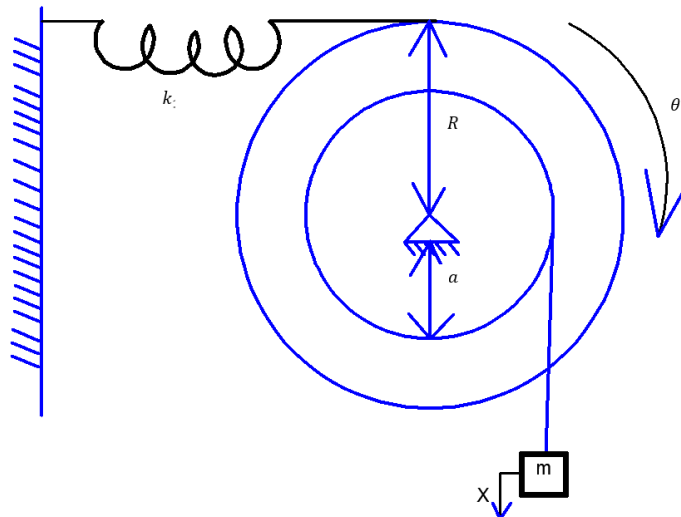
$$\text{Where, } \omega_0^2 = \frac{(k_1 R^2 + k_2 a^2)}{M R^2} \text{ (clean pulsation or proper pulsation)}$$

$$\ddot{\theta} + \omega_0^2 \theta = 0$$

$$\theta(t) = A \cos\left(\sqrt{\frac{(k_1 R^2 + k_2 a^2)}{M R^2}} t\right)$$

Exercise 07

Considering the following mechanical system, where it is formed by two pulleys (glued) fixed at the center and free to rotate around the fixation point. A mass m is placed to the system through an extensible wire.



1. Extract the energies of the proposed system
2. Establish the differential equation using the Lagrangian method
3. Deduce the solution

Correction:

$$T = \frac{1}{2} J_p \dot{\theta}^2 + \frac{1}{2} m \dot{x}^2 = \frac{1}{2} M R^2 \dot{\theta}^2 + \frac{1}{2} m a^2 \dot{\theta}^2$$

$$U = \frac{1}{2} k_1 R^2 \theta^2 - mgh = \frac{1}{2} k_1 R^2 \theta^2 - mgx = \frac{1}{2} k R^2 \theta^2 - mga \sin \theta = \frac{1}{2} k R^2 \theta^2 - mga \theta$$

$$L = T - U$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\begin{cases} \frac{\partial L}{\partial \dot{\theta}} = (MR^2 + ma^2) \dot{\theta} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = (MR^2 + ma^2) \ddot{\theta} \\ \frac{\partial L}{\partial \theta} = k R^2 \theta \end{cases}$$

$$(MR^2 + ma^2) \ddot{\theta} + k R^2 \theta = 0$$

Where, $\omega_0^2 = \frac{k R^2}{MR^2 + ma^2}$ (clean pulsation or proper pulsation)

$$\ddot{\theta} + \omega_0^2 \theta = 0$$

$$\theta(t) = A \cos\left(\sqrt{\frac{k R^2}{MR^2 + ma^2}} t\right)$$

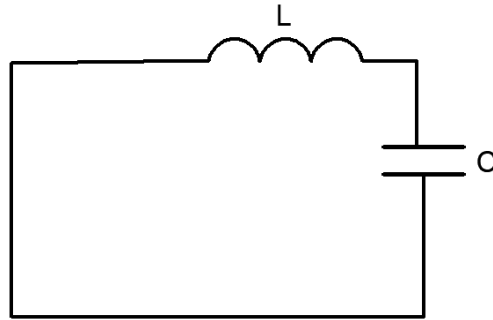
The period: $T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{MR^2 + ma^2}{k R^2}}$

Exercise 08

Considering an electrical circuit composed of inductance L and a capacitor C (placed in serial). We inject a quantity of charge q rapidly and then we stop.

1. Give the detailed electrical circuit model.
2. Establish the differential equation.

Correction:



$$u_l + u_c = 0$$

$$\text{Where: } u_l = L \frac{di}{dt}, u_c = \frac{1}{C} \int i dt, i = \dot{q} = \frac{dq}{dt}, \frac{di}{dt} = \frac{d^2q}{dt^2} = \ddot{q}$$

$$L\ddot{q} + \frac{1}{C}q = 0$$

$$\ddot{q} + \frac{1}{LC}q = 0$$

$$q(t) = A \cos(\omega_0 t + \varphi)$$

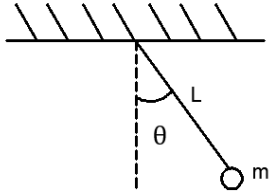
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\text{The period: } T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{LC}$$

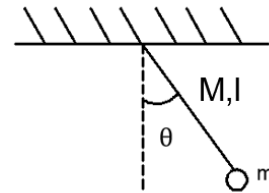
Exercise 09

Find the kinetic energy, the potential energy and solve the system using the Lagrangian method

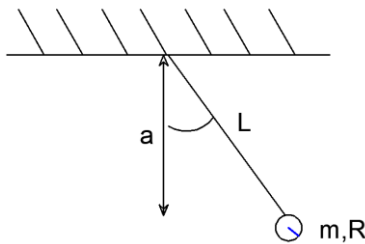
1. A point mass attached to a simple wire



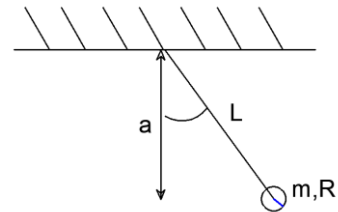
2. A simple mass attached to a bar (M, l)



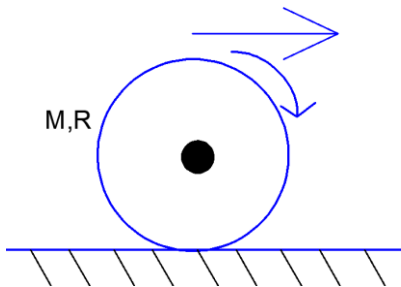
3. A sphere (m, R) attached to simple wire



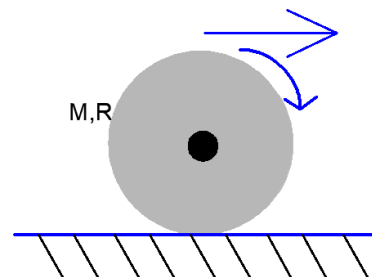
4. A sphere (m, R) attached to a bar (M, l)



5. A circle (M, R) moves without friction



6. A disc (M, R) moves without friction



Correction

- A point mass attached to a simple wire

$$T = \frac{1}{2} J \dot{\theta}^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = mgh = mgl - mgl \cos \theta$$

$$L = T - U$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\begin{cases} \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = ml^2 \ddot{\theta} \\ \frac{\partial L}{\partial \theta} = mgl \sin \theta = mgl \theta \end{cases}$$

$$ml^2 \ddot{\theta} + mgl \theta = 0$$

Where, $\omega_0^2 = \frac{g}{l}$ (clean pulsation or proper pulsation)

$$\ddot{\theta} + \omega_0^2 \theta = 0$$

- **A simple mass attached to a bar (M, l)**

$$T = \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} J_b \dot{\theta}^2 = \frac{1}{2} ml^2 \dot{\theta}^2 + \left(\frac{1}{2} \left[\frac{Ml^2}{12} + \frac{Ml^2}{4} \right] \dot{\theta}^2 \right) = \frac{1}{2} ml^2 \dot{\theta}^2 + \frac{1}{2} \frac{Ml^2}{3} \dot{\theta}^2$$

$$U = mgh + mg \frac{l}{2} \cos \theta = mgl - mgl \cos \theta - mg \frac{l}{2} \cos \theta$$

$$L = T - U$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\begin{cases} \frac{\partial L}{\partial \dot{\theta}} = (ml^2 + \frac{Ml^2}{3}) \dot{\theta} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = (ml^2 + \frac{Ml^2}{3}) \ddot{\theta} \\ \frac{\partial L}{\partial \theta} = mgl \sin \theta + mg \frac{l}{2} \sin \theta = mgl \theta + Mg \frac{l}{2} \theta \end{cases}$$

$$(ml^2 + \frac{Ml^2}{3}) \ddot{\theta} + (mgl + Mg \frac{l}{2}) \theta = 0$$

$$\ddot{\theta} + \omega_0^2 \theta = 0$$

- **A sphere (m, R) attached to simple wire**

$$T = \frac{1}{2} J \dot{\theta}^2 = \frac{1}{2} m(l + R)^2 \dot{\theta}^2$$

$$U = mgh = mg(l + R) - mg(l + R) \cos \theta$$

$$L = T - U$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\begin{cases} \frac{\partial L}{\partial \dot{\theta}} = (m(l+R)^2)\dot{\theta} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = (m(l+R)^2)\ddot{\theta} \\ \frac{\partial L}{\partial \theta} = mg(l+R)\sin\theta = mg(l+R)\theta \end{cases}$$

$$(m(l+R)^2)\ddot{\theta} + mg(l+R)\theta = 0$$

$$\ddot{\theta} + \omega_0^2 \theta = 0$$

- A sphere (m, R) attached to bar

$$T = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}J_b\dot{\theta}^2 = \frac{1}{2}m(l+R)^2\dot{\theta}^2 + \frac{1}{2}\frac{Ml^2}{3}\dot{\theta}^2$$

$$U = mgh + mg\frac{l}{2}\cos\theta = mg(l+R) - mg(l+R)\cos\theta - mg\frac{l}{2}\cos\theta$$

$$L = T - U$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\begin{cases} \frac{\partial L}{\partial \dot{\theta}} = (m(l+R)^2 + \frac{Ml^2}{3})\dot{\theta} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = (m(l+R)^2 + \frac{Ml^2}{3})\ddot{\theta} \\ \frac{\partial L}{\partial \theta} = mg(l+R)\sin\theta + mg\frac{l}{2}\sin\theta = mg(l+R)\theta + Mg\frac{l}{2}\theta \end{cases}$$

$$(m(l+R)^2 + \frac{Ml^2}{3})\ddot{\theta} + (mg(l+R) + Mg\frac{l}{2})\theta = 0$$

$$\ddot{\theta} + \omega_0^2 \theta = 0$$

- A circle (M, R) moves without friction

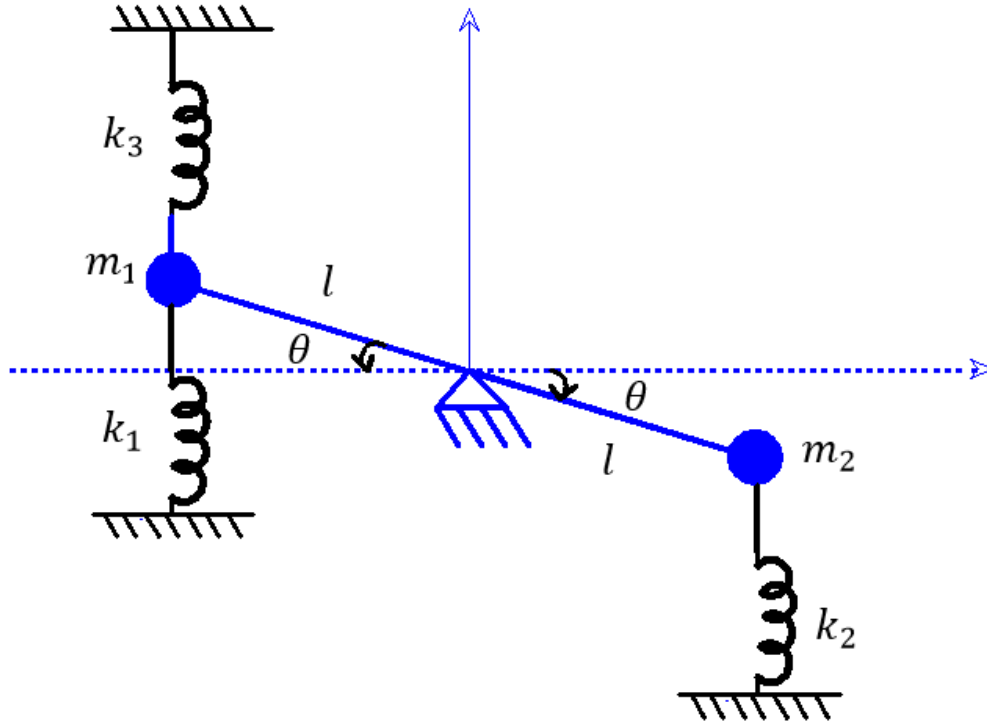
$$T = \frac{1}{2}J\dot{\theta}^2 = T = \frac{1}{2}MR^2\dot{\theta}^2$$

- A disc (M, R) moves without friction

$$T = \frac{1}{2}J\dot{\theta}^2 = T = \frac{1}{2}\frac{MR^2}{2}\dot{\theta}^2$$

Exercise 10

Considering the bellow mechanical system. The bar of length $2l$ and negligible mass. On its ends are fixed the masses (m_1 and m_2) and springs (k_1 , k_2 and k_3). At the equilibrium position, the bar was horizontal ($\theta(0) = 0$).



1. Extract the energies of the proposed system
2. Establish the differential equation using the Lagrangian method and find the solution
3. Give an equivalent electrical circuit

Correction

$$T = \frac{1}{2}J_{m1} \dot{\theta}^2 + \frac{1}{2}J_{m2} \dot{\theta}^2 = \frac{1}{2}m_1 l^2 \dot{\theta}^2 + \frac{1}{2}m_2 l^2 \dot{\theta}^2$$

$$U = \frac{1}{2}(k_1 + k_2 + k_3)l^2 \theta^2 + m_1 gh - m_2 gh$$

$$U = \frac{1}{2}(k_1 + k_2 + k_3)l^2 \theta^2 + m_1 gl \sin \theta - m_2 gl \sin \theta$$

$$U = \frac{1}{2}(k_1 + k_2 + k_3)l^2 \theta^2 + m_1 gl \theta - m_2 gl \theta$$

$$L = T - U$$

$$L = \frac{1}{2} m_1 l^2 \dot{\theta}^2 + \frac{1}{2} m_2 l^2 \dot{\theta}^2 - \frac{1}{2} (k_1 + k_2 + k_3) l^2 \theta^2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

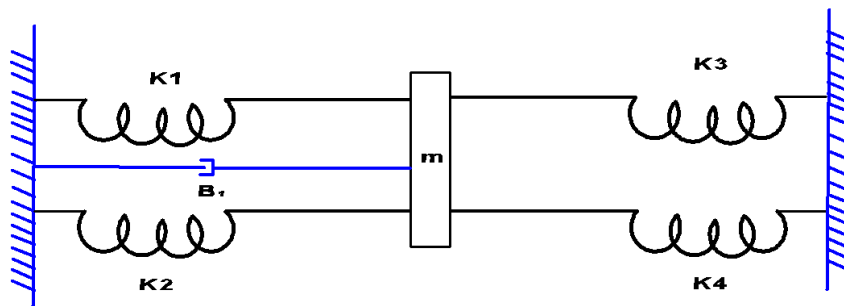
$$\begin{cases} \frac{\partial L}{\partial \dot{\theta}} = (m_1 + m_2) l^2 \dot{\theta} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = (m_1 + m_2) l^2 \ddot{\theta} \\ \frac{\partial L}{\partial \theta} = (k_1 + k_2 + k_3) l^2 \theta \end{cases}$$

$$((m_1 + m_2) l^2) \ddot{\theta} + (k_1 + k_2 + k_3) l^2 \theta = 0$$

$$\ddot{\theta} + \omega_0^2 \theta = 0$$

Exercise 11

Considering the above mechanical system.



1. Extract the energies of the proposed system
2. Establish the differential equation using the Lagrangian method and find the solution
3. Give an equivalent electrical circuit

Correction

$$T = \frac{1}{2} m \dot{x}^2$$

$$U = \frac{1}{2} (k_1 + k_2 + k_3 + k_4) x^2$$

$$D = \frac{1}{2} \beta \dot{x}^2$$

$$L = T - U$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} (k_1 + k_2 + k_3 + k_4) x^2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = - \frac{\partial D}{\partial \dot{x}}$$

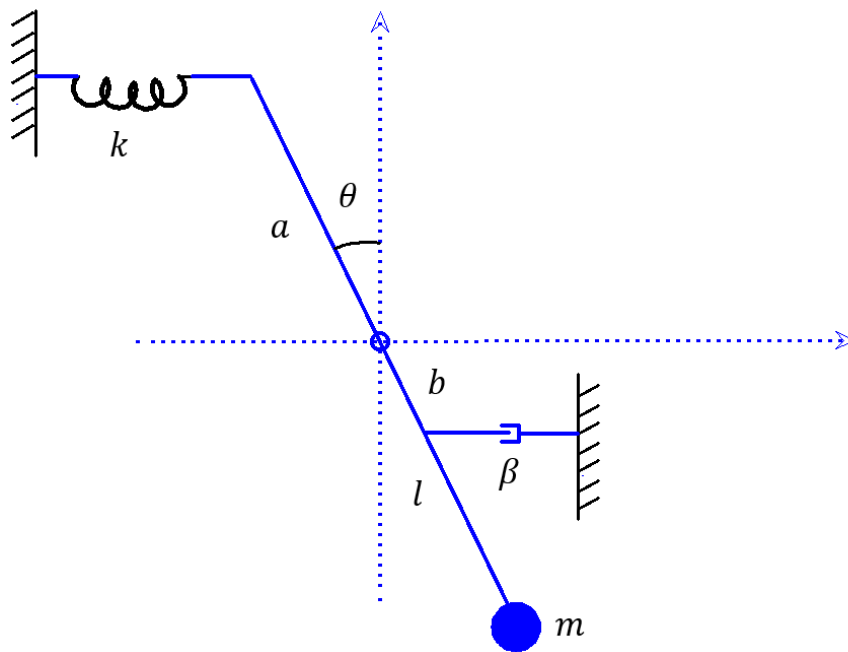
$$\begin{cases} \frac{\partial L}{\partial \dot{x}} = m \dot{x} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = m \ddot{x} \\ \frac{\partial L}{\partial x} = (k_1 + k_2 + k_3) x \end{cases}$$

$$m \ddot{x} + (k_1 + k_2 + k_3) x = 0$$

$$\ddot{x} + \omega_0^2 x = 0$$

Exercise 12

Considering the above mechanical system.



1. Extract the energies of the proposed system
2. Establish the differential equation using the Lagrangian method and find the solution
3. Give an equivalent electrical circuit

Correction

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = \frac{1}{2} k a^2 \theta^2 + mgl - mgl \cos \theta$$

$$D = \frac{1}{2} \beta b^2 \dot{\theta}^2$$

$$L = T - U$$

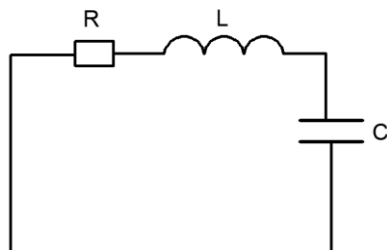
$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - \frac{1}{2} k a^2 \theta^2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = - \frac{\partial D}{\partial \dot{\theta}}$$

$$\left\{ \begin{array}{l} \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = m l^2 \ddot{\theta} \\ \frac{\partial L}{\partial \theta} = k a^2 \theta + mgl \theta \\ \frac{\partial D}{\partial \dot{\theta}} = \beta b^2 \dot{\theta} \end{array} \right.$$

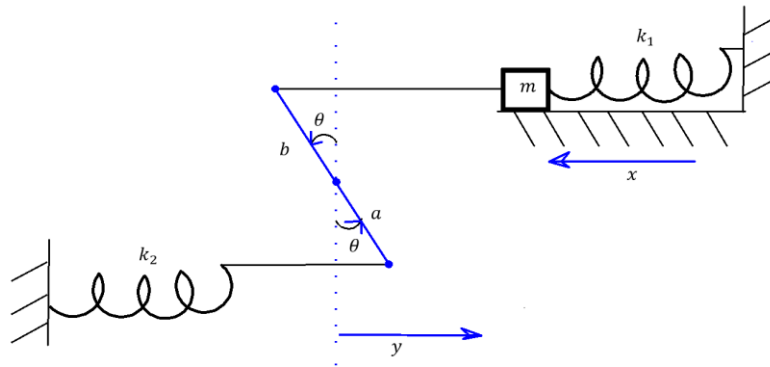
$$m l^2 \ddot{\theta} + \beta b^2 \dot{\theta} + (k a^2 + mgl) \theta = 0$$

$$\ddot{\theta} + 2\lambda \dot{\theta} + \omega_0^2 \theta = 0$$



Exercise 13

Considering the above mechanical system:



1. Extract the energies of the proposed system
2. Establish the differential equation using the Lagrangian method

Correction:

$$\begin{cases} T = T_m \\ U = U_{k1} + U_{k2} \end{cases}$$

$$\begin{cases} T = \frac{1}{2} m \dot{x}^2 \\ U = \frac{1}{2} k_1 b^2 \theta^2 + \frac{1}{2} k_2 a^2 \theta^2 = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2 \end{cases}$$

$$x = b\theta, y = a\theta \Rightarrow \frac{x}{y} = \frac{b}{a} \rightarrow y = \frac{ax}{b}$$

$$\begin{cases} T = \frac{1}{2} m \dot{x}^2 \\ U = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 \frac{a^2 x^2}{b^2} \end{cases}$$

$$L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} x^2 (k_1 + k_2 \frac{a^2}{b^2})$$

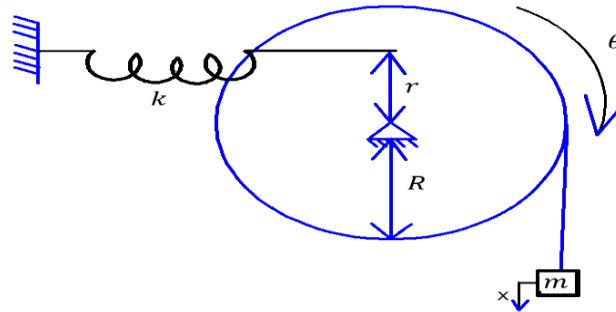
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

$$m \ddot{x} + (k_1 + k_2 \frac{a^2}{b^2}) x = 0$$

$$\ddot{x} + W_0^2 x = 0$$

Exercise 14

Considering the above mechanical system:



1. Extract the energies of the proposed system
2. Establish the differential equation using the Lagrangian method

Correction:

$$\begin{cases} T = T_{\text{rotation}} + T_m \\ U = U_k \end{cases}$$

$$\begin{cases} T = \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} m R^2 \dot{\theta}^2 \\ U = \frac{1}{2} k_1 r^2 \theta^2 \end{cases}$$

$$L = T - U = \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} m R^2 \dot{\theta}^2 - \frac{1}{2} k r^2 \theta^2$$

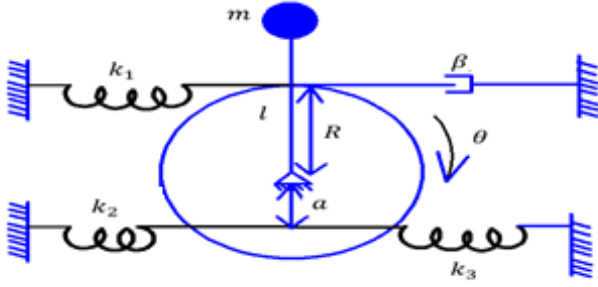
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

$$(J + m R^2) \ddot{\theta} + k r^2 \theta = 0$$

$$\ddot{\theta} + W_0^2 \theta = 0$$

Exercise 15

Considering the above mechanical system. Its consists of a solid pulley (M_p, R), where a bar (M_b, l) is connected to the center of the pulley and a mass m is connected to the extremity of the bar.



1. Extract the energies of the proposed system
2. Establish the differential equation using the Lagrangian method

Correction:

$$\begin{cases} T = T_{p1rotation} + T_{bar} + T_m \\ U = U_{k1} + U_{k2} + U_{k3} + U_m + U_{bar} \\ D = D_\beta \end{cases}$$

$$\begin{cases} T = \frac{1}{2}J_{p1}\dot{\theta}^2 + \frac{1}{2}J_{bar}\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\theta}^2 \\ U = \frac{1}{2}k_1R^2\theta^2 + \frac{1}{2}k_2a^2\theta^2 + \frac{1}{2}k_3a^2\theta^2 + mgl\cos\theta + M_bg\frac{l}{2}\cos\theta \\ D = \frac{1}{2}\beta R^2\dot{\theta}^2 \end{cases}$$

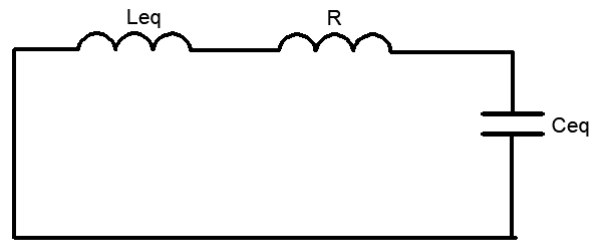
$$\begin{cases} T = \frac{1}{2}\frac{M_pR^2}{2} + \frac{1}{2}\frac{M_b l^2}{3}\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\theta}^2 \\ U = \frac{1}{2}k_1R^2\theta^2 + \frac{1}{2}k_2a^2\theta^2 + \frac{1}{2}k_3a^2\theta^2 + mgl\cos\theta + M_bg\frac{l}{2}\cos\theta \\ D = \frac{1}{2}\beta R^2\dot{\theta}^2 \end{cases}$$

$$L = T - U = \frac{1}{2}\frac{M_pR^2}{2} + \frac{1}{2}\frac{M_b l^2}{3}\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}k_1R^2\theta^2 - \frac{1}{2}k_2a^2\theta^2 - \frac{1}{2}k_3a^2\theta^2 - mgl\cos\theta - M_bg\frac{l}{2}\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = -\frac{\partial D}{\partial \dot{\theta}}$$

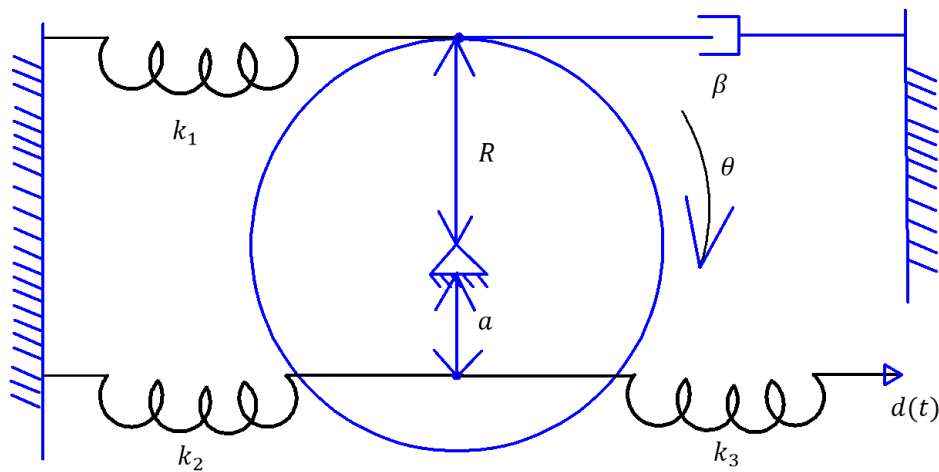
$$\left(\frac{M_pR^2}{2} + \frac{M_b l^2}{3} + ml^2\right)\ddot{\theta} + \beta R^2\dot{\theta} + (k_1R^2 + k_2a^2 + k_3a^2 - mlg - M_bg\frac{l}{2})\theta = 0$$

$$\ddot{\theta} + 2\lambda\dot{\theta} + W_0^2\theta = 0$$



Exercise 16

Considering the above mechanical system. Its consists of a solid pulley (M, R) , where the spring k_3 is subjected to a forced displacement $d(t) = A\cos(\omega t)$.



1. Extract the energies of the proposed system
2. Establish the differential equation using the Lagrangian method and find the solution
3. Give the final expression of the pulsation corresponding to the maximum angle of rotation.

Correction

$$T = \frac{1}{2} \frac{M R^2}{2} \dot{\theta}^2$$

$$U = \frac{1}{2} k_1 R^2 \theta^2 + \frac{1}{2} k_2 a^2 \theta^2 + \frac{1}{2} k_3 (a\theta - d)^2$$

$$D = \frac{1}{2} \beta R^2 \dot{\theta}^2$$

$$L = T - U$$

$$L = \frac{1}{2} \frac{M R^2}{2} \dot{\theta}^2 - \frac{1}{2} k_1 R^2 \theta^2 - \frac{1}{2} k_2 a^2 \theta^2 - \frac{1}{2} k_3 (a\theta - d)^2$$

$$L = \frac{1}{2} \frac{M R^2}{2} \dot{\theta}^2 - \frac{1}{2} k_1 R^2 \theta^2 - \frac{1}{2} k_2 a^2 \theta^2 - \frac{1}{2} k_3 a^2 \theta^2 + k_3 a \theta d - \frac{1}{2} k_3 a d^2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = - \frac{\partial D}{\partial \dot{\theta}}$$

$$\frac{M R^2}{2} \ddot{\theta} + \beta R^2 \dot{\theta} + (k_1 R^2 + k_2 a^2 + k_3 a^2) \theta - k_3 a d = 0$$

$$\frac{M R^2}{2} \ddot{\theta} + \beta R^2 \dot{\theta} + (k_1 R^2 + k_2 a^2 + k_3 a^2) \theta = k_3 a d$$

$$\frac{M R^2}{2} \ddot{\theta} + \beta R^2 \dot{\theta} + (k_1 R^2 + k_2 a^2 + k_3 a^2) \theta = k_3 a A \cos(\omega t)$$

$$\ddot{\theta} + 2\lambda \dot{\theta} + \omega_0^2 \theta = F_0 \cos(\omega t)$$

$$\begin{cases} \theta_0 = \frac{F_0}{\sqrt{(-\omega^2 + \omega_0^2)^2 + 4\lambda^2 \omega^2}} \\ \varphi = -\arctang \frac{2\lambda \omega}{-\omega^2 + \omega_0^2} \end{cases}$$

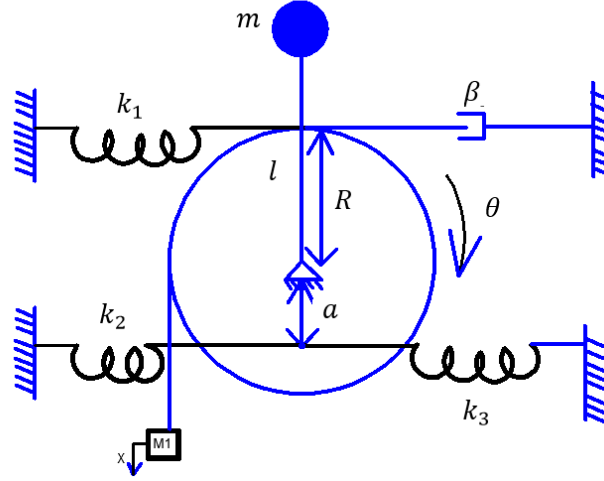
$$2\lambda = \frac{2\beta}{M} \Rightarrow \lambda = \frac{\beta}{M}$$

$$\omega_0^2 = \frac{2(k_1 R^2 + k_2 a^2 + k_3 a^2)}{M R^2}$$

$$\omega = \omega_r = \sqrt{\omega_0^2 - 2\lambda^2} = \sqrt{\frac{2(k_1 R^2 + k_2 a^2 + k_3 a^2)}{M R^2} - 2 \frac{\beta^2}{M^2}}$$

Exercise 17

Considering the above mechanical system. Its consists of a solid pulley (M_p, R) , where a bar (M_b, l) is connected to the center of the pulley and a mass M_1 in connected to the extremity of the pulley through an extensible wire.



1. Extract the energies of the proposed system
2. Establish the differential equation using the Lagrangian method and find the solution

Correction

$$\begin{cases} T = T_{P1rotation} + T_{bar} + T_m + T_M \\ U = U_{k1} + U_{k2} + U_{k3} + U_m + U_{bar} + U_M \\ D = D_\beta \end{cases}$$

$$\begin{cases} T = \frac{1}{2} J_{p1} \dot{\theta}^2 + \frac{1}{2} J_{bar} \dot{\theta}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} M_1 R^2 \dot{\theta}^2 \\ U = \frac{1}{2} k_1 R^2 \theta^2 + \frac{1}{2} k_2 a^2 \theta^2 + \frac{1}{2} k_3 a^2 \theta^2 + m g l \cos \theta + M_b g \frac{l}{2} \cos \theta + M_1 g R \theta \\ D = \frac{1}{2} \beta R^2 \dot{\theta}^2 \end{cases}$$

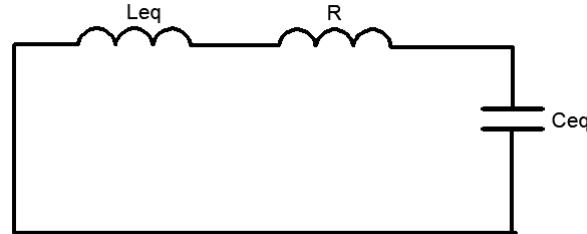
$$\begin{cases} T = \frac{1}{2} \frac{M_p R^2}{2} + \frac{1}{2} \frac{M_b l^2}{3} \dot{\theta}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} M_1 R^2 \dot{\theta}^2 \\ U = \frac{1}{2} k_1 R^2 \theta^2 + \frac{1}{2} k_2 a^2 \theta^2 + \frac{1}{2} k_3 a^2 \theta^2 + m g l \cos \theta + M_b g \frac{l}{2} \cos \theta \\ D = \frac{1}{2} \beta R^2 \dot{\theta}^2 \end{cases}$$

$$L = T - U = \frac{1}{2} \frac{M_p R^2}{2} + \frac{1}{2} \frac{M_b l^2}{3} \dot{\theta}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} M_1 R^2 \dot{\theta}^2 - \frac{1}{2} k_1 R^2 \theta^2 - \frac{1}{2} k_2 a^2 \theta^2 - \frac{1}{2} k_3 a^2 \theta^2 - m g l \cos \theta - M_b g \frac{l}{2} \cos \theta$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = - \frac{\partial D}{\partial \dot{\theta}}$$

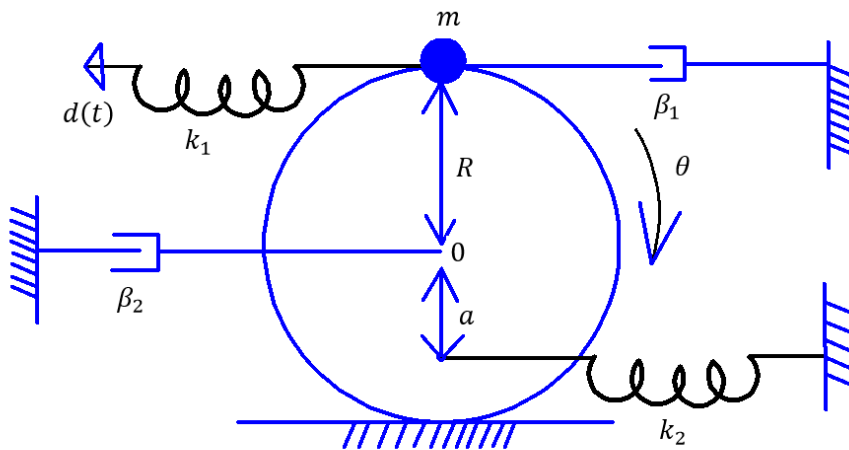
$$\left(\frac{M_p R^2}{2} + \frac{M_b l^2}{3} + m l^2 + M_1 R^2 \right) \ddot{\theta} + \beta R^2 \dot{\theta} + (k_1 R^2 + k_2 a^2 + k_3 a^2 - m l g - M_b g \frac{l}{2}) \theta = 0$$

$$\ddot{\theta} + 2\lambda \dot{\theta}^2 + W_0^2 \theta = 0$$



Exercise 18

Considering the above mechanical system. Its consists of a solids pulley (M_p, R), which ensures translation and rotation movement. The spring k_1 is subjected to a forced displacement $d(t) = A \cos(wt)$.



1. Extract the energies of the proposed system
2. Establish the differential equation using the Lagrangian method and find the solution

Correction

$$\begin{cases} T = T_{rotation} + T_{translation} + T_m \\ U = U_{k1} + U_{k2} + U_m \\ D = D_{\beta} \end{cases}$$

$$\left\{ \begin{array}{l} T = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(2R)^2\dot{\theta}^2 \\ U = \frac{1}{2}k_1(d - 2R\theta)^2 + \frac{1}{2}k_2((R - a)\theta)^2 - (mgl - mgl\cos\theta) \\ D = \frac{1}{2}\beta((R - a)\dot{\theta})^2 \end{array} \right.$$

$$\left\{ \begin{array}{l} T = \frac{1}{2}\left(\frac{3MR^2}{2} + m4R^2\right)\dot{\theta}^2 \\ U = \frac{1}{2}k_1(d - 2R\theta)^2 + \frac{1}{2}k_2((R - a)\theta)^2 + mgl\cos\theta \\ D = \frac{1}{2}\beta((R - a)\dot{\theta})^2 \end{array} \right.$$

$$L = T - U = \frac{1}{2}\left(\frac{3MR^2}{2} + m4R^2\right)\dot{\theta}^2 - \frac{1}{2}k_1(d - 2R\theta)^2 - \frac{1}{2}k_2((R - a)\theta)^2 - mgl\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = -\frac{\partial D}{\partial \dot{\theta}}$$

$$\left(\frac{3MR^2}{2} + m4R^2\right)\ddot{\theta} + \beta(R - a)\dot{\theta} + (4k_1R^2 + k_2(R - a) - mgl)\theta = 2k_1Rd$$

$$\left(\frac{3MR^2}{2} + m4R^2\right)\ddot{\theta} + \beta(R - a)\dot{\theta} + (4k_1R^2 + k_2(R - a) - mgl)\theta = 2k_1RA\cos(wt)$$

$$\ddot{\theta} + 2\lambda\dot{\theta} + w_0^2\theta = F_0\cos(wt)$$

$$\left\{ \begin{array}{l} \theta_0 = \frac{F_0}{\sqrt{(-w^2 + w_0^2)^2 + 4\lambda^2 w^2}} \\ \varphi = -\arctang \frac{2\lambda w}{-w^2 + w_0^2} \end{array} \right.$$

$$2\lambda = \frac{\beta(R - a)}{\frac{3MR^2}{2} + m4R^2} \Rightarrow \lambda = \frac{\beta(R - a)}{2\left(\frac{3MR^2}{2} + m4R^2\right)}$$

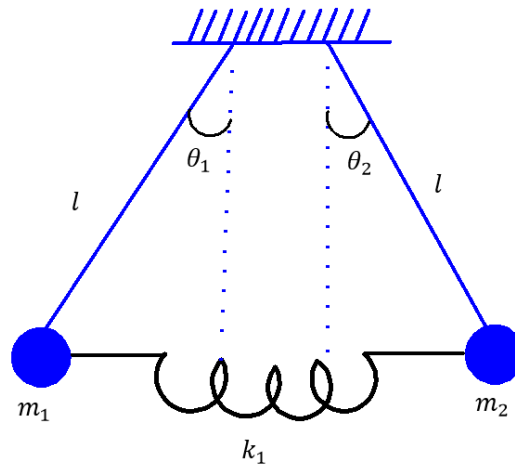
$$w_0^2 = \frac{4k_1R^2 + (k_2(R - a) - mgl)}{\frac{3MR^2}{2} + m4R^2}$$

$$w = w_r = \sqrt{w_0^2 - 2\lambda^2} = \sqrt{\frac{4k_1 R^2 + (k_2(R-a) - mgl)}{\frac{3MR^2}{2} + m4R^2} - 2\left(\frac{\beta(R-a)}{2(\frac{3MR^2}{2} + m4R^2)}\right)^2}$$

$$\ddot{\theta} + 2\lambda\dot{\theta} + W_0^2\theta = 0$$

Exercise 19

Considering the above mechanical system.



1. Extract the energies of the proposed system
2. Establish the differential equation using the Lagrangian method and find the solution

Correction

$$\begin{cases} T = T_{m1} + T_{m2} \\ U = U_k + U_{m1} + U_{m2} \end{cases}$$

$$\begin{cases} T = \frac{1}{2}m_1 l^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 l^2 \dot{\theta}_2^2 \\ U = \frac{1}{2}k (l\theta_1 - l\theta_2)^2 + (m_1 gl - m_1 gl \cos\theta_1) + m_2 gl - m_2 gl \cos\theta_2 \end{cases}$$

$$\begin{cases} T = \frac{1}{2}m_1 l^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 l^2 \dot{\theta}_2^2 \\ U = \frac{1}{2}k (l\theta_1 - l\theta_2)^2 - m_1 g l \cos\theta_1 - m_2 g l \cos\theta_2 \end{cases}$$

$$L = T - U = \frac{1}{2}m_1 l^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 l^2 \dot{\theta}_2^2 - \frac{1}{2}k (l\theta_1 - l\theta_2)^2 + m_1 g l \cos\theta_1 + m_2 g l \cos\theta_2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\begin{cases} m_1 l^2 \ddot{\theta}_1 + (k l^2 + m g l) \theta_1 - k l^2 \theta_2 = 0 \\ m_2 l^2 \ddot{\theta}_2 + (k l^2 + m g l) \theta_2 - k l^2 \theta_1 = 0 \end{cases}$$

By taking the following simplification: $m_1 = m_2 = m$

$$\begin{cases} \ddot{\theta}_1 + \left(\frac{k}{m} + \frac{g}{l}\right) \theta_1 - \frac{k}{m} \theta_2 = 0 \\ \ddot{\theta}_2 + \left(\frac{k}{m} + \frac{g}{l}\right) \theta_2 - \frac{k}{m} \theta_1 = 0 \end{cases} \Rightarrow \begin{cases} \ddot{\theta}_1 = -\left(\frac{k}{m} + \frac{g}{l}\right) \theta_1 + \frac{k}{m} \theta_2 = 0 \\ \ddot{\theta}_2 = \frac{k}{m} \theta_1 - \left(\frac{k}{m} + \frac{g}{l}\right) \theta_2 = 0 \end{cases}$$

Step01: Extracting matrix

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -\frac{k}{m} - \frac{g}{l} & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m} - \frac{g}{l} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = [A] \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

Step02: Compute the Eigen values

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} -\frac{k}{m} - \frac{g}{l} & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m} - \frac{g}{l} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\frac{k}{m} - \frac{g}{l} - \lambda & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m} - \frac{g}{l} - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0 \Rightarrow \det \begin{bmatrix} -\frac{k}{m} - \frac{g}{l} - \lambda & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m} - \frac{g}{l} - \lambda \end{bmatrix} = 0$$

$$\left(-\frac{k}{m} - \frac{g}{l} - \lambda\right)^2 - \left(\frac{k}{m}\right)^2 = 0 \Rightarrow \left(-\frac{k}{m} - \frac{g}{l} - \lambda - \frac{k}{m}\right) \left(-\frac{k}{m} - \frac{g}{l} - \lambda + \frac{k}{m}\right) = 0$$

$$\lambda = -\frac{g}{l} - 2\frac{k}{m} \quad \text{and} \quad \lambda = -\frac{g}{l}$$

Step03: compute the Eigen vectors $\vec{V}_1(\alpha_1, \beta_1)$, $\vec{V}_2(\alpha_2, \beta_2)$

For: $\lambda = -\frac{g}{l} - 2\frac{k}{m}$:

$$[A - \lambda I] \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -\frac{k}{m} - \frac{g}{l} + \frac{g}{l} + 2\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m} - \frac{g}{l} + \frac{g}{l} + 2\frac{k}{m} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = 0$$

$$[A - \lambda I] \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & \frac{k}{m} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = 0$$

$$\begin{cases} \frac{k}{m}\alpha_1 + \frac{k}{m}\beta_1 = 0 \\ \frac{k}{m}\alpha_1 + \frac{k}{m}\beta_1 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_1 + \beta_1 = 0 \rightarrow \alpha_1 = -\beta_1 \\ \alpha_1 + \beta_1 = 0 \rightarrow \alpha_1 = -\beta_1 \end{cases} \Rightarrow \alpha_1 = 1, \beta_1 = -1$$

$$\vec{V}_1(1, -1)$$

For: $\lambda = -\frac{g}{l}$

$$[A - \lambda I] \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -\frac{k}{m} - \frac{g}{l} + \frac{g}{l} & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m} - \frac{g}{l} + \frac{g}{l} \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = 0$$

$$[A - \lambda I] \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = 0$$

$$\begin{cases} -\frac{k}{m}\alpha_2 + \frac{k}{m}\beta_2 = 0 \\ \frac{k}{m}\alpha_2 - \frac{k}{m}\beta_2 = 0 \end{cases} \Rightarrow \begin{cases} -\alpha_2 + \beta_2 = 0 \rightarrow \alpha_2 = \beta_2 \\ \alpha_2 - \beta_2 = 0 \rightarrow \alpha_2 = \beta_2 \end{cases} \Rightarrow \alpha_2 = 1, \beta_2 = 1$$

$$\vec{V}_2(1, 1)$$

Step04: resolution system basing on the Eigen vectors

$$\begin{cases} \ddot{\phi}_1 = \lambda_1 \phi_1 \\ \ddot{\phi}_2 = \lambda_2 \phi_2 \end{cases} \Rightarrow \begin{cases} \ddot{\phi}_1 + \frac{g}{l} \phi_1 = 0 \\ \ddot{\phi}_2 + (\frac{g}{l} + 2\frac{k}{m}) \phi_2 = 0 \end{cases} \Rightarrow \begin{cases} \phi_1 = A \cos(\sqrt{\frac{g}{l}} t + \varphi_1) \\ \phi_2 = A \cos(\sqrt{\frac{g}{l} + 2\frac{k}{m}} t + \varphi_2) \end{cases}$$

Step05: resolution system basing on the initial base (θ_1, θ_2)

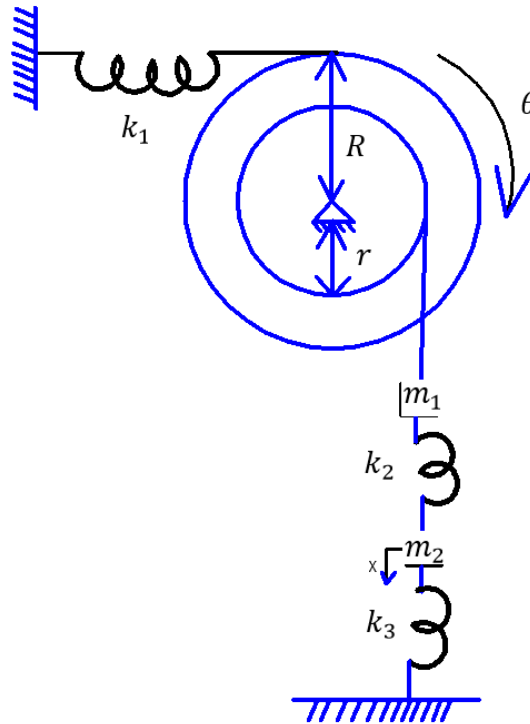
$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = [V_1 \quad V_2] \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \Rightarrow \begin{cases} \theta_1 = \alpha_1 \phi_1 + \alpha_2 \phi_2 \\ \theta_2 = \beta_1 \phi_1 + \beta_2 \phi_2 \end{cases}$$

$$\begin{cases} \theta_1 = \alpha_1 A \cos(w_1 t + \varphi_1) + \alpha_2 B \cos(w_2 t + \varphi_2) \\ \theta_2 = \beta_1 A \cos(w_1 t + \varphi_1) + \beta_2 B \cos(w_2 t + \varphi_2) \end{cases}$$

$$\begin{cases} \theta_1 = A \cos\left(\sqrt{\frac{g}{l}} t + \varphi_1\right) + B \cos\left(\sqrt{\frac{g}{l} + 2\frac{k}{m}} t + \varphi_2\right) \\ \theta_2 = -A \cos\left(\sqrt{\frac{g}{l}} t + \varphi_1\right) + B \cos\left(\sqrt{\frac{g}{l} + 2\frac{k}{m}} t + \varphi_2\right) \end{cases}$$

Exercise 20

Considering the above mechanical system.



1. Extract the energies of the proposed system
2. Establish the differential equation using the Lagrangian method and find the solution

Correction

$$\begin{cases} T = T_p + T_{m1} + T_{m2} \\ U = U_{k1} + U_{k2} + U_{k3} + U_{m1} + U_{m2} \end{cases}$$

$$\begin{cases} T = \frac{1}{2}MR^2\dot{\theta}^2 + \frac{1}{2}m_1r^2\dot{\theta}^2 + \frac{1}{2}m_2\dot{x}^2 \\ U = \frac{1}{2}k_1 R^2\theta^2 + \frac{1}{2}k_2 (r\theta - x)^2 + \frac{1}{2}k_3 x^2 - m_1glr\theta - m_2gx \end{cases}$$

$$\begin{cases} T = \frac{1}{2}MR^2\dot{\theta}^2 + \frac{1}{2}m_1r^2\dot{\theta}^2 + \frac{1}{2}m_2\dot{x}^2 \\ U = \frac{1}{2}k_1 R^2\theta^2 + \frac{1}{2}k_2 (r\theta - x)^2 + \frac{1}{2}k_3 x^2 \end{cases}$$

$$L = T - U = \frac{1}{2}MR^2\dot{\theta}^2 + \frac{1}{2}m_1r^2\dot{\theta}^2 + \frac{1}{2}m_2\dot{x}^2 - \frac{1}{2}k_1R^2\theta^2 - \frac{1}{2}k_2(r\theta - x)^2 - \frac{1}{2}k_3x^2$$

$$L = \frac{1}{2}MR^2\dot{\theta}^2 + \frac{1}{2}m_1r^2\dot{\theta}^2 + \frac{1}{2}m_2\dot{x}^2 - \frac{1}{2}k_1R^2\theta^2 - \frac{1}{2}k_2r^2\theta^2 + k_2r\theta x - \frac{1}{2}k_2x^2 - \frac{1}{2}k_3x^2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

$$\begin{cases} (MR^2 + m_1r^2)\ddot{\theta} + (k_1R^2 + k_2r^2)\theta - k_2rx = 0 \\ m_2\ddot{x} - k_2r\theta + (k_2 + k_3)x = 0 \end{cases}$$

By taking the following simplification: $m_1 = M$, $m_2 = 2M$, $k_1 = k_2 = k$, $k_3 = 2k$

$$\begin{cases} (MR^2 + M r^2)\ddot{\theta} + (kR^2 + kr^2)\theta - kr x = 0 \\ 2M\ddot{x} - kr\theta + (k + 2k)x = 0 \end{cases}$$

By taking the following simplification: $R = \sqrt{2}r$

$$\begin{cases} (Mr^2 + M r^2)\ddot{\theta} + (k2r^2 + kr^2)\theta - kr x = 0 \\ 2M\ddot{x} - kr\theta + (k + 2k)x = 0 \end{cases}$$

$$\begin{cases} \ddot{\theta} + \frac{3kr^2}{2M r^2}\theta - \frac{kr}{2M r^2}x = 0 \\ \ddot{x} - \frac{kr}{2M}\theta + \frac{3k}{2M}x = 0 \end{cases}$$

$$\begin{cases} \ddot{\theta} + \frac{3k}{2M}\theta - \frac{k}{2Mr}x = 0 \\ \ddot{x} - \frac{kr}{2M}\theta + \frac{3k}{2M}x = 0 \end{cases} \Rightarrow \begin{cases} \ddot{\theta} = -\frac{3k}{2M}\theta + \frac{k}{2Mr}x \\ \ddot{x} = \frac{kr}{2M}\theta - \frac{3k}{2M}x \end{cases} \Rightarrow$$

Step01: Extracting matrix

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} -\frac{3k}{2M} & \frac{k}{2Mr} \\ \frac{kr}{2M} & -\frac{3k}{2M} \end{bmatrix} \begin{bmatrix} \theta \\ x \end{bmatrix} \Rightarrow \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} = [A] \begin{bmatrix} \theta \\ x \end{bmatrix}$$

Step02: Compute the Eigen values

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} -\frac{3k}{2M} & \frac{k}{2Mr} \\ \frac{kr}{2M} & -\frac{3k}{2M} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\frac{3k}{2M} - \lambda & \frac{k}{2Mr} \\ \frac{kr}{2M} & -\frac{3k}{2M} - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0 \Rightarrow \det \begin{bmatrix} -\frac{3k}{2M} - \lambda & \frac{k}{2Mr} \\ \frac{kr}{2M} & -\frac{3k}{2M} - \lambda \end{bmatrix} = 0$$

$$\left(-\frac{3k}{2M} - \lambda\right)^2 - \left(\frac{k}{2M}\right)^2 = 0 \Rightarrow \left(-\frac{3k}{2M} - \lambda - \frac{k}{2M}\right) \left(-\frac{3k}{2M} - \lambda + \frac{k}{2M}\right) = 0$$

$$\lambda = -2\frac{k}{M} \quad \text{and} \quad \lambda = -\frac{k}{M}$$

Step03: compute the Eigen vectors $\vec{V}_1(\alpha_1, \beta_1)$, $\vec{V}_2(\alpha_2, \beta_2)$

$$\text{For: } \lambda = -2\frac{k}{M}$$

$$[A - \lambda I] \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -\frac{3k}{2M} + 2\frac{k}{M} & \frac{k}{2Mr} \\ \frac{kr}{2M} & -\frac{3k}{2M} + 2\frac{k}{M} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = 0$$

$$[A - \lambda I] \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \frac{k}{2M} & \frac{k}{2Mr} \\ \frac{kr}{2M} & \frac{k}{2M} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = 0$$

$$\begin{cases} \frac{k}{2M} \alpha_1 + \frac{k}{2Mr} \beta_1 = 0 \\ \frac{kr}{2M} \alpha_1 + \frac{k}{2M} \beta_1 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_1 + \frac{1}{r} \beta_1 = 0 \rightarrow \alpha_1 = -\frac{1}{r} \beta_1 \\ \alpha_1 r + \beta_1 = 0 \rightarrow \alpha_1 = -\frac{1}{r} \beta_1 \end{cases} \Rightarrow \alpha_1 = 1, \beta_1 = -r$$

$$\vec{V}_1(1, -r)$$

For: $\lambda = -\frac{k}{M}$

$$[A - \lambda I] \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -\frac{3k}{2M} + \frac{k}{M} & \frac{k}{2Mr} \\ \frac{kr}{2M} & -\frac{3k}{2M} + \frac{k}{M} \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = 0$$

$$[A - \lambda I] \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -\frac{k}{2M} & \frac{k}{2Mr} \\ \frac{kr}{2M} & -\frac{k}{2M} \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = 0$$

$$\begin{cases} -\frac{k}{2M} \alpha_2 + \frac{k}{2Mr} \beta_2 = 0 \\ \frac{kr}{2M} \alpha_2 - \frac{k}{2M} \beta_2 = 0 \end{cases} \Rightarrow \begin{cases} -\alpha_2 + \frac{1}{r} \beta_2 = 0 \rightarrow \alpha_2 = \frac{1}{r} \beta_2 \\ \alpha_2 r - \beta_2 = 0 \rightarrow \alpha_2 = \frac{1}{r} \beta_2 \end{cases} \Rightarrow \alpha_2 = 1, \beta_2 = r$$

$$\vec{V}_2(1, r)$$

Step04: resolution system basing on the initial base (θ_1, θ_2)

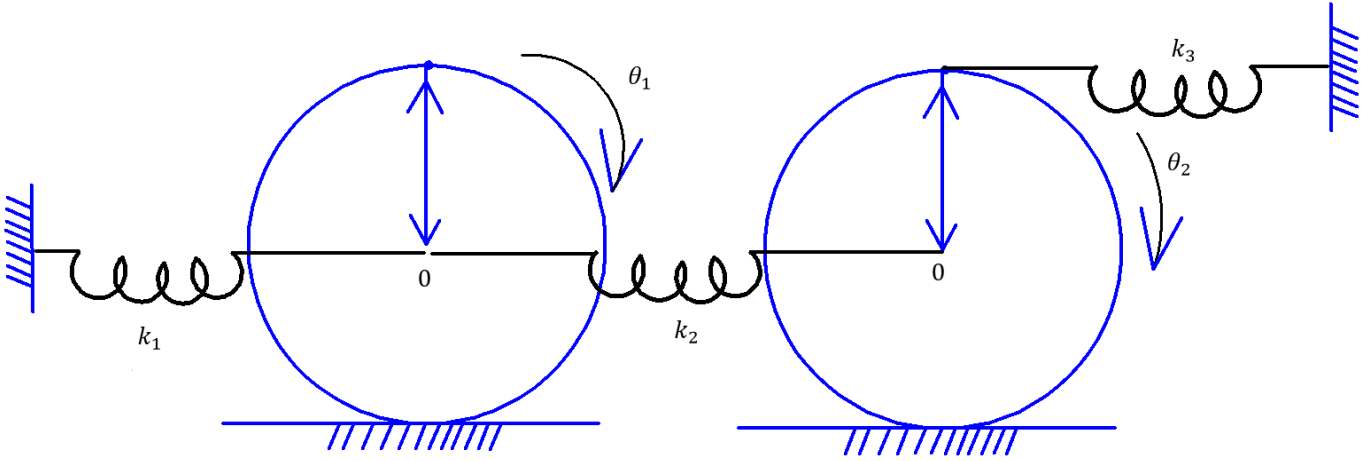
$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = [V_1 \quad V_2] \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \Rightarrow \begin{cases} \theta_1 = \alpha_1 \phi_1 + \alpha_2 \phi_2 \\ \theta_2 = \beta_1 \phi_1 + \beta_2 \phi_2 \end{cases}$$

$$\begin{cases} \theta_1 = \alpha_1 A \cos(w_1 t + \varphi_1) + \alpha_2 B \cos(w_2 t + \varphi_2) \\ \theta_1 = \beta_1 A \cos(w_1 t + \varphi_1) + \beta_2 B \cos(w_2 t + \varphi_2) \end{cases}$$

$$\begin{cases} \theta_1 = A \cos\left(\sqrt{2 \frac{k}{M}} t + \varphi_1\right) + B \cos\left(\sqrt{\frac{k}{M}} t + \varphi_2\right) \\ \theta_2 = -r A \cos\left(\sqrt{2 \frac{k}{M}} t + \varphi_1\right) + r B \cos\left(\sqrt{\frac{k}{M}} t + \varphi_2\right) \end{cases}$$

Exercise 21

Considering the above mechanical system. It is composed of two pulleys (M, R).



1. Describe the model
2. Extract the energies of the proposed system
3. Establish the differential equation using the Lagrangian method and find the solution

Correction:

$$\begin{cases} T = T_{p1rotation} + T_{p1translation} + T_{p2rotation} + T_{p2translation} \\ U = U_{k1} + U_{k2} + U_{k3} \end{cases}$$

$$\begin{cases} T = \frac{1}{2}M_1R^2\dot{\theta}_1^2 + \frac{1}{2}M_1R^2\dot{x}_1^2 + \frac{1}{2}M_2R^2\dot{\theta}_2^2 + \frac{1}{2}M_2R^2\dot{x}_2^2 \\ U = \frac{1}{2}k_1R^2\theta_1^2 + \frac{1}{2}k_2(R\theta_1 - R\theta_2)^2 + \frac{1}{2}k_34R^2\theta_2^2 \end{cases}$$

$$\begin{cases} T = \frac{1}{2}M_1R^2\dot{\theta}_1^2 + \frac{1}{2}\frac{M_1R^2}{2}\dot{\theta}_1^2 + \frac{1}{2}M_2R^2\dot{\theta}_2^2 + \frac{1}{2}\frac{M_2R^2}{2}\dot{\theta}_2^2 \\ U = \frac{1}{2}k_1R^2\theta_1^2 + \frac{1}{2}k_2R^2\theta_1^2 - k_2R^2\theta_1\theta_2 + k_2R^2\theta_2^2 + \frac{1}{2}k_34R^2\theta_2^2 \end{cases}$$

$$L = T - U = \frac{3}{4}M_1R^2\dot{\theta}_1^2 + \frac{3}{4}M_2R^2\dot{\theta}_2^2 - \frac{1}{2}k_1R^2\theta_1^2 - \frac{1}{2}k_2R^2\theta_1^2 + k_2R^2\theta_1\theta_2 - k_2R^2\theta_2^2 - \frac{1}{2}k_34R^2\theta_2^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\begin{cases} \frac{3}{2}M_1R^2\ddot{\theta}_1 + (k_1R^2 + k_2R^2)\theta_1 - k_2R^2\theta_2 = 0 \\ \frac{3}{2}M_1R^2\ddot{\theta}_2 + (k_2R^2 + k_34R^2)\theta_2 - k_2R^2\theta_1 = 0 \end{cases}$$

Part B

Exercise 01

The function that describe the propagation wave along a rope is given by the following expression:

$$y = 15\cos(0.25x + 75t)$$

1. Extract the following parameters: the amplitude, number of wave, the pulse, the wave length, the frequency, the period, the speed
2. Give the type of the wave, justify your choice

Exercise 02

Considering the following wave function $y(x, t) = Ae^{-(2x+3)^2}$

1. Find the type of direction? Justify your choice

Exercise 03

Prove that the bellow function waves are the solutions of the propagation wave equation:

1. $y(x, t) = Ae^{j(\omega t + kx)}$
2. $y(x, t) = A\sin 2\pi f(t - \frac{x}{v})$
3. $y(x, t) = A\sin(k_1x + k_2y + k_3z - \omega t)$

Références

- [1] H. GEORGI, THE PHYSICS OF WAVES, Harvard University, 1993.
- [2] R. Lefort, Ondes et Vibrations, 2017.
- [3] J. Bruneaux, Vibrations, ondes, Ellipses, 2008.
- [4] n. m. e. amine, vibrations et ondes mecaniques cours et exercices corrigés, office des publications universitaires, 2020.
- [5] T. Becherrawy, Vibrations, ondes et optique, 2010.
- [6] J. Brac, Propagation d'ondes acoustiques et élastiques, Hermès science Publ. Lavoisier, 2003.
- [7] H. Djelouah, Vibrations et Ondes Mécaniques – Cours & Exercices, USTH, algerie.
- [8] S.boudrahem, physique des vibrations et des ondes mecaniques, office des publications universitaires, 2015.
- [9] B. Crowell, Vibrations and waves, Fullerton, California: Book 3 in the Light and Matter series of introductory physics textbooks, 1999.