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Study of some Roman domination parameters

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This thesis is dedicated to my family, teachers, friends

and for all lovers of graph theory.

ABSTRACT

Let $G = (V, E)$ be a simple graph. A *Roman dominating function* (RDF for short) on G is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight $w(f)$ of an RDF f is defined as $w(f) = \sum_{u \in V} f(u)$. The minimum weight of an RDF on a graph G is called the *Roman domination number* of G, denoted $\gamma_R(G)$.

A *double Roman dominating function* (DRDF) of a graph G is a function $f : V \to \{0, 1, 2, 3\}$ for which the following conditions are satisfied.

i) If $f(v) = 0$, then the vertex v must have at least two neighbors assigned 2 under f or one neighbor assigned 3 under f.

ii) If $f(v) = 1$, then the vertex v must have at least one neighbor u with $f(u) \geq 2$.

The weight $w(f)$ of an DRDF f is the value $w(f) = \sum_{u \in V} f(u)$. The minimum weight of an DRDF on a graph G is called the *double Roman domination number* of G, denoted $\gamma_{dR}(G)$.

In this thesis, we will extend the study of double Roman domination by presenting new results on the Nordhaus-Gaddum type inequality and providing a characterization of all graphs G satisfying $\gamma_{dR}(G) = 2\gamma_R(G) - 1$. We will also explore the concept of *criticality*, and solve some problems from various papers in this area.

RESUME

Soit $G = (V, E)$ un graphe simple. Une *fonction de domination romaine* (RDF) sur G est une fonction $f : V \longrightarrow \{0, 1, 2\}$ vérifiant la condition suivante: chaque sommet u pour lequel $f(u) = 0$ est adjacent à au moins un sommet v tel que $f(v) = 2$. Le poids $w(f)$ d'une fonction de domination romaine f est la valeur $w(f) = \sum_{u \in V} f(u)$. Le poids minimal d'une fonction de domination romaine de G est appelé le *nombre de domination romaine* de G, noté $\gamma_R(G)$.

Une *fonction de domination romaine double* (DRDF) d'un graphe G est une fonction $f : V \to \{0, 1, 2, 3\}$ vérifiant les conditions suivantes:

i) Si $f(v) = 0$, alors le sommet v doit avoir au moins deux voisins u_1, u_2 tels que $f(u_1) = f(u_2) = 2$ ou un voisin u tel que $f(u) = 3$.

ii) Si $f(v) = 1$, alors le sommet v doit avoir au moins un voisin u tel que $f(u) \geq 2$.

Le poids $w(f)$ d'une fonction de domination romaine double f est la valeur $w(f) = \sum_{u \in V} f(u)$. Le poids minimal d'une fonction de domination romaine double de G est appelé le *nombre de domination romaine double* de G, noté $\gamma_{dR}(G)$.

Dans cette thèse, nous étendrons l'étude de la domination romaine double en présentant de nouveaux résultats sur l'inégalité de type Nordhaus-Gaddum et en fournissant une caractérisation de tous les graphes G satisfaisant $\gamma_{dR}(G) = 2\gamma_R(G) - 1$. Nous explorerons également le concept de *criticité* et résoudrons certains problèmes tirés de divers articles dans le domaine.

الملخص

لنفترض أن G=(V,E) رسمًا بيانيًا بسيطًا. دالة السيطرة الرومانية (يُشار إليها اختصارًا بـ RDF) على G هي دالة f:V→{0,1,2} تستوفي الشرط الذي ينص على أن كل قمة u حيث f(u)=0 تكون مجاورة لقمة واحدة على الأقل v حيث f(v)=2. يُعرّف الوزن w(f) لدالة السيطرة الرومانية f على أنه $w(f)=\Sigma_{u\in V}f(u)$. يُطلق على أقل وزن لدالة سيطرة رومانية RDF على رسم البياني G اسم عدد السيطرة الرومانية للرسم G، ويُرمز له بالرمز $\gamma_R(G)$.

دالة السيطرة الرومانية المزدوجة (DRDF) للرسم البياني G هي دالة f:V→{0,1,2,3} تستوفي الشروط التالية:

i. إذا كانت f(v)=0، فإن القمة v يجب أن يكون لها على الأقل جاران قيمتهما 2 حسب f أو جار واحد قيمته 3 حسب f.
ii. إذا كانت f(v)=1، فإن القمة v يجب أن يكون لها على الأقل جار u واحد حيث $f(u) \geq 2$.

الوزن w(f) لدالة السيطرة الرومانية المزدوجة f هو القيمة $w(f)=\Sigma_{u\in V}f(u)$. يُطلق على أقل وزن لدالة سيطرة رومانية مزدوجة DRDF على الرسم البياني G اسم عدد السيطرة الرومانية المزدوجة للرسم G، ويُرمز له بالرمز $\gamma_{dR}(G)$.

في هذه الأطروحة، سنوسع نطاق دراسة السيطرة الرومانية المزدوجة من خلال تقديم نتائج جديدة حول متباينة نوع نوردهاوس-جادوم وتقديم توصيف لجميع الرسوم البيانية G التي تحقق $\gamma_{dR}(G)=2\gamma_R(G)-1$. سنستكشف أيضًا مفهوم الحرجية، ونحل بعض المسائل من أوراق بحثية مختلفة في هذا المجال.

ACKNOWLEDGMENTS

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I would also like to express my heartfelt gratitude to the committee members for their invaluable contributions and guidance. I am deeply honored to have Prof. Mustapha Chellali as the President of the committee, and I extend my sincere thanks for his insightful leadership. I also wish to thank Prof. Noureddine Ikhlef-Eschouf, Dr. Nacéra Meddah, Dr. Samia Kerdjoudj, and Dr. Mohamed Zamime for their valuable time, thoughtful reviews, and constructive feedback that greatly enriched my work. Your expertise and encouragement have been instrumental in completing this thesis. Thank you all for your dedication and support.

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INTRODUCTION

Graph theory is a prominent area of discrete mathematics, encompassing both theoretical developments and practical applications. Its origins trace back to 1736 when Euler addressed the Königsberg bridge problem [34], exploring whether it was possible to traverse each of the seven bridges exactly once. Graphs provide a powerful framework for modeling and simplifying a wide range of problems by reducing them to the study of vertices and edges. In recent years, computer scientists have driven many advancements in graph theory, particularly due to the growing importance of algorithmic aspects.

Among the fundamental concepts in graph theory is domination in graphs. Historically, the first domination-type problems emerged from chess. For instance, the chess master C.F. de Jaenisch [49] and other chess enthusiasts studied how pieces like queens could be placed on an $n \times n$ chessboard such that every square either contains a queen or is attacked by a queen. For example, five queens are required to dominate an 8×8 chessboard (four queens leave at least two squares unattacked). It has been observed by Yaglom and Yaglom [83] that there are exactly 4860 such placements of five queens (such as placing them along the main diagonal at squares a1, c3, e5, f6, and g7).

The formal study of domination in graphs is often attributed to Claude Berge in 1958 [16], who introduced the concept of the domination number (though he did not use this term). In 1962, Oystein Ore published Theory of Graphs [68], the first graph theory book in English, where he formally coined the term "domination". This marked the beginning of domination as a theoretical area of graph theory. However, it was not until 1977, with the publication of the seminal survey paper Towards a Theory of Domination in Graphs by Cockayne and Hedetniemi, that the field experienced significant growth. Since 1998, research in domination has expanded rapidly, with over $4,000$ papers published to date.

Another intriguing concept is graph protection, which involves placing mobile guards on graph vertices to defend against attacks. This idea has historical roots in the military strategies of the Roman Empire. Modern research on graph protection began in the late 20th century, inspired by four publications referencing the strategies of Emperor Constantine the Great ($274-337$ AD). Ian Stewart's paper Defend the Roman Empire! in Scientific American [76] was particularly influential, responding to C. S. ReVelle's question, Can you protect the Roman Empire?

During the third century, the Roman Empire dominated much of Europe, North Africa, and the Near East. Its defense relied on a forward strategy, with approximately fifty legions securing even the most remote regions. However, by the fourth century, the empire's power waned, and the number of legions decreased significantly. According to E. N. Luttwak's The Grand Strategy of the Roman Empire [72], Emperor Constantine devised a new strategy to address this decline. He decreed that no more than two legions should be stationed in any city, and any city without stationed legions must be within proximity of a city with two legions. This ensured that one legion could be moved to defend an attacked city.

At the time, the empire's connectivity resembled Figure 1, and Constantine faced the challenge of allocating only four legions to defend the entire empire. He stationed two legions in Rome and two in Constantinople, the empire's capitals. While this deployment secured most regions, it left Britain vulnerable, and it was eventually the first to be lost. Modern analyses suggest alternative solutions, such as deploying one legion in Britain, two in Rome, and one in Asia Minor. This historical context inspired the mathematical concept of Roman domination, which involves protecting a graph through strategic resource allocation.

While the classical problem remains relevant in military operations research [10], it can also be applied to model and solve issues requiring time-critical assistance needs to be provided with some reserve. For example, first-aid services should not deploy their entire team for a single emergency call. Studying these types of domination problems is crucial for optimizing and efficiently organizing emergency services.

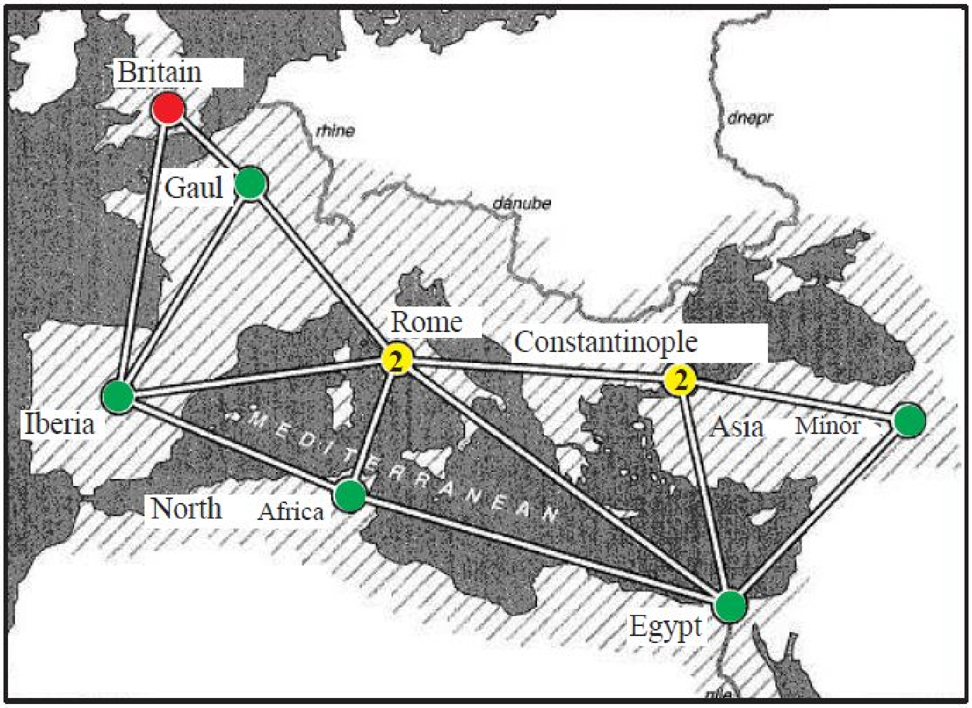

Figure 1: The Roman Empire, fourth century AD.

This thesis is organized as follows:

Chapter 1 introduces fundamental definitions and illustrative examples essential for the subsequent chapters.

Chapter 2 explores Roman domination further, surveying its various extensions and discussing associated challenges.

Chapter 3 focuses on double Roman domination, a stronger version of Roman domination where three legions can be deployed at a single location, and presents improvements to existing results.

Chapter 4 extends the concept of supercriticality to double Roman domination, building on studies initiated by Sumner and Blitch (1983) [78], and addresses open problems in the field.

Finally, the thesis ends with a conclusion summarizing the main contributions and outlining possible directions for future research.

Chapter 1

Basic concepts and notation in graphs

In this chapter, we need to define some terminology and notation for the purpose of this thesis. Additional terms will be introduced whenever necessary. Several illustrative examples are provided to help the reader understand the ideas more clearly. Unless stated otherwise, the notation and definitions follow those in Haynes, Hedetniemi, and Henning [44, 45].

1.1 Fundamental definitions on graphs

A *graph* G is an ordered pair $(V(G), E(G))$ consisting of a set of *vertices* $V = V(G)$ together with a set $E = E(G)$ of unordered pairs of vertices called *edges*. For notational simplicity, we write the edge uv for the unordered pair $\{u, v\}$. We denote the numbers of vertices and edges in G by $n = n(G) = |V|$ and $m = m(G) = |E|$; these two basic parameters are called the *order* and *size* of G, respectively (Note that there are many numbers, referred to as parameters, associated with a graph G). We will assume that all graphs are *simple*, i.e. there is at most one edge between any two distinct vertices, and no edge connects a vertex to itself. If $e = uv$ is an edge in a graph G, we say that u and v are *adjacent* in G. In this case, we say that each of u and v is incident with the edge e. Two edges are adjacent if they have a common vertex. Two vertices in a graph G are *independent* if they are not adjacent. Similarly, two edges are independent if they are not adjacent. A neighbor of a vertex v in G is a vertex that is adjacent to v. For every vertex $v \in V$, the *open* *neighborhood* of v in G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. For a set of vertices $S \subseteq V$, the open neighborhood of S is the set $N_G(S) = \bigcup_{v \in S} N(v)$ and its closed neighborhood is the set $N_G[S] = \bigcup_{v \in S} N(v) \cup S$. The *degree* $\deg_G(v)$ of a vertex v is the number of neighbors v has in G, that is, $\deg_G(v) = |N_G(v)|$. For a subset of vertices $S \subseteq V$, the degree of v in S, denoted $\deg_S(v)$, is the number of vertices in S adjacent to the vertex v. In particular, if $S = V$, then $\deg_S(v) = \deg_G(v)$. An *isolated vertex* is a vertex of degree 0 in G. A *leaf* is a vertex of degree one, while its neighbor is a *support* vertex. A support vertex with two or more leaf neighbors is called a *strong support* vertex. A *weak support* vertex is a support vertex that is not a strong support. When there is no ambiguity, we omit the subscript G from graph-theoretic symbols, and write, for example, $N(v), N[v], N(S), N[S]$ and $\deg(v)$ instead of $N_G(v), N_G[v], N_G(S), N_G[S]$ and $\deg_G(v)$, respectively.

A graph is isolate-free if it does not contain an isolated vertex. Any graph with just one vertex is referred to as *trivial*. All other graphs are *nontrivial*. By $\Delta(G) = \Delta$ and $\delta(G) = \delta$ we denote the *maximum degree* and the *minimum degree* of G, respectively. The set of leaves is denoted by $L(G)$ and the set of support vertices is denoted by $S(G)$.

Two graphs G and H are *isomorphic*, denoted $G \cong H$, if there exists a bijection $\varphi : V(G) \to V(H)$ such that two vertices u and v are adjacent in G if and only if the two vertices $\varphi(u)$ and $\varphi(v)$ are adjacent in H (see Figure 1.1).

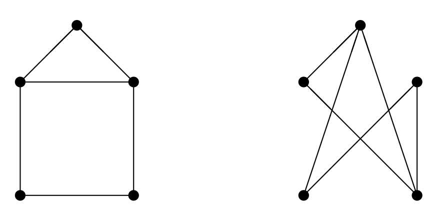

Figure 1.1: Two isomorphic graphs.

1.2 Special families of graphs

Certain types of graphs play prominent roles in graph theory, so it is necessary to mention some of them, which we will also need throughout this thesis.

Let $V(H)$, $E(H)$, $V(G)$ and $E(G)$ be the vertex set and the vertex edge set of H and G, respectively. If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then we say that H is a *subgraph* of G. A subgraph H of a graph G is called a *spanning subgraph* of G if $V(H) = V(G)$. If $U \neq \emptyset$ is a subset of $V(G)$, then the subgraph of G induced by U, denoted $G[U]$, is defined to be the graph having vertex set U and edge set consisting of those edges of G that have both ends in U. If $S \subsetneq V$ is a set of vertices, then we write $G - S$ for the subgraph of G induced by $V - S$. Also, if $S = \{x\}$ then we write $G - x$ instead of $G - \{x\}$. If $F \subseteq E$ is a set of edges then we write $G - F = (V, E - F)$. If $F = \{e\}$ then we write $G - e$ instead of $G - \{e\}$.

In a graph, a *path* of length k from vertex v_0 to vertex v_k is a collection of edges, denoted with $P = v_0v_1...v_{k-1}v_k$. A *cycle* is a closed path where $v_0 = v_k$. A path (cycle) of order n is denoted by P_n (C_n). A cycle C_3 is often called a *triangle*. A graph G is called a *cactus* graph if each edge of G is contained in at most one cycle. A *unicyclic* graph is a graph with exactly one cycle.

The *complement* graph $\overline{G}$ of G is the graph defined in the same vertex set of G, where an edge belongs to $\overline{G}$ if and only if it does not belong to G. The cycle C_6 and its complement are shown in Figure 1.2.

Figure 1.2: The cycle C_6 (on the left) and its complement $\overline{C_6}$ (on the right).

A graph G is *connected* if for any two distinct vertices, there is a path between them. The *components* of G are the maximal connected subgraphs of G. Let u and v be two vertices of G. If u and v are in the same component of G, we define the *distance* between u and v, denoted by $d(u, v)$, to be the length of a shortest $u - v$ path. The *diameter* of a graph G, denoted $\operatorname{diam}(G)$, is the greatest distance between two vertices of G. A connected graph and a disconnected graph are shown in Figure 1.3.

A graph G is *complete* if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n.

A vertex v of G is called a *cut vertex* of G if $G - v$ has more components than G. A *block* of a

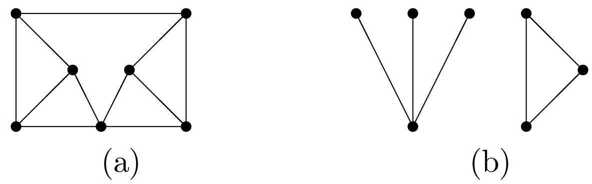

Figure 1.3: (a) A connected graph, and (b) a disconnected graph.

graph is a maximal induced subgraph without cut vertex. A *block graph* is a graph all blocks of which are complete.

A graph G is called *bipartite*, if V can be partitioned into two subsets X and Y such that each edge $uv \in E(G)$ connects a vertex of X and a vertex of Y. A bipartite graph G is complete, if $|X| = p$, $|Y| = q$, and $uv \in E(G)$ for all $u \in X$ and $v \in Y$, and it is denoted by $K_{p,q}$.

A *tree* is a connected graph with no cycles. A *star* $K_{1,p}$ for $p \geq 1$, is a tree of order $p+1$ having at least p leaves. For $r, s \geq 1$, a *double star* $S(r,s)$ is a tree with exactly two adjacent vertices that are not leaves, one of which has r leaf neighbors and the other has s leaf neighbors. Figure 1.4 shows a complete graph, a complete bipartite graph and a double star.

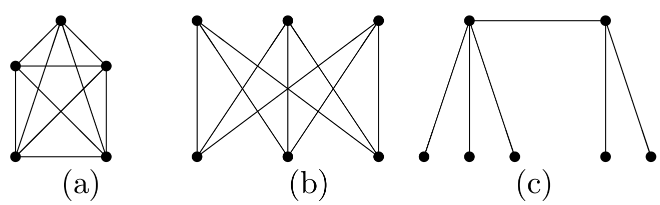

Figure 1.4: (a) The complete graph K_5, (b) the complete bipartite graph $K_{3,3}$ and (c) the double star $S(3,2)$.

A graph G of order at least two is called *regular* if its vertices have the same degree and *semiregular* if $\Delta(G) - \delta(G) = 1$. We say that, a graph G is r-regular if $\delta(G) = \Delta(G) = r$. A 3-regular graph is also referred to as a *cubic graph*. For example, the graph K_4 is cubic graph.

Let H be any graph. A graph G is called *H-free* if it does not contain H as an induced subgraph. This idea helps define types of graphs by saying which subgraphs are not allowed. For example, trees are graphs with no cycles, and *claw-free* graphs do not contain the claw, which is the graph $K_{1,3}$.

A *planar graph* is a graph that can be drawn on a plane without any of its edges crossing each other. In other words, it is possible to place the graph on a flat surface such that no two edges intersect except at their endpoints.

For classes of graphs not defined here, we refer the reader to the survey [20] by Brandstädt, Le, and Spinrad.

1.3 Basic graph operations

Let $G_1 = (U_1, E_1)$ and $G_2 = (U_2, E_2)$ be two graphs. The *union* of G_1 and G_2 written as $G_1 \cup G_2$ is the graph $G = (U_1 \cup U_2, E_1 \cup E_2)$. In particular, if G is a disjoint union of k copies of a graph H, we write $G = kH$. The *join* of G_1 and G_2, denoted $G_1 \vee G_2$, is a graph formed from disjoint copies of G_1 and G_2 by connecting each vertex of G_1 to each vertex of G_2. The *Cartesian product* of G_1 and G_2, denoted $G_1 \Box G_2$, is the graph with vertex set $\{(u, v) : u \in G_1, v \in G_2\}$. Two vertices (v_1, u_1) and (v_2, u_2) are adjacent in $G_1 \Box G_2$ if and only if one of the following is true: $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2 or $v_1 = v_2$ and u_1 is adjacent to u_2 in G_1. In particular, the Cartesian product $P_m \Box P_n$ is called the $m \times n$ *grid graph* and is denoted by $G_{m,n}$. Figure 1.5 shows the graph cartesian product $C_4 \Box P_3$.The *corona* $cor(G)$ of a graph G is the graph obtained from G by adding

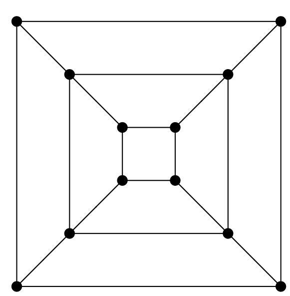

Figure 1.5: The graph $C_4 \Box P_3$.

for each vertex $v \in V$ a new vertex v' and the edge vv'. Figure 1.6 shows the corona of $\overline{C_6}$.

Let uv be an edge of G. By subdividing the edge uv we mean removing it, and adding a new vertex, say x, along with two new edges ux and xv. In this case, we say that the edge uv has been subdivided. Now, for a positive integer t, a *healthy spider* is a star $K_{1,t}$ with all its edges subdivided. A *wounded spider* is a star $K_{1,t}$ with at most $t - 1$ of its edges subdivided.

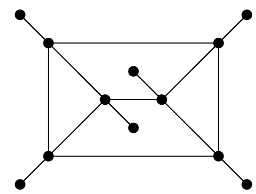

Figure 1.6: The $cor(\overline{C_6})$.

1.4 Domination in graphs

In 1977, Cockayne and Hedetniemi [32] published a survey paper, in which the notation $\gamma(G)$ was first used for the domination number of a graph G. Since the publication of this paper, domination in graphs has been studied extensively and several additional research papers have been published on this topic.

Now, we present the definition of dominating sets in graphs. Let $G = (V, E)$ be a simple graph. A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A minimum dominating set with such cardinality is called $\gamma(G)$-set. Every graph has a dominating set, since $S = V$ is such a set, and so $\gamma(G) \leq n$. We note that a graph G can have several $\gamma(G)$-sets.

For example in the graph $K_{2,3}$ in Figure 1.7, $\{a, b\}$ and $\{a, x\}$ are examples of $\gamma(K_{2,3})$-sets, and thus $\gamma(K_{2,3}) = 2$.

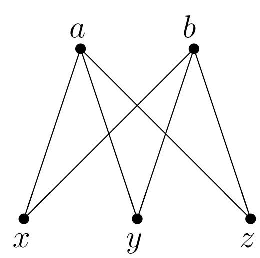

Figure 1.7: The graph $K_{2,3}$.

The concept of domination, in both theoretical and applied sense, has received the attention of many researchers. It has been used to study the optimal location of facilities such as radar stations, hardware or software resources, and communication networks. The practical utility of domination often prompts the development of additional parameters. Many domination parameters have arisen when an additional condition is imposed on domination. This condition can be internal to the dominating set, external to the dominating set, or both internal and external simultaneously. In the following chapters, we focus on some of these parameters.

The decision problem to determine the domination number of a graph is known to be $\mathcal{NP}$-*complete* (see [38]). Hence, researchers are interested in exploring simple upper and lower bounds that are easy to verify. Characterizing the graphs for which these bounds are attained becomes essential. Also, they aim to establish inequalities between certain parameters and identify conditions under which equality is achieved.

In 1975, Cockayene et al. [33] introduced the first linear algorithm to determine the domination number in trees.

In the literature, there is another way that domination and the domination number of a graph G has been looked: A *dominating function* (abbreviated *DF*) on a graph G is a function $f : V \longrightarrow \{0, 1\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 1$. The weight $w(f)$ of a dominating function f is the value $w(f) = \sum_{u \in V} f(u)$. The minimum weight of a dominating function on a graph G is called the *domination number* of G, denoted by $\gamma(G)$. It can be readily seen that a DF f, generates two sets S and $V - S$ such that $S = \{v \in V(G) : f(v) = 1\}$ and $V - S = \{v \in V(G) : f(v) = 0\}$. Thus $w(f) = |S|$.

Moreover, some researchers have studied domination functions with codomains other than $\{0, 1\}$, which led to new domination parameters.

Chapter 2

A survey of selected Roman domination parameters

The concept of domination is extended to Roman domination, a topic that has garnered significant attention in recent research. This chapter will focus on providing a brief overview on the various parameters related to Roman domination, rather than an exhaustive examination of each one. However, before we can explore these parameters, it is essential to first define Roman domination and provide some results related to it.

2.1 Roman domination

A *Roman dominating function* (abbreviated *RDF*) on a graph G is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight $w(f)$ of a Roman dominating function f is the value $w(f) = \sum_{u \in V} f(u)$. The minimum weight of an RDF on a graph G is called the *Roman domination number* of G, denoted by $\gamma_R(G)$ (see Figure 2.1). It can be readily seen that an RDF f, generates three sets V_0, V_1, V_2 such that $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0, 1, 2\}$. We can equivalently write $f = (V_0, V_1, V_2)$. Moreover, we observe that $w(f) = |V_1| + 2|V_2|$. The concept of Roman domination was introduced by Cockayne, Dreyer, Hedetniemi, and Hedetniemi [31]. For more details, see [25, 26] and the survey [27].

It is mentioned in [31] that the Roman domination problem on trees can be solved in linear time and it remains $\mathcal{NP}$-complete when restricted to split graphs, bipartite graphs, and planar graphs (see [57]). Applications of Roman domination is also shown in [24].

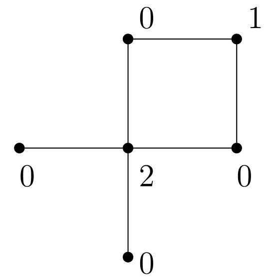

Figure 2.1: A graph G with $\gamma_R(G) = 3$.

The Roman domination number can be determined for certain graph families, as shown in the following result.

Proposition 2.1.1 ([31]) *For the classes of paths P_n, cycles C_n and the grid graph $G_{2,n}$, $\gamma_R(P_n) = \gamma_R(C_n) = \lceil \frac{2n}{3} \rceil$ and $\gamma_R(G_{2,n}) = n + 1$.*

Since 2004, research on Roman domination has grown rapidly. Summarizing key results and highlighting open problems would benefit the community.

2.1.1 Bounds on Roman domination number

An upper bound on the Roman domination number for connected graphs in terms of their order was established by Chambers et al. [24]. They also characterized the graphs that achieve this upper bound. Let $\mathcal{H}$ be denote the family of connected graphs G of order n constructed from a connected graph H such that each vertex of H is identified with the central vertex of a P_5.

Let $\mathcal{H}$ be the family of connected graphs G of order n (a multiple of 5), constructed from $\frac{n}{5}$ copies of P_5 by adding a connected subgraph induced by the central vertices of these paths.

Theorem 2.1.1 ([24]) *If G is a connected graph of order n, then $\gamma_R(G) \leq \frac{4}{5}n$, with equality if and only if $G \in \mathcal{H} \cup \{C_5\}$.*

This bound has been improved for graphs with minimum degree at least 2 or 3, as shown by the following three results. Let $\mathcal{B} = \{C_4, C_5, C_8, H_1, H_2\}$, where H_1 and H_2 are the graphs illustrated in Figure 2.2.

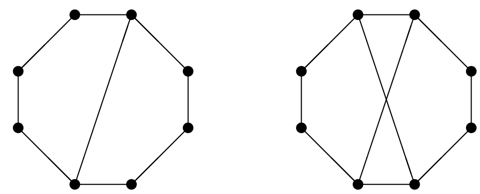

Figure 2.2: Graphs H_1 (on the left) and H_2 (on the right).

Theorem 2.1.2 ([24]) *If G is a graph of n vertices with $\delta(G) \geq 2$ and $G \notin \mathcal{B}$, then $\gamma_R(G) \leq \frac{8n}{11}$. Moreover, if $n \geq 9$, then $\gamma_R(G) = \frac{8n}{11}$ if and only if*

1. *If $n = 11$, then G is isomorphic to F (see Figure 2.3) plus a subset of one of $\{y_1y_3, y_1y_4, y_2y_3, y_2y_4\}$, $\{wz_1, y_1y_3, y_1y_4\}$, or $\{wz_1, wz_3, y_1y_3\}$ added as edges.*

2. *If $n > 11$, then G consists of disjoint copies of the graphs $F, F + wz_1$, and $F + wz_1 + wz_3$ with additional edges connecting copies of w.*

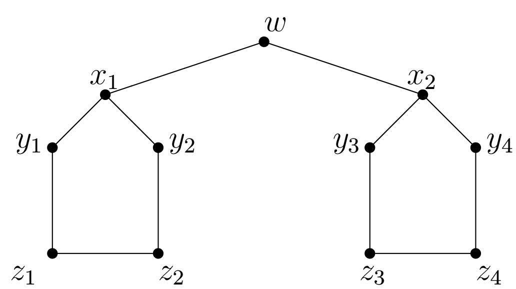

Figure 2.3: The graph F.

Bermudo [14] improved the previous bound by considering graphs with order at least 15 and minimum degree at least two.

Theorem 2.1.3 ([14]) *Let G be a graph of order $n \geq 15$, with $\delta(G) \geq 2$, which does not contain any induced subgraph isomorphic to F_1 or F_2 (see Figure 2.4). Then, $\gamma_R(G) \leq \frac{12n}{17}$.*

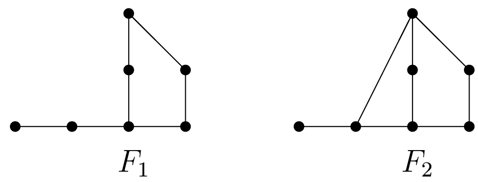

Figure 2.4: Graphs F_1 and F_2.

Bermudo [14] provided an infinite family of connected graphs that achieve the previous bound. Consider the graph $G^i = (V^i, E^i)$ of order 17 shown in Figure 2.5, where $\gamma_R(G^i) = 12$. Now, consider a connected graph $G_k = (V_k, E_k)$ such that $V_k = \cup_{i=1}^{k} V^i$, $E_k = \cup_{i=1}^{k} E^i \cup M$, where $M \subseteq \{v_i, v_j : 1 \leq i < j \leq k\}$. It can be checked that G_k with minimum degree two, without induced subgraph isomorphic to F_1 or F_2 and $\gamma_R(G_k) = 12k$. For example, $G_2 = (V^1 \cup V^2, E^1 \cup E^2 \cup \{v_1 v_2\})$ and $\gamma_R(G_2) = 24$.

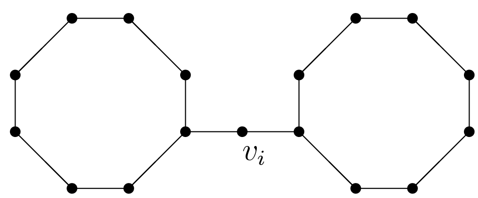

Figure 2.5: The graph G^i.

Theorem 2.1.4 ([58]) *If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma_R(G) \leq \frac{2n}{3}$.*

Liu and Chang [58] provided an infinite family of connected graphs G of order n with $\gamma_R(G) = \frac{2n}{3}$. For any integer $t \geq 3$, construct graph H_t from the union of two disjoint $3t$-cycles $x_1, x_2, ..., x_{3t}, x_1$ and $y_1, y_2, ..., y_{3t}, y_1$ by adding edges $x_i y_{j_i}$ for $1 \leq i \leq 3t$, where $j_i = i$ if $i \equiv 1 (\bmod 3)$, $j_i = i+1$ if $i \equiv 2 (\bmod 3)$ and $j_i = i-1$ if $i \equiv 0 (\bmod 3)$; Figure 2.6 shows the graph H_3.

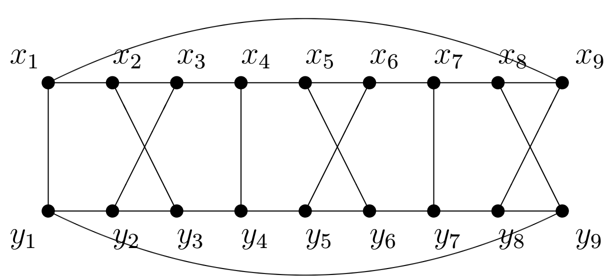

Figure 2.6: A graph H_3 with $\gamma_R(H_3) = 12$.

It is natural to expect a relationship between $\gamma_R(G)$ and $\gamma(G)$ for a graph G. In what follows, we present some results in this direction.

Proposition 2.1.2 ([31]) *For any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.*

To see the sharpness of the bounds in Proposition 2.1.2, consider the following examples. If G is a nontrivial star $K_{1,n-1}$, then $\gamma_R(G) = 2\gamma(G) = 2$. On the other hand, the empty graphs $\overline{K_n}$ are the only graphs for which $\gamma_R(G) = \gamma(G) = n$. Moreover, a graph G with $\gamma_R(G) = 2\gamma(G)$ is called a *Roman graph*. This raises the following interesting problem:

Problem 2.1.1 ([31]) *Can you find some classes of Roman graphs?*

Henning [47] characterized Roman trees, but a characterization Roman graphs remains open. Cockayne et al. [31] characterized the connected graphs G with $\gamma_R(G) \in \{\gamma(G)+1, \gamma(G)+2\}$.

Proposition 2.1.3 ([31]) *If G is a connected graph of order n, then:*

1. *$\gamma_R(G) = \gamma(G) + 1$ if and only if there is a vertex $v \in V(G)$ of degree $n - \gamma(G)$.*
2. *$\gamma_R(G) = \gamma(G) + 2$ if and only if*
   - **a.** *G does not have a vertex $v \in V(G)$ of degree $n - \gamma(G)$;*
   - **b.** *either G has a vertex of degree $n - \gamma(G) - 1$ or G has two vertices v and w such that $|N[v] \cup N[w]| = n - \gamma(G) + 2$.*

Xing, Chen and Chen [82] presented the following theorem as a solution to the open question posed in [31].

Theorem 2.1.5 ([82]) *Let G be a connected graph of order n with $\gamma(G) \geq 2$. If k is an integer such that $2 \leq k \leq \gamma(G)$, then $\gamma_R(G) = \gamma(G) + k$ if and only if:*

a. *for any integer s with $1 \leq s \leq k-1$, G does not have a set U_t of t $(1 \leq t \leq s)$ vertices such that $|\cup_{v \in U_t} N[v]| = n - \gamma(G) - s + 2t$;*

b. *there exists an integer l with $1 \leq l \leq k$, and G has a set W_l of l vertices such that* $|\cup_{v \in W_l} N[v]| = n - \gamma(G) - k + 2l$.

Remark 2.1.1 *The proof of Theorem 2.1.5 contains a gap that has been corrected in [81] by Wu and Xing.*

Wu [80] and Favaron et al. [35] also provided relations involving $\gamma_R(G)$ and $\gamma(G)$ for any connected graph as follows. We recall that $\mathcal{R}$ is the family of graphs G obtained from a connected graph H such that each vertex of H is identified with the central vertex of a P_5 or with an internal vertex of a path P_4 where the $|V(H)|$ paths are vertex-disjoint.

Theorem 2.1.6 ([80]) *For any graphs G and H, $\gamma_R(G \square H) \geq \gamma(G)\gamma(H)$.*

Theorem 2.1.7 ([35]) *For any graph G of order $n \geq 3$, $\gamma_R(G) + \frac{\gamma(G)}{2} \leq n$, with equality if and only if G is $C_4, C_5, cor(C_4)$ or $G \in \mathcal{R}$.*

In [15], Bermudo et al. stated the following conjecture, which is still open.

Conjecture 2.1.1 ([15]) *If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma_R(G) + \gamma(G) \leq n$.*

2.1.2 Nordhaus-Gaddum type results for Roman domination

A *Nordhaus–Gaddum*-type result provides either a lower or an upper bound on the sum (or product) of a parameter of a graph and its complement in terms of the number of vertices, honoring the classic paper by Nordhaus and Gaddum (1956). Since then, similar types of relations have been proposed for various other graph invariants, including domination (see the survey [1]).

Firstly, Chambers et al. [24] proved the Nordhaus–Gaddum inequalities for γ_R.

Theorem 2.1.8 ([24]) *If G is a graph of order $n \geq 3$, then $5 \leq \gamma_R(G) + \gamma_R(\overline{G}) \leq n + 3$. The lower bound is achieved if and only if G (or $\overline{G}$) contains a vertex of degree $n-1$ and $\overline{G}$ (or G) contains a vertex of degree $n-2$. The upper bound is achieved if and only if G or $\overline{G}$ is either C_5 or $\frac{n}{2}K_2$.*

Furthermore, they proved the following result.

Theorem 2.1.9 ([24]) *if $n \geq 160$, then $\gamma_R(G) \cdot \gamma_R(\overline{G}) \leq \frac{16n}{5}$, with equality if and only if G or $\overline{G}$ is $\frac{n}{5}C_5$.*

Subsequently, Jafari Rad and Rahbani [51] also investigated Nordhaus–Gaddum type bounds for Roman domination. In the following, a cycle C_n is represented by $v_1v_2, ..., v_nv_1$, where $V(C_n) = \{v_1, v_2, ..., v_n\}$. Furthermore by $C_n + v_iv_j$, where $|v_i - v_j| > 1$, we mean a graph obtained from C_n by adding the chord v_iv_j. Similarly, $C_n + v_iv_j + v_{i'}v_{j'}$ and $C_n + v_iv_j + v_{i'}v_{j'} + v_{i''}v_{j''}$ denote C_n with two or three such chords added, respectively. Now, we will recall some relevant families of graphs:

- $\mathcal{G}_0$. The class of all graphs G of order $n \geq 2$ with $\Delta(G) = n - 1$ and $\delta(G) \geq n - 2$.
- $\mathcal{G}_1$. The class of graphs $P_i + sK_2$ $(3 \leq i \leq 5, s \geq 0)$, $2K_3, C_3 + K_2, C_3 + 2K_2$, $C_4 + K_2, C_4 + C_3$, $C_5 + K_2, C_5, C_6, C_6 + v_3v_5$, $C_6 + v_3v_6, C_6 + v_3v_6 + v_1v_4$, $C_7, C_7 + v_1v_5$, $C_7 + v_1v_5 + v_2v_6, C_7 + v_4v_6 + v_3v_7 + v_3v_5$, $C_8, C_8 + v_1v_5, C_8 + v_1v_5 + v_2v_6$.
- $\mathcal{G}_2$. The class of seven specific graphs depicted in Figure 2.7.

Let $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$.

Theorem 2.1.10 ([51]) *For a graph G of order $n \geq 2$, $\gamma_R(G) + \gamma_R(\overline{G}) = n + 2$ if and only if $G \in \mathcal{G}$ or $\overline{G} \in \mathcal{G}$.*

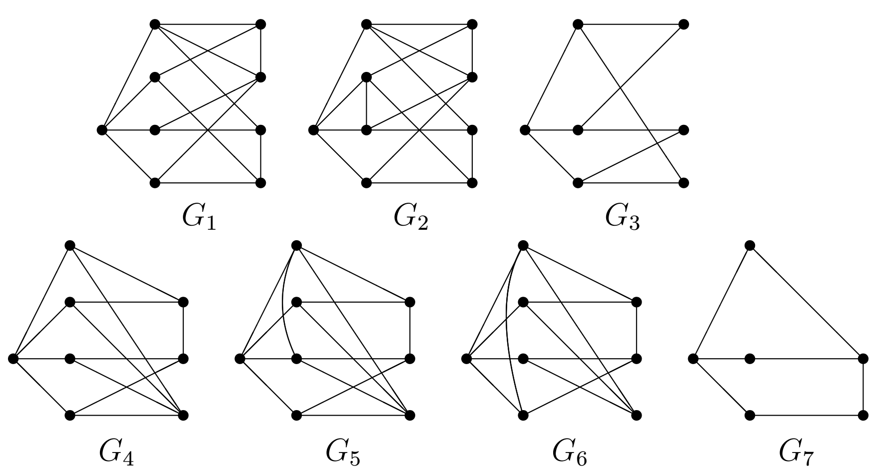

Figure 2.7: The Family $\mathcal{G}_2$.

Before studying Nordhaus–Gaddum type bounds for another Roman domination parameter, we must report an error in the previous characterization, as stated in Theorem 2.1.10. This will lead us to inquire about the validity of the results presented by the authors and which have been published.

$\gamma_R(G_1) + \gamma_R(\overline{G_1}) = \gamma_R(G_2) + \gamma_R(\overline{G_2}) = 5 + 4 < 9 + 2.$

$\gamma_R(G_3) + \gamma_R(\overline{G_3}) = 4 + 4 < 7 + 2.$

$\gamma_R(G_4) + \gamma_R(\overline{G_4}) = \gamma_R(G_5) + \gamma_R(\overline{G_5}) = \gamma_R(G_6) + \gamma_R(\overline{G_6}) = 5 + 4 < 8 + 2.$

$\gamma_R(C_7 + v_4v_6 + v_3v_7 + v_3v_5) + \gamma_R(\overline{C_7 + v_4v_6 + v_3v_7 + v_3v_5}) = 4 + 4 < 7 + 2.$

In the same year, Bouchou et al. [19] also independently provided a characterization of extremal graphs of a Nordhaus-Gaddum bound for $\gamma_R(G) + \gamma_R(\overline{G}) = n + 2$.

Theorem 2.1.11 ([19]) *For a graph G of order $n \geq 3$, $\gamma_R(G) + \gamma_R(\overline{G}) = n + 2$ if and only if*

G or $\overline{G} \in \{K_n\} \cup \mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$, where:

- $\mathcal{H}_0 = \{C_6, C_7, C_8, C_i \cup C_j$, where $i, j \in \{3, 4, 5\}\}$.
- $\mathcal{H}_1 = \{pK_1 \cup qK_2 : p \geq 1, q \geq 1$ and $p + 2q = n\}$.
- $\mathcal{H}_2 = \{qK_2 \cup H$ with $2q + |V(H)| = n$, where $H \in \{P_3, P_4, P_5, C_3, C_4, C_5\}$ if $q \neq 0$ and $H \in \{P_3, P_4, P_5\}$ if $q = 0\}$.
- $\mathcal{H}_3 = \{F_1, F_2, M_1, M_2\}$, (shown in Figure 2.8).

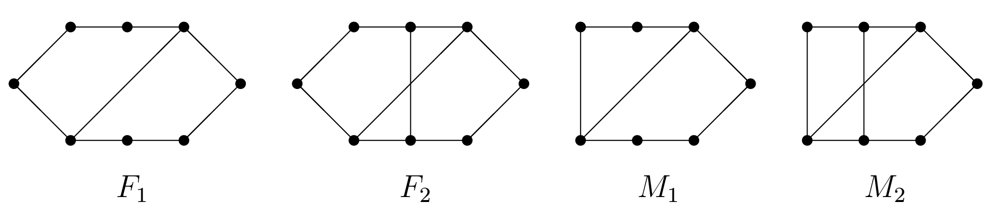

Figure 2.8: The Family $\mathcal{H}_3$.

2.1.3 Critical concepts for Roman Domination

When investigating a graph parameter μ, it is often useful to study a more restricted class of graphs known as critical graphs. In these graphs, the addition of a set of edges or the removal of a set of vertices/edges can either increase or decrease μ, or leave μ unchanged. The study of *criticality* in graphs is a very active area in graph theory. In the following, we present some important results where γ_R changes when removing a vertex or removing/adding an edge of the graph. A good compilation of the criticality properties can be found in [73].

Vertex removal

Jafari Rad and Volkmann [53] proved the following result.

Theorem 2.1.12 ([53]) *Let v be a vertex of a graph G. Then $\gamma_R(G-v) < \gamma_R(G)$ if and only if there is a γ_R-function f on G such that $v \in V_1^f$. If $\gamma_R(G-v) < \gamma_R(G)$ then $\gamma_R(G-v) = \gamma_R(G)-1$. If $\gamma_R(G-v) > \gamma_R(G)$ then for every γ_R-function f on G, $f(v) = 2$.*

According to the effects of vertex removal on the Roman domination number of a graph G, we say that G is Roman domination vertex critical, or just γ_R-*vertex critical*, if for any vertex v of $V(G)$, $\gamma_R(G-v) < \gamma_R(G)$. If G is γ_R-vertex critical and $\gamma_R(G) = k$, then we call G a k-γ_R-vertex critical graph. Similarly, we say that G is γ-vertex critical, if for any vertex v of $V(G)$, $\gamma(G-v) < \gamma(G)$.

Proposition 2.1.4 ([43]) *For any vertex v in a γ_R-vertex critical graph G, $\gamma_R(G-v) = \gamma_R(G)-1$.*

Theorem 2.1.13 ([43]) *A block graph G is γ_R-vertex critical if and only if $G = K_2$.*

Theorem 2.1.14 ([53]) *A graph G of order $n \geq 4$ is 3-γ_R-vertex critical if and only if n is even, and G is an $(n-2)$-regular graph.*

Theorem 2.1.15 ([53]) *For any γ_R-vertex critical graph G, $diam(G) \leq \left\lceil \frac{3\gamma_R(G)-5}{2} \right\rceil$.*

The authors in [53] conjectured that every γ-vertex critical graph is γ_R-vertex critical. However, Blidia and Chellali [18] disproved this conjecture by providing the following counterexample.

Let G_n be a family of connected cactus graphs obtained by $n \geq 2$ disjoint cycles C_4 sharing a common vertex, say x (the graph G_4 is shown in Figure 2.9). The authors demonstrated that $\gamma(G_n) = n + 1$, $\gamma(G_n - x) = n$, $\gamma_R(G_n) = n + 2$ and $\gamma_R(G_n - x) = 2n$. Therefore, G_n is γ-vertex critical but not γ_R-vertex critical.

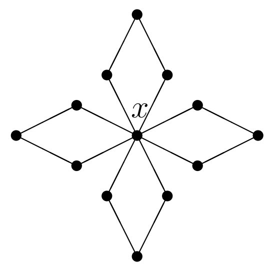

Figure 2.9: Connected cactus graph G_4.

The characterization of the connected γ_R-vertex critical unicyclic graphs was given in [42]. Let $\mathcal{E}$ be the class of all graphs G which either $G = cor(C_m)$, where $m \equiv 1 \pmod 3$ or $G = C_n$, where $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$.

Theorem 2.1.16 ([42]) *A connected unicyclic graph G is γ_R-vertex critical if and only if $G \in \mathcal{E}$.*

Edge removal

It has already been mentioned that the removal of an edge from G cannot decrease $\gamma_R(G)$, however, it can increase it by at most one as shown in [53].

Proposition 2.1.5 ([53]) *If e is an edge of a graph G, then $\gamma_R(G) \leq \gamma_R(G - e) \leq \gamma_R(G) + 1$.*

In [53], it was indicated that if G is a graph with $\Delta(G) \leq 1$, then there does not exist any edge such that $\gamma_R(G - e) > \gamma_R(G)$, and the following theorem was also presented.

Theorem 2.1.17 ([53]) *Let G be a graph with $\Delta(G) \geq 2$. Then $\gamma_R(G - e) = \gamma_R(G) + 1$ for each edge $e \in E(G)$ if and only if G is a forest in which each component is an isolated vertex or a star of order at least 3.*

Corollary 2.1.1 ([53]) *$\mathcal{R}_{CVR} \cap \mathcal{R}_{CER} \neq \emptyset$, where $\mathcal{R}_{CVR}$ and $\mathcal{R}_{CER}$ are the classes of graphs G such that $\gamma_R(G - v) \neq \gamma_R(G)$ and $\gamma_R(G - e) \neq \gamma_R(G)$, respectively, for all $v \in V(G)$ and $e \in E(G)$, respectively.*

Edge addition

We begin by recalling the following results of Hansberg, Jafari Rad and Volkmann:

Theorem 2.1.18 *Let G be a graph and x and y be non-adjacent vertices of G. Then $\gamma_R(G)-1 \leq \gamma_R(G+xy) \leq \gamma_R(G)$. Moreover, $\gamma_R(G+xy) = \gamma_R(G)-1$ if and only if there is a γ_R-function f on G such that $\{f(x), f(y)\} = \{1,2\}$.*

Now, we say that G is *Roman domination edge critical*, or just γ_R-*edge critical*, if for any $e \in E(\overline{G})$; $\gamma_R(G+e) < \gamma_R(G)$.

Proposition 2.1.6 ([43]) *Let G be a γ_R-edge critical graph, and let $e \in E(\overline{G})$. Then $\gamma_R(G+e) = \gamma_R(G)-1$.*

The authors in [43] provided a characterization of γ_R-edge critical trees. Let P_6 be the path v_1 - v_2 -v_3 - v_4 - $v_5 - v_6$. We add two new vertices x and y, and join x to v_3, and join y to v_4, to obtain a tree T. Let H_1 be a tree obtained from T by adding a vertex x_1 and joining x_1 to x. Also let H_2 be a tree obtained from H_1 by adding a new vertex y_1 and joining y_1 to y.

Theorem 2.1.19 ([43]) *A tree T is γ_R-edge critical if and only if $T \in \{H_1, H_2\}$.*

Indeed, the authors in [42] provided a characterization of the connected γ_R-edge critical unicyclic graphs.

Theorem 2.1.20 ([42]) *A connected unicyclic graph G is γ_R-edge critical if and only if $G \in \{C_4, C_5, H_1, H_{2,}, H_3, H_4, H_5, H_6\}$ (see Figure 2.10).*

Definition 2.1.1 *A matching in a graph G is a subset of pair-wise non-incident edges. A matching M is said to be perfect if $|M| = \frac{|V(G)|}{2}$.*

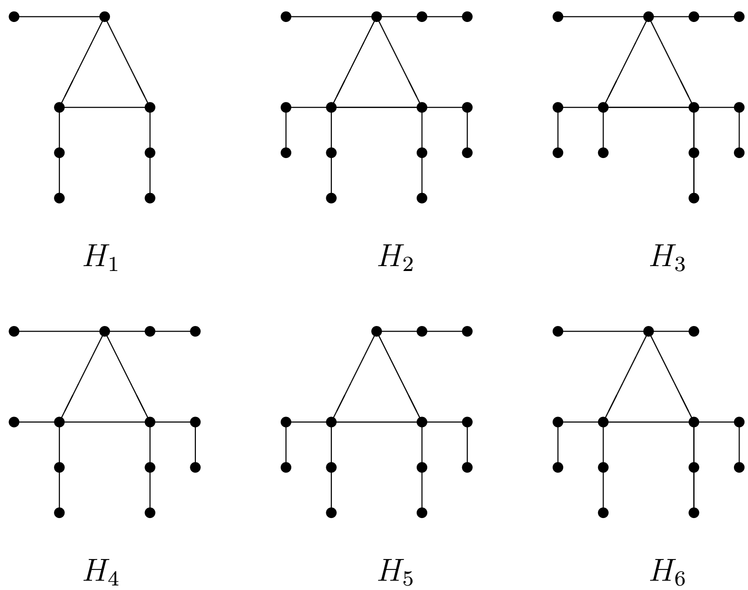

Figure 2.10: Connected γ_R-edge critical unicyclic graphs.

Let H be the graph constructed from K_{2r} as follows: Label the vertices of K_{2r} as $u_1, u_2, ..., u_r$, $w_1, w_2, ..., w_r$, and remove from K_{2r} the perfect matching u_iw_i where $1 \leq i \leq r$. Let $\mathcal{F}$ be the class of all graphs G constructed as follows: Start with a complete graph K_m (where $m \geq 2$) and join each vertex of $K_m$to every vertex of H. Then, add a path $P_2$by connecting one of its end vertices to every vertex in H. Figure 2.11 shows the smallest example of a graph belonging to $\mathcal{F}$.

Chellali et al. [28] provided a characterization of γ_R-edge critical graphs G where $\gamma_R(G) = 4$ and $diam(G) = 3$.

Theorem 2.1.21 ([28]) *If G is a connected 4-γ_R-edge critical graph, then $diam(G) \leq 3$, with equality if and only if $G \in \mathcal{F}$.*

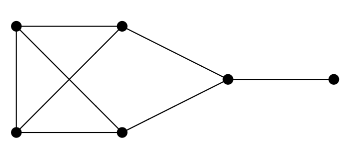

Figure 2.11: A graph $G \in \mathcal{F}$.

Remark 2.1.2 *The authors in [28] did not mention the existence of a γ_R-edge critical connected graph G with $\gamma_R(G) = 4$ and $diam(G) = 2$. Therefore, it is necessary to provide an example of a graph G, and let it be $K_{3,3}$.*

The concepts of criticality and the Nordhaus-Gaddum inequality, among others, are also prominent in various parameters of Roman domination. However, we will refrain from discussing them further here, as they have already been addressed in the context of Roman domination, which we believe provides sufficient understanding.

The value of each of the following Roman domination parameters is defined as the minimum weight of a function of the given type, where the weight $w(f)$ of such a function f is the sum of all assigned values, $w(f) = \sum_{u \in V} f(u)$.

2.2 Total Roman domination

A *total Roman dominating function* of a graph G with no isolated vertex (TRDF), is a Roman dominating function f on G with the additional property that every vertex $x \in V$ for which $f(x) \geq 1$ is adjacent to at least one vertex $y \in V$ such that $f(y) \geq 1$. The *total Roman domination number* is $\gamma_{tR}(G)$, let $V_i = \{v \in V : f(v) = i\}$ where $0 \leq i \leq 2$, and $V_f^+ = V_1 \cup V_2$. Thus, we write $f = (V_0, V_1, V_2)$. A TRDF of G with weight $\gamma_{tR}(G)$ is called a $\gamma_{tR}(G)$-function. As a new variant of the Roman domination, the concept of the total Roman domination was introduced by Liu and Chang [57].

Ahangar et al. [5] showed that for any graph G without isolated vertices,

$$2\gamma(G) \leq \gamma_{tR}(G) \leq 3\gamma(G), \tag{2.1}$$

and had established an upper bound on the total Roman domination number in terms of the Roman domination number.

Theorem 2.2.1 ([5]) *If G is a graph of order n with no isolated vertex, then $\gamma_{tR}(G) \leq 2\gamma_R(G) - 1$. Further, $\gamma_{tR}(G) = 2\gamma_R(G) - 1$ if and only if $\Delta(G) = n - 1$.*

Also, they raised the following problems.

Problem 2.2.1 ([5]) *Characterize the graphs G satisfying $\gamma_{tR}(G) = 2\gamma(G)$.*

Problem 2.2.2 ([5]) *Characterize the graphs G satisfying $\gamma_{tR}(G) = 3\gamma(G)$.*

Jafar Amjadi et al. [6] provided a constructive characterization of trees T whith $\gamma_{tR}(T) = 2\gamma(T)$ and $\gamma_{tR}(T) = 3\gamma(T)$, resolving the problems mentioned earlier for trees. However, the problems remain unsolved in general.

Cabrera Martinez et al. [21] improved the lower and upper bounds given in inequality chain 2.1. For this purpose, they used the following variant of the concept of domination. A semitotal dominating set of a graph G without isolated vertices, is a dominating set D of G such that every vertex in D is within distance two of another vertex of D. The *semitotal domination number*, denoted by $\gamma_{t2}(G)$, is the minimum cardinality among all semitotal dominating sets of G (see [41]).

Theorem 2.2.2 ([21]) *For any graph G with neither isolated vertex nor components isomorphic to K_2, $\gamma_{t2}(G) + \gamma(G) \leq \gamma_{tR}(G) \leq \gamma_R(G) + \gamma(G)$.*

They then presented the following conjecture.

Conjecture 2.2.1 ([21]) *Let G be a graph with no isolated vertex. Then $\gamma_{tR}(G) = 3\gamma(G)$ if and only if $\gamma_{tR}(G) = \gamma_R(G) + \gamma(G)$.*

Note that according to the bound $\gamma_{tR}(G) \leq \gamma_R(G) + \gamma(G)$ and Proposition 2.1.2, we conclude that the conjecture only requires proving the sufficiency part.

Ahangar [2] proved Conjecture 2.2.1 for nontrivial trees. However, the Conjecture remains unsolved in general.

2.3 Outer-independent Roman Domination

Ahangar et al. [3] combined Roman domination with vertex independence and introduced the outer independent Roman domination. The Roman dominating function f is an *outer-independent Roman dominating function* (OIRDF) on G if the set of vertices labeled with zero under f is an independent set. The *outer-independent Roman domination number* is $\gamma_{oiR}(G)$. An OIRDF of minimum weight is called a γ_{oiR}-function.

After the paper [3] was published, the topic attracted many researchers. Poureidi et al. [70] proposed an algorithm to compute $\gamma_{oiR}(G)$ in $O(|V|)$ time. Martínez et al. [22] obtained some bounds on $\gamma_{oiR}(G)$ in terms of other parameters. Nazari-Moghaddam et al. [62] provided a constructive characterization of trees T with $\gamma_{oiR}(T) = \gamma_R(T)$. Gao et al. [37] determined the exact values of $\gamma_{oiR}(C_3 \Box C_n)$ and $\gamma_{oiR}(C_m \Box C_n)$ for $m \equiv 0 \pmod 4$ and $n \equiv 0 \pmod 4$.

Ahangar et al. [3], established the following results.

Proposition 2.3.1 ([3]) *For $n \geq 2$, $\gamma_{oiR}(P_n) = \gamma_{oiR}(C_n) = 3\left\lfloor \frac{n}{4} \right\rfloor + i$, where $n \equiv i \pmod 4$ and $i \in \{0,1,2\}$, and $\gamma_{oiR}(P_n) = \gamma_{oiR}(C_n) = 3\left\lfloor \frac{n}{4} \right\rfloor + 2$ otherwise.*

Proposition 2.3.2 ([3]) *Let G be a connected graph of order n. Then $\gamma_{oiR}(G) = n$ if and only if $G = K_n$.*

Theorem 2.3.1 ([3]) *Let G be a connected graph of order $n \geq 2$. Then the following conditions are equivalent:*

(i) $\gamma_{oiR}(G) = n - 1$.

(ii) *G is a $(K_{1,3}, 2K_{1,2})$-free graph different from K_n.*

(iii) *G has a $\gamma_{oiR}(G)$-function $f = (V_0, V_1, V_2)$ such that $|V_2| = 1$ and $|V_0| = 1$.*

A *vertex cover* of a graph G is a set of vertices that covers all the edges of G. The minimum cardinality of a vertex cover is denoted by $\beta(G)$.

Proposition 2.3.3 ([3]) *If G is a graph without isolated vertices, then $\beta(G) + 1 \leq \gamma_{oiR}(G) \leq 2\beta(G)$. Both bounds are tight for trees.*

Martínez et al. [22] characterized the trees that achieve the lower bound. For this purpose, they constructed the following family: Let $\mathcal{T}$ be the family of trees $T_{r,s}$ of order $r + s + 1$ with $r \geq 1$ and $r - 1 \geq s \geq 0$, obtained from a star $K_{1,r}$ by subdividing s edges exactly once.

Theorem 2.3.2 ([22]) *Let T be a nontrivial tree. Then $\gamma_{oiR}(T) = \beta(T) + 1$ if and only if $T \in \mathcal{T}$*

.

Theorem 2.3.3 ([37]) *For* $m \equiv 0 \pmod 4$ *and* $n \equiv 0 \pmod 4$, $\gamma_{oiR}(C_m \Box C_n) = \frac{5mn}{8}$.

Theorem 2.3.4 ([37]) *For any integer* $n \geq 4$, $\gamma_{oiR}(C_3 \Box C_n) = \lceil \frac{7n}{3} \rceil$.

A graph G is a vertex cover Roman graph if $\gamma_{oiR}(G) = 2\beta(G)$. Martínez et al. [23] provide a constructive characterization of vertex cover Roman trees.

Now, recall that, a set $S \subseteq V(G)$ is an independent dominating set of G if S is an independent and dominating set at the same time. The *independent domination number* of G is the minimum cardinality among all independent dominating sets of G and is denoted by $i(G)$ (see [68, 17, 40]).

Theorem 2.3.5 ([22]) *For any graph* G *with no isolated vertex, order* n, $\gamma_{oiR}(G) \leq n - i(G) + \gamma(G)$.

The authors in [22] noted that the upper bound is achieved in the case of complete graphs. Motivated by this observation, they raised the following question:

Question 2.3.1 ([22]) *Is it the case that* $\gamma_{oiR}(G) = n - i(G) + \gamma(G)$ *if and only if* G *is a complete graph?*

2.4 Double Roman domination

A *double Roman dominating function* (DRDF) on a graph G is a function $f : V \longrightarrow \{0, 1, 2, 3\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 3$ or two vertices v_1 and v_2 for which $f(v_1) = f(v_2) = 2$, and every vertex u for which $f(u) = 1$ is adjacent to at least one vertex v for which $f(v) \geq 2$. The *double Roman domination number* is $\gamma_{dR}(G)$ (see Figure *2.12*). A DRDF of minimum weight is called a γ_{dR}-function. Any DRDF f on a graph G induces four sets V_0, V_1, V_2, V_3 where $V_i = \{v \in V : f(v) = i\}$. Thus, we write $f = (V_0, V_1, V_2, V_3)$. A vertex $u \in V_0$ is said to be double Roman dominated if $|N_G(u) \cap V_2| \geq 2$ or $|N_G(u) \cap V_3| \geq 1$. This definition was first introduced in 2016 by Beeler et al. [13], for references on double Roman domination, see for example, [4, 8, 9, 52].

It is known that the decision version of the double Roman domination problem (MIN-DOUBLE-RDF) is $\mathcal{NP}$-complete, even when restricted to some classes of graphs, for example see [4, 12, 71].

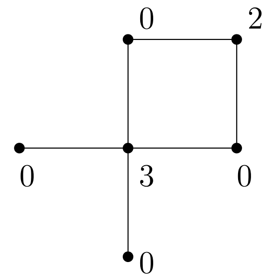

Figure 2.12: A graph G with $\gamma_{dR}(G) = 5$.

Remark 2.4.1 *In Figure* 2.12, *the numbers indicate the values of the corresponding vertices assigned by a* γ_{dR}*-function.*

In [13], Beeler et al. obtained the following results.

Proposition 2.4.1 ([13]) *In a DRDF of weight* $\gamma_{dR}(G)$*, no vertex needs to be assigned the value* 1.

By Proposition 2.4.1, we now consider the DRDF of a graph G in which there exists no vertex assigned with 1 in the following.

For a DRDF f of a graph G, let (V_0, V_2, V_3) be the ordered partition of $V(G)$ induced by f such that $V_i = \{x : f(x) = i\}$ for $i = 0, 2, 3$. It can be seen that there exists a $1 - 1$ correspondence between the function f and the partition (V_0, V_2, V_3) of $V(G)$, we write $f = (V_0, V_2, V_3)$.

Also, they presented the following result.

Proposition 2.4.2 ([13]) *For any graph* G, $2\gamma(G) \leq \gamma_{dR}(G) \leq 3\gamma(G)$.

Remark 2.4.2 ([13]) *For any graph* G,

$$\begin{array}{c} \gamma_R(G) < \gamma_{dR}(G) \leq 2\gamma_R(G) \\ \gamma(G) \leq \gamma_R(G) \leq 2\gamma(G) \leq \gamma_{dR}(G) \leq 3\gamma(G) \end{array}.$$

The characterization of the *double Roman trees* T; that is, $\gamma_{dR}(T) = 3\gamma(T)$, was given by Henning et al. in [48].

Proposition 2.4.3 **([4])** *For any integer $n \geq 1$,*

$$\gamma_{dR}(P_n) = \begin{cases} n & \text{if } n \equiv 0 \pmod 3 \\ n+1 & \text{if } n \equiv 1, 2 \pmod 3. \end{cases}$$

Proposition 2.4.4 **([4])** *For any integer $n \geq 3$,*

$$\gamma_{dR}(C_n) = \begin{cases} n & \text{if } n \equiv 0, 2, 3, 4 \pmod 6 \\ n+1 & \text{if } n \equiv 1, 5 \pmod 6. \end{cases}$$

Proposition 2.4.5 **([4])** *Let G be a connected graph of order $n \geq 3$. Then*

1. *$\gamma_{dR}(G) = 3$ if and only if $\Delta(G) = n - 1$.*

2. *$\gamma_{dR}(G) = 4$ if and only if $G = \overline{K_2} \vee H$, where H is a graph with $\Delta(H) \leq |V(H)| - 2$.*

3. *$\gamma_{dR}(G) = 5$ if and only if $\Delta(G) = n - 2$ and $G \neq \overline{K_2} \vee H$ for any graph H of order $n - 2$.*

Remark 2.4.3 *There are no graphs G with a double Roman domination number $\gamma_{dR}(G) = 1$. Additionally, for any graph G, $\gamma_{dR}(G) = 2$ if and only if G is K_1.*

Anu and Lakshmanan [9] proved the following existence result.

Theorem 2.4.1 **([9])** *Given any two positive integers $a, b \geq 3$, there exist a graph G and an induced subgraph H of G such that $\gamma_{dR}(G) = a$ and $\gamma_{dR}(H) = b$.*

Remark 2.4.4 *As seen in the previous theorem, no general relationship exists between the double Roman domination number of a graph and that of its induced subgraphs; in other words, they are incomparable.*

Now, we are focusing on bounding the double Roman domination number in terms of the order of the graph. Khoeilar et al. [54] established the following result.

Theorem 2.4.2 ([54]) *Let G be a graph of order $n \geq 5$, $\delta(G) \geq 2$ and with no component isomorphic to C_5 or C_7. Then $\gamma_{dR}(G) \leq \frac{11n}{10}$.*

Moreover, the authors in [54] presented an infinite family $\mathcal{G}$ of graphs that demonstrates the sharpness of the upper bound in their theorem. Let H be a graph obtained from two cycles of C_5 by adding an edge between them. For any graph G, let G_H be the graph obtained from G by adding $|V(G)|$ copies $H_1, ..., H_{|V(G)|}$ of H, where x_i denotes a vertex of degree three in H_i, by identifying x_i with the ith vertex of G. Let $\mathcal{G} = \{G_H : G \text{ is a graph}\}$. They conjectured that $\mathcal{G}$ is the only family of extremal graphs achieving the bound $\frac{11}{10}n$.

Shao et al. [75] disproved this conjecture by characterizing all extremal graphs for this bound. Let H' be a graph obtained from two cycles of C_5 by adding two edges joining a vertex of one cycle to two non-adjacent vertices of the other cycle, and H'' be the graph illustrated in Figure 2.13. Moreover, for any graph G, let $G_{H,H'}$ be the graph obtained from G by identifying the ith vertex of G with either a vertex of degree three of a copy of H or a vertex of degree four of a copy of H'. Let $\mathcal{A} = \{G_{H,H'} : G \text{ is a connected graph}\}$.

Theorem 2.4.3 ([75]) *Let G be a connected graph of order $n \geq 5$ with minimum degree two different from C_5 and C_7. Then $\gamma_{dR}(G) = \frac{11}{10}n$ if and only if $G \in \mathcal{A} \cup \{H''\}$.*

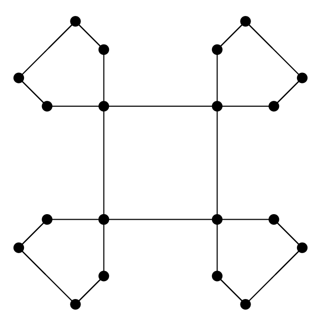

Figure 2.13: The graph H''.

Beeler et al. [13] noted that for every connected graph G with a minimum degree of at least three, the inequality $\gamma_{dR}(G) \leq \frac{9n}{8}$ holds, and they posed the question of whether this bound could be improved. In response to this question, Ahangar et al. [4] presented the following result.

Proposition 2.4.6 ([4]) *If G is a graph of order n and minimum degree $\delta(G) \geq 3$, then $\gamma_{dR}(G) \leq n$. This bound is sharp for the complement of the cycle C_6.*

It is not known whether this bound can be improved.

Problem 2.4.1 ([69]) *Let G be a graph with minimum degree at least three, different from $\overline{C_6}$. Is $\gamma_{dR}(G) \leq n$ the best possible?*

The examples mentioned above are far from encompassing all the variants. Several new variations of Roman domination have been introduced, reflecting the flexibility of the field and the potential to explore different mathematical contexts based on practical or theoretical applications.

Chapter 3

Further results on the double Roman domination

In this chapter we provide a characterization of extremal graphs of a Nordhaus-Gaddum type bound for $\gamma_{dR}(G)$ improving the corresponding results given in [52] and [79]. Moreover, we give a characterization of graphs G for which the equality $\gamma_{dR}(G) = 2\gamma_R(G) - 1$ holds, improving the corresponding results given in [84].

3.1 Graphs G of order n with $2(n-\Delta)-1 \leq \gamma_{dR}(G) \leq 2(n-\Delta)+1$

In this section we provide a characterization of some classes of graphs G with $\gamma_{dR}(G) \geq 2(n-\Delta) - 1$, including regular graphs, semiregular graphs and graphs with $\Delta - \delta = 2$.

Using Propositions 2.4.3 and 2.4.4, we have the following straightforward observation for nontrivial graphs with $\Delta \leq 2$.

Observation 3.1.1 *Let G be a graph of order n and maximum degree $\Delta \leq 2$. Then*

1. $\gamma_{dR}(G) = 2(n-\Delta)+1$ *if and only if*

   $G = pK_1 \cup H$*, where* $H \in \{K_2, P_3, C_3, P_4\}$ *and* $n = p + |V(H)|$.

2. $\gamma_{dR}(G) = 2(n-\Delta)$ *if and only if* $G = \overline{K_n}$ *or* $G = pK_1 \cup H$, *where* $H \in \{2K_2, K_2 \cup P_3, K_2 \cup C_3, K_2 \cup P_4, C_4, C_5, P_5\}$ *and* $n = p + |V(H)|$.

3. $\gamma_{dR}(G) = 2(n-\Delta) - 1$ *if and only if* $G = pK_1 \cup K_2 \cup H$, *where* $H \in \{C_4, C_5, P_5\}$ *or* $G = pK_1 \cup 2K_2 \cup H$, *where* $H \in \{K_2, P_3, C_3, P_4\}$.

Jafari Rad and Rahbani [52] presented a family of graphs G with $\gamma_{dR}(G) = 2(n-\Delta)+1$ as follows: A vertex that belongs to a minimum dominating set of G called a *good vertex*. The set of all good vertices of G is denote by $good(G)$, and $G - good(G)$ denotes the subgraph of G induced by $V(G) - good(G)$. For a graph H, an H-partition is a partition of $V(H)$ into $p+1$ nonempty sets $A_0, A_1, ..., A_p$ for some integer $p < n$ such that the following hold:

1. If $p \geq 2$, then for $i \geq 1$ the subgraph of H induced by $V(H) - A_i$ has domination number at least two, or a $\gamma(H[V(H) - A_i])$-set is contained in A_0.

2. If $p \leq 1$, then $1 \leq \gamma(H) \leq 2$. Moreover;

   - If $\gamma(H) = 1$, then $good(H) \subseteq A_0$; and every $\gamma(H - good(H))$-set has at most one common vertex with $\bigcup_{i=1}^{p} A_i$ whenever $\gamma(H - good(H)) = 2$.

   - If $\gamma(H) = 2$, then $\bigcup_{i=1}^{p} A_i$ contains at most one vertex of a $\gamma(H)$-set, for $i = 1, 2, ..., p$; otherwise a $\gamma(H)$-set is contained in A_i for $i \in \{1, ..., p\}$ and no $\gamma(H)$-set is contained in $\bigcap_{u \in A_0} N(u)$.

Remark 3.1.1 *For any graph H, the set $A_0 = V(H)$ itself forms an H-partition. Therefore, every graph H has an H-partition.*

Definition 3.1.1 *Let $A_0, A_1, ..., A_p$ be an H-partition of a graph H. Let $\mathcal{F}$ be the family of graphs G that can be obtained from a graph H by adding $p+1$ new vertices $v_1, v_2, ..., v_p, u$, joining v_i to all of the vertices of A_i for $i = 1, 2, ..., p$, and joining u to all of the vertices of H (see Figure 3.1).*

Theorem 3.1.1 ([52]) *If G is graph of order n with maximum degree Δ, then $\gamma_{dR}(G) \leq 2(n-\Delta) + 1$, with equality if and only if $G \in \mathcal{F}$.*

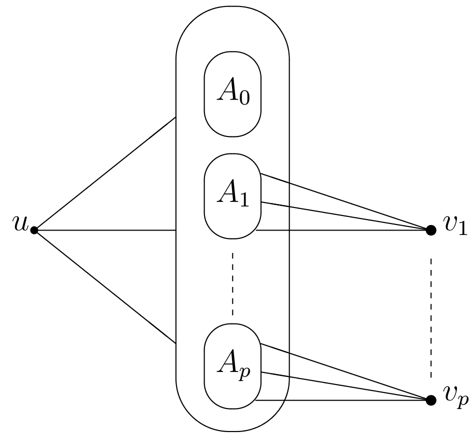

Figure 3.1: Structure of graphs in the family $\mathcal{F}$.

For any vertex $v \in V(G)$, we write $\overline{N}[v] = V(G) - N[v]$. We also denote by t the number of edges joining the vertices of $N(v)$ to the vertices of $\overline{N}[v]$.

Proposition 3.1.1 ([66]) *Let G be a graph of order n with maximum degree Δ and p a positive integer, such that $\Delta - \delta \leq 2$. Then $\gamma_{dR}(G) = 2(n-\Delta)+1$ if and only if either $\Delta = n-1$, or $\Delta = n-2$ and $G \neq \overline{K_2} \vee H$ for any graph H of order $n-2$, or $G \in pK_1 \cup H$,where $H \in \{K_2, P_3, C_3, P_4\} \cup \{cor(P_3), cor(C_3)\}$.*

Proof. Let G be a graph of order n with maximum degree Δ and minimum degree δ such that $\Delta - \delta = k \in \{0,1,2\}$ and $\gamma_{dR}(G) = 2(n-\Delta)+1$. If $\Delta \leq 2$, then from Observation 3.1.1 we obtain $G = pK_1 \cup H$ where $H \in \{K_2, P_3, C_3, P_4\}$ and $n = p + |V(H)|$. Now assume that $\Delta \geq 3$. According to the construction of Family $\mathcal{F}$ described above in Definition 3.1.1, every vertex in $\overline{N}[v]$ has at least $\Delta - k$ neighbors in $N(v)$, and every vertex in $N(v)$ has at most one neighbor in $\overline{N}[v]$, but at least one vertex which has no neighbor in $\overline{N}[v]$. So we have $(\Delta - k)\left|\overline{N}[v]\right| \leq t \leq |N(v)| - 1$, which provides $(\Delta-k)(n-\Delta-1) \leq \Delta - 1$, and thus $n \leq \Delta + 2 + \frac{k-1}{\Delta-k}$. Clearly, for $\Delta \geq 2k$, we have $\Delta \geq n-2$, and by Proposition 2.4.5, $G \neq \overline{K_2} \vee H$ for any graph H of order $n-2$. Assume now that $\Delta \leq 2k-1$. Since $\Delta \geq 3$ and $k \leq 2$, we obtain that $k = 2$ and $\Delta = 3$, and thus $n \in \{4,5,6\}$. If $n \in \{4,5\}$, then $\Delta \geq n-2$, again by Proposition 2.4.5, $G \neq \overline{K_2} \vee H$ for any graph H of order $n-2$. If $n = 6$, then $t = 2$. It is a simple matter to check that $G = cor(P_3)$ or $cor(C_3)$.

The converse is easy to show. ■

Next, we present a necessary conditions for connected graphs G of order n and maximum degree

Δ, where $2(n-\Delta)-1 \leq \gamma_{dR}(G) \leq 2(n-\Delta)$.

Lemme 3.1.1 ([66]) *Let G be a graph of order n with maximum degree Δ. If $\gamma_{dR}(G) = 2(n-\Delta) - p$, where $p \in \{0,1\}$, then for every vertex v of maximum degree we have:*

1. *Every vertex of $N(v)$ has at most two neighbors in $\overline{N}[v]$.*

2. *$\overline{N}[v] \neq \emptyset$ and every component of $G\left[\overline{N}[v]\right]$ has at most two vertices. Moreover*

   i) If $p = 0$, then $G\left[\overline{N}[v]\right]$ contains at most one edge.

   ii) If $p = 1$, then $G\left[\overline{N}[v]\right]$ contains at most two independent edges.

Proof. Let G be a graph with $\gamma_{dR}(G) = 2(n-\Delta) - p$ where $p \in \{0,1\}$. Let v be a vertex of maximum degree Δ. If some vertex $u \in N(v)$ has at least three neighbors in $\overline{N}[v]$, then $f = (N(u) \cup N(v) - \{u,v\}, V(G) - (N(u) \cup N(v)), \{u,v\})$ is a DRDF with weight at most $2(n-\Delta) - 2$, a contradiction. Hence (1) follows. If $\overline{N}[v] = \emptyset$, then $\Delta = n-1$, and so $\gamma_{dR}(G) = 3 = 2(n-\Delta)+1$, a contradiction. Thus we may assume that $\overline{N}[v] \neq \emptyset$. Suppose there is a component of $G[\overline{N}[v]]$, say F, has at least three vertices. Let $x \in V(F)$, with $|N_F(x)| \geq 2$. Clearly $f = (N(\{v,x\}), V(G) - N[\{v,x\}], \{v,x\})$ is a DRDF, with weight at most $2(n-\Delta) - 2$, a contradiction. Now suppose that $p = 0$ and $G\left[\overline{N}[v]\right]$ contains two independent edges xy and $x'y'$. Then clearly $g = (N(v) \cup \{y,y'\}, V(G) - (N[v] \cup \{x,x',y,y'\}), \{v,x,x'\})$ is a DRDF, with weight at most $2(n-\Delta) - 1$, a contradiction. Finally suppose that $p = 1$ and $G\left[\overline{N}(v)\right]$ contains at least three independent edges xy, $x'y'$ and $x''y''$. Then clearly $g = (N(v) \cup \{y,y',y''\}, V(G) - (N[v] \cup \{x,x',x'',y,y',y''\}), \{v,x,x',x''\})$ is a DRDF, with weight at most $2(n-\Delta) - 2$, a contradiction. Hence (2) follows. ■

Proposition 3.1.2 ([66]) *Let G be a graph of order n with maximum degree Δ such that $\Delta - \delta \leq 1$. Then $\gamma_{dR}(G) = 2(n-\Delta)$ if and only if either $G \in \{\overline{K_n}, C_4, C_5, (n-4)K_1 \cup 2K_2, K_2 \cup P_3, K_2 \cup C_3, K_2 \cup P_4, P_5\}$, or $\Delta = n-3$ and $\Delta \geq 3$, or $\Delta = n-2$, $\Delta \geq 3$ and $G = \overline{K_2} \vee H$, where H is a graph with $\Delta(H) \leq |V(H)| - 2$.*

Proof. Let G be a graph of order n with maximum degree Δ such that $\Delta - \delta = k \in \{0, 1\}$, and let $v \in V(G)$ be a vertex of maximum degree. Assume that $\gamma_{dR}(G) = 2(n - \Delta)$. If $\Delta \leq 2$, then from Observation 3.1.1 we obtain $G \in \{\overline{K_n}, 2K_2, C_4, C_5\}$, or $G \in \{(n-4)K_1 \cup 2K_2; n \geq 5\}$, or $G \in \{K_2 \cup P_3, K_2 \cup C_3, K_2 \cup P_4, P_5\}$. Now assume that $\Delta \geq 3$. By Lemma 3.1.1, every vertex in $\overline{N}[v]$ has at least $\Delta - k - 1$ neighbors in $N(v)$, and every vertex in $N(v)$ has at most two neighbors in $\overline{N}[v]$, and $\left|\overline{N}[v]\right| \neq 0$. We proceed according to the value of $\left|\overline{N}[v]\right|$.

Case 1. If $\left|\overline{N}[v]\right| \geq 5$, then $2(\Delta - k - 1) + 3(\Delta - k) \leq t \leq 2\Delta$, which provides $\Delta \leq \left\lfloor \frac{5k+2}{3} \right\rfloor \leq 2$, a contradiction.

Case 2. $\left|\overline{N}[v]\right| = 4$. Then $\Delta = n-5$, and thus $2(\Delta - k - 1) + 2(\Delta - k) \leq t \leq 2\Delta$, which provides $\Delta \leq 2k + 1$, and thus $k = 1$, $\Delta = 3$ and $n = 8$. By Theorem 2.4.2, $\gamma_{dR}(G) \leq \frac{11n}{10} < 2(n - \Delta)$, a contradiction.

Case 3. $\left|\overline{N}[v]\right| = 3$. Then $\Delta = n-4$, and thus $2(\Delta - k - 1) + (\Delta - k) \leq t \leq 2\Delta$, which provides $\Delta \leq 3k + 2$. So $k = 1$ and $\Delta \in \{3, 4, 5\}$. Set $\overline{N}[v] = \{x, y, z\}$, we have three possibilities.

Subcase 3.1. $\Delta = 5$. Then $n = 9$, which gives $t = 10$. Thus $\overline{N}[v]$ has exactly one edge and every vertex in $\overline{N}[v]$ has degree 4. Let $N(v) = \{a, b, c, d, e\}$. Without loss of generality, we assume that $xy \in E(G)$. Since $t = 10$, $|N(x) \cap N(v)| = |N(y) \cap N(v)| = 3$, and $|N(z) \cap N(v)| = 4$. Let $N(z) = \{a, b, c, d\}$. Clearly, x and y have no common neighbor in $\{a, b, c, d\}$, and so x and y have e as a unique common neighbor in $N(v)$. The function $f = (\{x, y, a, b, c, d, v\}, \emptyset, \{z, e\})$ is an DRDF on G of weight 6, which contradicts the fact that $\gamma_{dR}(G) = 2(n - \Delta)$.

Subcase 3.2. $\Delta = 4$. Then $n = 8$, which gives $t \in \{7, 8\}$. Clearly, $\overline{N}[v]$ is not independent. Without loss of generality, assume that $xy \in E(G)$. Let $N(v) = \{a, b, c, d\}$. Since $|N(z) \cap N(v)| \geq 3$, we may assume that $\{a, b, c\} \subseteq N(z)$. Clearly, xd or $yd \in E(G)$, say $xd \in E(G)$. The function $f = (\{a, b, c, d, y\}, \{v, z\}, \{x\})$ is an DRDF on G of weight 7, which contradicts the fact that $\gamma_{dR}(G) = 2(n - \Delta)$.

Subcase 3.3. $\Delta = 3$. Then $n = 7$. Note that $\delta \geq 2$. Again by Theorem 2.4.2, $\gamma_{dR}(G) \leq \frac{11n}{10} < 2(n - \Delta)$ a contradiction.

Case 4. $\left|\overline{N}[v]\right| = 2$. Then $\Delta = n - 3$ holds.

Case 5. $\left|\overline{N}\left[v\right]\right| = 1$. Then $\Delta = n-2$, and thus by Proposition 2.4.5, $\gamma_{dR}(G) = 2\left(n-\Delta\right)$ leads $G = \overline{K_2} \vee H$, where H is a graph with $\Delta\left(H\right) \leq \left|V\left(H\right)\right| - 2$.

The converse is easy to show. ■

Proposition 3.1.3 ([66]) *Let G be a Δ-regular graph of order $n \geq 2$. Then $\gamma_{dR}(G) = 2\left(n-\Delta\right) - 1$ if and only if $G = 3K_2$.*

Proof. Let G be a Δ-regular graph of order $n \geq 2$. Assume that $\gamma_{dR}(G) = 2\left(n-\Delta\right) - 1$. If $\Delta \geq 3$, then by Lemma 3.1.1, every vertex in $\overline{N}[v]$ has at least $\Delta - 1$ neighbors in $N(v)$, and every vertex in $N(v)$ has at most two neighbors in $\overline{N}[v]$. If $\left|\overline{N}\left[v\right]\right| \geq 3$, then $2\left(\Delta-1\right) + \Delta \leq t \leq 2\Delta$, which provides $\Delta \leq 2$, a contradiction. Therefore $\left|\overline{N}[v]\right| \leq 2$, and so $\Delta \geq n-3$. By Propositions 3.1.1 and 3.1.2, we have $\gamma_{dR}(G) \geq 2\left(n-\Delta\right)$, a contradiction. Now assume that $\Delta \leq 2$, then by Observation 3.1.1, we have $G = 3K_2$.

The converse is easy to show. ■

3.2 Nordhaus-Gaddum type inequality for double Roman domination

Jafari Rad and Rahbani [52], and Volkmann [79] presented Nordhaus-Gaddum type inequalities for the double Roman domination number in terms of the order of the graph G.

Theorem 3.2.1 ([52]) *For any graph G of order $n \geq 2$, $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 2n+3$, with equality if and only if $G \in \left\{K_n, \overline{K_n}\right\}$.*

In the following, let $K_n - e$ and $K_n - \{e_1, e_2\}$ represent the complete graph minus an edge and the complete graph minus two independent edges, respectively. Additionally, let $\mathcal{H}_1$ =$\{2K_2, C_4, P_4, C_5,$ $K_n - e, \overline{K_n - e}$ for $n \geq 3\}$.

Theorem 3.2.2 ([52]) *Let G be a graph of order $n \geq 3$ such that $G \notin \left\{K_n, \overline{K_n}\right\}$. Then $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 2n+2$, with equality if and only if $G \in \mathcal{H}_1$.*

Theorem 3.2.3 ([79]) *Let G be a graph of order $n \geq 4$ such that $G \notin \{K_n, \overline{K_n}\} \cup \mathcal{H}_1$. Then $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 2n + 1$, with equality if and only if $G \in \left\{K_n - \{e_1, e_2\}, \overline{K_n - \{e_1, e_2\}}\right\}$ and $n \geq 5$ or $G \in \{P_5, 3K_2, \overline{P_5}, \overline{3K_2}\}$.*

According to Theorems 3.2.1, 3.2.2 and 3.2.3, if G is a graph such that $G \notin \mathcal{H} = \{K_n, \overline{K_n}\} \cup \mathcal{H}_1 \cup \mathcal{H}_2$, then $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 2n$, where $\mathcal{H}_2 = \left\{K_n - \{e_1, e_2\}, \overline{K_n - \{e_1, e_2\}}, P_5, 3K_2, \overline{P_5}, \overline{3K_2}; n \geq 5\right\}$. In the sequel, we provide a characterization of graphs G of order $n \geq 4$ for which $\gamma_R(G) + \gamma_R(\overline{G}) = 2n$. For this purpose, We introduce the following families of graphs:

- $\mathcal{F}_0 = \{4K_2, 2C_3, C_6, C_7\}$.
- $\mathcal{F}_1 = \{(n-6)\, K_1 \cup 3K_2; n \geq 7, K_2 \cup P_3, K_2 \cup C_3, K_2 \cup P_4\} \cup$
  $\{F : F$ is semiregular with $n(F) = 6$ and $\Delta(F) = 3\}$.
- $\mathcal{F}_2 = \{(n-3)\, K_1 \cup P_3, (n-3)\, K_1 \cup C_3, (n-4)\, K_1 \cup P_4; n \geq 4\} \cup$
  $\{cor(P_3), cor(C_3), F_1, F_2, F_3\}$, where F_1, F_2 and F_3 are the graphs illustrated in Figure 3.2.

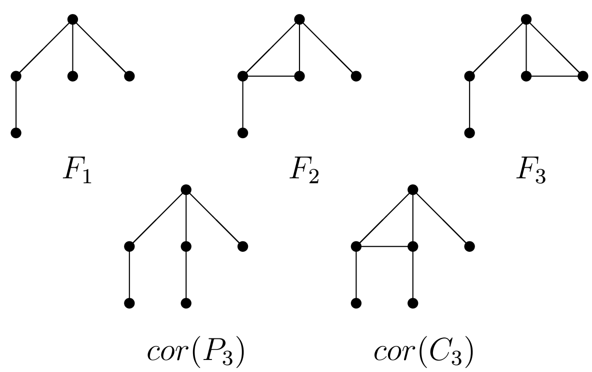

Figure 3.2: Graphs G in $\mathcal{F}_2$ with $\Delta(G) = 3$.

Theorem 3.2.4 ([66]) *Let G be a graph of order $n \geq 4$ such that $G \notin \mathcal{H}$. Then $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 2n$, with equality if and only if G or $\overline{G} \in \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2$.*

Proof. Clearly, the upper bound follows from Theorems 3.2.1, 3.2.2 and 3.2.3, since $G \notin \mathcal{H}$.

Assume now that $\gamma_{dR}(G)+\gamma_{dR}(\overline{G})=2n$. By Theorem 3.1.1, we have

$$\begin{aligned}
2n &= \gamma_{dR}(G)+\gamma_{dR}(\overline{G}) \\
&\leq 2\left(n-\Delta\left(G\right)\right)+1+2\left(n-\Delta\left(\overline{G}\right)\right)+1 \\
&\leq 2\left(n-\Delta\left(G\right)\right)+1+2\left(n-\left(n-1-\delta(G)\right)\right)+1 \\
&\leq 2n-2\left(\Delta(G)-\delta(G)\right)+4.
\end{aligned}$$

Hence $\Delta(G)-\delta(G)\leq 2$. Therefore G is either regular or semiregular or $\Delta(G)-\delta(G)=2$. We distinguish three cases.

Case 1. G is regular. Then without loss of generality we consider three possibilities:

Subcase 1.1. $\gamma_{dR}(G)=2\left(n-\Delta\left(G\right)\right)+1$ and $\gamma_{dR}(\overline{G})=2\left(n-\Delta\left(\overline{G}\right)\right)-3$. By Proposition 3.1.1, we have $G=K_n$, excluded, since $K_n\in\mathcal{H}$.

Subcase 1.2. $\gamma_{dR}(G)=2\left(n-\Delta\left(G\right)\right)$ and $\gamma_{dR}(\overline{G})=2\left(n-\Delta\left(\overline{G}\right)\right)-2$. By Proposition 3.1.2, and since $G\notin\left\{\overline{K_n},C_4,2K_2,C_5\right\}\subset\mathcal{H}$, we have $\Delta(G)=n-3$ or $n-2$ with $\Delta(G)\geq 3$. Clearly, if $\Delta(G)=n-3$, then $\overline{G}$ is the disjoint union of p copies of cycles of order n_i, where $p\geq 1$ and $n=\sum_{i=1}^{p}n_i$. Using the fact that $\gamma_{dR}(C_{n_i})\leq n_i+1$ (see Proposition 2.4.4), we have $2n-6=\gamma_{dR}(\overline{G})=\sum_{i=1}^{p}\gamma_{dR}(C_{n_i})\leq n+p$, which gives $n\leq p+6$. On the other hand, since $n_i\geq 3$, for $i\in\{1,...,p\}$, we have $n\geq 3p$, so, $p\leq 3$. Now, it is easy to check that if $p=1$, then $\overline{G}\in\{C_6,C_7\}$, and if $p=2$, then $\overline{G}\in\{2C_3,C_3\cup C_4\}$, finally if $p=3$ then $\overline{G}=3C_3$. So far, we obtained $\overline{G}\in\{C_6,C_7,2C_3,C_3\cup C_4,3C_3\}$. However, since $\gamma_{dR}(C_3\cup C_4)=7$ and $\gamma_{dR}(3C_3)=9$, while $2\left(n\left(C_3\cup C_4\right)-\Delta\left(C_3\cup C_4\right)\right)-2=8$ and $2\left(n\left(3C_3\right)-\Delta\left(3C_3\right)\right)-2=12$, the graphs $C_3\cup C_4$ and $3C_3$ must be excluded, as it does not satisfy the equality $\gamma_{dR}(\overline{G})=2\left(n-\Delta\left(\overline{G}\right)\right)-2$. Therefore, we conclude that $\overline{G}\in\{2C_3,C_6,C_7\}\subset\mathcal{F}_0$. Now assume that $\Delta(G)=n-2$. Then each component of $\overline{G}$ is a K_2. For such graphs we have $\gamma_{dR}(\overline{G})=\frac{3n}{2}$ and $\Delta\left(\overline{G}\right)=1$, where n is the order of $\overline{G}$. By applying the equality $\gamma_{dR}(\overline{G})=2\left(n-\Delta\left(\overline{G}\right)\right)-2$, we obtain $n=8$, which uniquely yields $\overline{G}=4K_2$. Hence, $\overline{G}\in\mathcal{F}_0$.

Subcase 1.3. $\gamma_{dR}(G)=2\left(n-\Delta\left(G\right)\right)-1$ and $\gamma_{dR}(\overline{G})=2\left(n-\Delta\left(\overline{G}\right)\right)-1$. By Proposition 3.1.3, we have $G=3K_2$, excluded, since $3K_2\in\mathcal{H}$.

Case 2. G is semi-regular. Then, without loss of generality, we have two possibilities:

Subcase 2.1. $\gamma_{dR}(G) = 2\left(n - \Delta\left(G\right)\right) + 1$ and $\gamma_{dR}(\overline{G}) = 2\left(n - \Delta\left(\overline{G}\right)\right) - 1$. By Proposition 3.1.1, we have $G = (n-2)\,K_1 \cup K_2$, $\Delta(G) = n-1$, or $\Delta(G) = n-2$ and $G \neq \overline{K_2} \vee H$ for any graph H of order $n-2$. The graph $(n-2)\,K_1 \cup K_2$ is excluded, since it is in $\mathcal{H}$. If $\Delta\left(G\right) = n-1$, then $\Delta\left(\overline{G}\right) = 1$, and so $\gamma_{dR}(\overline{G}) = 2\left(n - \Delta\left(\overline{G}\right)\right) - 1$ leaves $\overline{G} = (n-6)\,K_1 \cup 3K_2$. Hence $\overline{G} \in \mathcal{F}_1$. Now assume that $\Delta\left(G\right) = n-2$. Then $\Delta\left(\overline{G}\right) = 2$. By Observation 3.1.1, we have $\overline{G} = K_2 \cup H$, where $H \in \{K_2 \cup P_3, K_2 \cup C_3, K_2 \cup P_4, C_4, C_5, P_5\}$, contradicting the fact that $G \neq \overline{K_2} \vee \overline{H}$.

Subcase 2.2. $\gamma_{dR}(G) = 2\left(n - \Delta\left(G\right)\right)$ and $\gamma_{dR}(\overline{G}) = 2\left(n - \Delta\left(\overline{G}\right)\right)$. By Proposition 3.1.2, we have $G \in \{pK_1 \cup 2K_2$ where $p \geq 1, K_2 \cup P_3, K_2 \cup C_3, K_2 \cup P_4, P_5\}$, or $\Delta(G) = n-3$ and $\Delta\left(G\right) \geq 3$, or $\Delta(G) = n-2$, $\Delta\left(G\right) \geq 3$ and $G = \overline{K_2} \vee H$, where G is a graph with $\Delta\left(G\right) \leq \left|V\left(G\right)\right| - 2$. The graphs $pK_1 \cup 2K_2$ where $p \geq 1$ and P_5 are excluded, since they are in $\mathcal{H}$. So for $\Delta(G) \leq 2$, $\gamma_{dR}(\overline{G}) = 2\left(n - \Delta\left(\overline{G}\right)\right)$ leaves $G \in \{K_2 \cup P_3, K_2 \cup C_3, K_2 \cup P_4\} \subset \mathcal{F}_1$. Now suppose that $\Delta(G) \geq 3$. If $\Delta(G) = n-2$, then $\Delta(\overline{G}) = 2$, and so $\overline{G} \in \{K_2 \cup P_3, K_2 \cup C_3, K_2 \cup P_4\} \subset \mathcal{F}_1$. Now assume that $\Delta\left(G\right) = n-3$. Then $\Delta(\overline{G}) = 3$, which means that $\Delta(\overline{G}) = n-3$, and thus $n = 6$. Therefore G and $\overline{G}$ are semi regular with maximum degree 3. Hence $\overline{G}$ and G are in $\mathcal{F}_1$.

Case 3. $\Delta(G) - \delta(G) = 2$. Then we have the only possibility: $\gamma_{dR}(G) = 2\left(n - \Delta\left(G\right)\right) + 1$ and $\gamma_{dR}(\overline{G}) = 2\left(n - \Delta\left(\overline{G}\right)\right) + 1$. By Proposition 3.1.1, we have either $M \in \{pK_1 \cup H$, where $H \in \{P_3, C_3, P_4\}$, $p \geq 1\} \cup \{cor(P_3), cor(C_3)\}$, or $\Delta(M) = n-1$, or $\Delta(M) = n-2$ and $M \neq \overline{K_2} \vee H$ for any graph H of order $n-2$, where $M \in \left\{G, \overline{G}\right\}$. Without loss of generality, if $\Delta(G) \leq 2$, then $G \in \{pK_1 \cup H$, where $H \in \left\{P_3, C_3, P_4\right\}, p \geq 1\}$. Therefore $\overline{G}$ has a vertex with degree $\Delta(\overline{G}) = n-1$. Hence $G \in \mathcal{F}_2$. Now suppose that $\Delta(G) \geq 3$. If $\Delta(G) = n-1$, then $\overline{G}$ has an isolated vertex, and so $\overline{G} \in \{pK_1 \cup H$, where $H \in \{P_3, C_3, P_4\}$ and $p \geq 1\}$. Hence $\overline{G} \in \mathcal{F}_2$. Assume that $\Delta(G) = n-2$, then $\Delta\left(\overline{G}\right) = 3$. By the construction of Family $\mathcal{F}$ described above, we get $n \in \{5, 6\}$. It is a simple matter to check that $G \in \{F_1, F_2, F_3, cor(P_3), cor(C_3)\} \subset \mathcal{F}_2$.

The converse is easy to see and we omit the details. ■

3.3 Graph with $\gamma_{dR}(G) = 2\gamma_R(G) - 1$

In this section, we give a characterization of connected graphs with $\gamma_{dR}(G) = 2\gamma_R(G) - 1$. We begin by recalling some important results that will be useful.

Theorem 3.3.1 ([31]) *For any graph G, $\gamma(G) \leq \gamma_R(G)$, with equality if and only if $G = \overline{K_n}$.*

Theorem 3.3.2 ([13]) *For any graph G, $\gamma_{dR}(G) \leq 2\gamma_R(G)$, with equality if and only if $G = \overline{K_n}$.*

From Theorem 3.3.2, if G is a nontrivial connected graph, then $\gamma_{dR}(G) \leq 2\gamma_R(G) - 1$. In what follows, we provide a characterization of graphs G satisfying the equality $\gamma_{dR}(G) = 2\gamma_R(G) - 1$, which extends the corresponding result given in [84] for trees.

Proposition 3.3.1 ([66]) *If G is a connected graph of order n with maximum degree Δ, then $\gamma_{dR}(G) = 2\gamma_R(G) - 1$ if and only if $\gamma_{dR}(G) = 2(n - \Delta) + 1$.*

Proof. Let $f = (V_0, V_1, V_2)$ be an RDF with minimum weight and $\gamma_{dR}(G) = 2w(f) - 1$. So $\gamma_{dR}(G) = 2|V_1| + 4|V_2| - 1$. It is clear that $g = (V_0, \emptyset, V_1, V_2)$ is a DRDF on G of weight $2|V_1| + 3|V_2|$. A simple calculation shows that $|V_2| \leq 1$. We have two cases:

Case 1. $V_2 = \emptyset$. Then $V_1 = V$. However, it is observed that $\gamma_R(G) = n$ if and only if $G = pK_2 \cup qK_1$ where $2p + q = n$. Since G is connected, $\gamma_{dR}(G) = 2\gamma_R(G) - 1$ leaves only $G = K_2$. Hence $\gamma_{dR}(G) = 2(n - \Delta) + 1$.

Case 2. $V_2 = \{v\}$. Since no edge of G joins V_1 and $\{v\}$, and $\{v\}$ dominates V_0, we have

$$\deg(v) = |V_0| = n - (|V_1| + |V_2|) = n - \gamma_R(G) + 1 = n - \frac{\gamma_{dR}(G) + 1}{2} + 1$$

and so $\Delta \geq \frac{2n - \gamma_{dR}(G) + 1}{2}$. Hence $\gamma_{dR}(G) \geq 2(n - \Delta) + 1$. Equality holds from the fact that $\gamma_{dR}(G) \leq 2(n - \Delta) + 1$.

Conversely, assume $\gamma_{dR}(G) = 2(n - \Delta) + 1$, and let v be a vertex of G with maximum degree Δ. We define $V_0 = N(v)$, $V_1 = V - N[v]$, and $V_2 = \{v\}$, then $f = (V_0, V_1, V_2)$ is an RDF with

$w(f) = n - \Delta + 1 = \frac{\gamma_{dR}(G)+1}{2}$. Since $\gamma_R(G) \geq \frac{\gamma_{dR}(G)+1}{2}$ for connected graphs, f is an RDF for G with $w(f) = \gamma_R(G)$. ■

The following result is an immediate consequence of Theorem 3.1.1 and Propositions 3.3.1.

Corollary 3.3.1 ([66]) *Let G be a connected graph of order n with maximum degree Δ. Then the following statements are equivalent:*

(i) $\gamma_{dR}(G) = 2\gamma_R(G) - 1.$

(ii) $\gamma_{dR}(G) = 2(n - \Delta) + 1.$

(iii) $G \in \mathcal{F}.$

We note that if $\gamma_{dR}(G) = 2\gamma(G) + 1$ and $\gamma_R(G) = \gamma(G) + 1$, then $\gamma_{dR}(G) = 2\gamma_R(G) - 1$. But the converse is not true as shown by the graph in Figure 3.3, where $\gamma(G) = 3$, $\gamma_R(G) = 5$ and $\gamma_{dR}(G) = 9$.

Remark 3.3.1 *If one of the following equalities $\gamma_{dR}(G) = 2\gamma(G) + 1$ and $\gamma_R(G) = \gamma(G) + 1$ is not hold, then clearly $\gamma_{dR}(G) \neq 2\gamma_R(G) - 1$.*

Now in the class of trees, from the construction of Family $\mathcal{F}$, described above, we observe that wounded spiders are the only trees in $\mathcal{F}$. On the other hand wounded spiders are the only trees T such that $\gamma_{dR}(T) = 2\gamma_R(T) - 1$, $\gamma_R(T) = \gamma(T) + 1$, or $\gamma_{dR}(T) = 2\gamma(T) + 1$, as shown by Zhang et al. [84], Cockayne et al. [31] and Ahangar et al. [4], respectively.

The following result is an immediate consequence of Corollary 3.3.1.

Corollary 3.3.2 ([66]) *Let T be a tree of order n with maximum degree Δ. Then the following statements are equivalent:*

(i) $\gamma_{dR}(T) = 2\gamma_R(T) - 1.$

(ii) $\gamma_{dR}(T) = 2\gamma(T) + 1.$

(iii) $\gamma_R(T) = \gamma(T) + 1$.

(iv) $\gamma_{dR}(T) = 2(n - \Delta) + 1$.

(v) T *is a wounded spider.*

3.4 Counterexamples to a published result

Mojdeh, Parsian and Masoumi [60] attempted to improve the bound $\gamma_{dR}(G) \le 2\gamma_R(G)$, where they proved that $\gamma_{dR}(G) \le \gamma_R(G) + \gamma(G)$. In the following, we will show that this result is incorrect [65].

Firstly, Recall that $B(X)$ is the set of vertices in $V - X$ that have a neighbor in the set X for every $X \subseteq V$. The *differential* of a set X is defined to be $\partial(X) = |B(X)| - |X|$, and the differential of G to be $\partial(G) = \max\{\partial(X) : X \subseteq V\}$. An *enclaveless number* (or *B-differential*) of G is $\Psi(G) = \max\{|B(X)| : X \subseteq V\}$.

It has been shown by Mojdeh, Parsian and Masoumi [60] that for every graph G of order n having no isolated vertices,

$$\gamma_{dR}(G) \le 2n - \Psi(G) - \partial(G) \tag{3.1}$$

It is worth noting that this result, whose invalidity will be shown, is presented in two separate papers by the same authors. The following Gallai theorems have been established in [15] and [56] for the differential of a graph and the enclaveless number, respectively.

Theorem 3.4.1 ([15]) *If G is a graph of order n, then $\partial(G) = n - \gamma_R(G)$.*

Theorem 3.4.2 ([56]) *For any graph G of order n, then $\Psi(G) = n - \gamma(G)$.*

Note that according to Theorems 3.4.1 and 3.4.2, the inequality 3.1 becomes $\gamma_{dR}(G) \le \gamma_R(G) + \gamma(G)$. Now, we will provide an infinite family of graphs showing that inequality 3.1 ($\gamma_{dR}(G) \le \gamma_R(G) + \gamma(G)$) is erroneous.

Let $\mathcal{G}$ be the family of trees T obtained from a double star $S(r,s)$ with $r \geq s \geq 2$, by subdividing twice the central edge and once any other edge of the double star $S(r,s)$. Figure 3.3 shows the smallest example of a tree belonging to $\mathcal{G}$. We can easily see that any tree T in $\mathcal{G}$ has order $n = 2(r+s)+4$, further $\gamma(T) = r+s+1$ and $\gamma_R(T) = r+s+4$, and thus leading to $\Psi(T) = r+s+3$ and $\partial(T) = r+s$. Now since $\gamma_{dR}(T) = 2(r+s)+6$, we consequently have $\gamma_{dR}(T) > 2n-\Psi(T)-\partial(T)$.

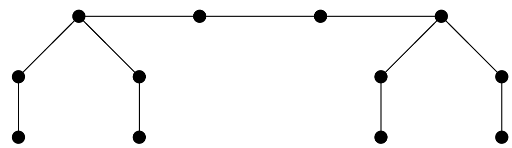

Figure 3.3: The tree T in $\mathcal{G}$.

In the following, we define another class of graphs different from trees for which 3.1 is not also valid. Let $\mathcal{H}$ be the family of graphs G obtained from a star $K_{1,p}$, with $p \geq 3$, by first subdividing once each edge of the star and then adding a new vertex attached to the center vertex and one of its neighbors. Figure 3.4 shows the smallest example of a graph belonging to $\mathcal{H}$. One can easily see that any graph G in $\mathcal{H}$ has order $n = 2p+2$, further $\gamma(G) = p$ and $\gamma_R(G) = p+2$, and thus leading to $\Psi(G) = p+2$ and $\partial(G) = p$. Now since $\gamma_{dR}(G) = 2p+3$, we consequently have $\gamma_{dR}(G) > 2n - \Psi(G) - \partial(G)$.

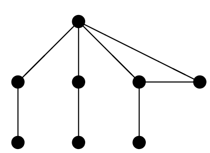

Figure 3.4: The graph G in $\mathcal{H}$.

We conclude by mentioning that inequality 3.1 is used in [59], which therefore calls into question the validity of certain results.

Chapter 4

Critical graphs for total and double Roman domination

In this chapter, we concentrate on *edge-critical graphs* with respect to graph parameter μ where $\mu \in \{\gamma_{tR}, \gamma_{dR}\}$ (that is, μ decreases when any missing edge is added).

4.1 Total Roman domination edge critical graphs

Sumner and Blitch [78] remarked that while adding an edge can decrease the domination number by at most one, that is $\gamma(G)-1 \leq \gamma(G+e) \leq \gamma(G)$ for any $e \in E(\overline{G})$, and they studied graphs for which $\gamma(G+e) = \gamma(G) - 1$ for each $e \in E(\overline{G})$, and called these graphs domination *edge critical.* A domination edge critical graph G with $\gamma(G) = k$ is called *k-γ-edge critical.*

We consider the behavior of the total Roman domination number of a graph G upon the addition of edges to G. In [55], Lampman et al. showed that for any graph G with no isolated vertices, if $e \in E(\overline{G})$, then $\gamma_{tR}(G) - 2 \leq \gamma_{tR}(G+e) \leq \gamma_{tR}(G)$. Define a graph G with no isolated vertices to be *γ_{tR}-edge-critical* if $\gamma_{tR}(G+e) < \gamma_{tR}(G)$ for every edge $e \in E(\overline{G}) \neq \emptyset$, and to be *$\gamma_{tR}$-edge-supercritical* if $\gamma_{tR}(G+e) = \gamma_{tR}(G) - 2$ for every edge $e \in E(\overline{G}) \neq \emptyset$. We say that G is *k-γ_{tR}-edge-supercritical* if $\gamma_{tR}(G) = k$ and G is γ_{tR}-edge-supercritical. Also in [55] the authors posed the following problems:

Question 4.1.1 ([55]) *Are the disjoint unions of two or more complete graphs, each having order at least* 3*, the only* γ_{tR}*-edge-supercritical graphs?*

We define vertex $u \in V$ as "*dead*" if every γ_{tR}-function f on G satisfies $f(u) = 0$.

Question 4.1.2 ([55]) *Do there exist* γ_{tR}*-edge-critical graphs containing dead vertices?*

Mynhardt et al. in [61] answered the first question by constructing the following class of graphs: Let G_r be the graph constructed from the complete graph K_{2r} as follows: Label the vertices of K_{2r} as $x_1, x_2, ..., x_r$, $y_1, y_2, ..., y_r$, and remove from K_{2r} a perfect matching x_iy_i where $1 \leq i \leq r$. Add a vertex disjoint K_3 component to K_{2r}, and label the added vertices u, v, w. Let w be adjacent to both x_i and y_i, and v be adjacent to x_i, for $1 \leq i \leq r$. Finally, add two more vertices x_0 and y_0, such that $x_0u, x_0x_i, y_0v, y_0y_i \in E(G_r)$ for $1 \leq i \leq r$. See Figure 4.1 for $r = 3$.

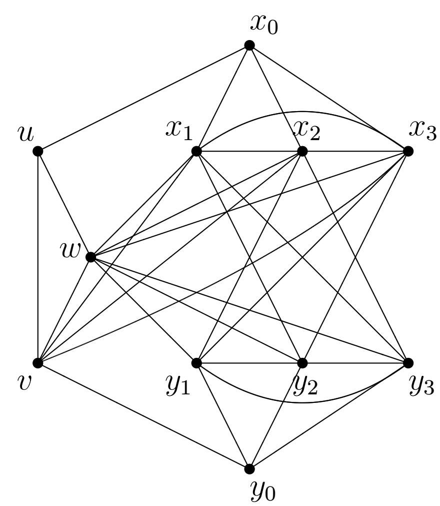

Figure 4.1: The graph G_3

They also posed the following conjectures:

Conjecture 4.1.1 ([61]) *If* G *is a* γ_{tR}*-edge-supercritical graph and* $u \in V(G)$*, then there exists a* γ_{tR}*-function* $f = (V_0, V_1, V_2)$ *such that* $u \in V_f^+$*, where* $V_f^+ = V_1 \cup V_2$*.*

Conjecture 4.1.2 ([61]) *If* G *is a* k*-*γ_{tR}*-edge-supercritical graph, then* $G \cup K_n$ *is* $(k+3)$*-*γ_{tR}*-edge-critical, for* $n \geq 3$*.*

In the next we settle the Question 4.1.2 and present proofs of Conjectures 4.1.1 and 4.1.2.

4.1.1 Answer to Question 4.1.2

Recall that a set S of vertices in a graph G is a total dominating set (TDS) of G if every vertex of G is adjacent to some vertex in S. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a TDS of G. See [30].

We make use of the following observation:

Observation 4.1.1 ([61]) *If G is a connected graph of order $n \geq 3$ such that $\Delta \leq n-2$, then $\gamma_t(G) + 2 \leq \gamma_{tR}(G) \leq 2\gamma_t(G)$.*

The next result demonstrates the existence of an infinite class of γ_{tR}-edge-critical graphs containing a *dead vertex*, which answers the second question posed by Lampman et al. in [55].

Let $\mathcal{G}$ be the class of all graphs G_p that are obtained first from the 7-cycle, $C = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ and join x_2 to x_4, x_5 to x_7, x_2 to x_7 and x_3 to x_6, and then adding a complete graph K_p for some $p \geq 1$ by joining each of its vertices to every vertex in $\{x_3, x_4, x_5, x_6\}$. The graph G_2 in $\mathcal{G}$ is shown in Figure 4.2.

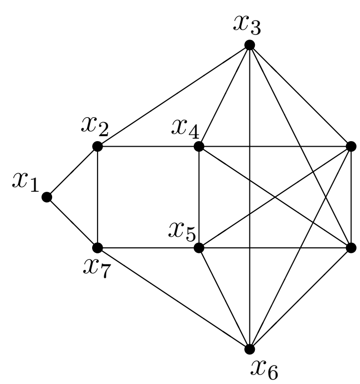

Figure 4.2: Example of a graph in $\mathcal{G}$ for $p = 2$.

Proposition 4.1.1 ([64]) *Every graph $G_p \in \mathcal{G}$ is a γ_{tR}-edge-critical graph. Moreover, x_1 is a dead vertex.*

Proof. Let $G_p \in \mathcal{G}$ of order n and maximum degree $\Delta(G_p)$. It is clear that $\Delta(G_p) \leq n-2$, and since no pair of adjacent vertices dominates G_p, $\gamma_t(G_p) \geq 3$. Thus by Observation 4.1.1, we have $\gamma_{tR}(G_p) \geq 5$. On the other hand, the function $f = (V(G_p) - \{x_2, x_3, x_6\}, \{x_3\}, \{x_2, x_6\})$ is a TRDF on G_p of weight 5, implying that $\gamma_{tR}(G_p) \leq 5$. Hence $\gamma_{tR}(G_p) = 5$. It is clear that x_1 is a dead vertex, otherwise $\gamma_{tR}(G_p) \geq 6$. Now, we shall show that G_p is a γ_{tR}-edge-critical graph. Let u an arbitrary vertex of the copy K_p in G_p. Without loss of generality, we can consider, in $E\left(\overline{G}_p\right)$, only the set of edges $E^* = \{x_1x_3, x_1x_4, x_1u, x_2x_5, x_2x_6, x_2u, x_3x_5\}$. For each e in E^*in the listed order, it is a simple matter to check that $\{x_3, x_6\}$, $\{x_4, x_5\}$, $\{x_1, u\}$, $\{x_2, x_5\}$, $\{x_2, x_6\}$, $\{x_2, u\}$, $\{x_2, x_3\}$ are total dominating sets of $G_p + e$. Thus by Observation 4.1.1, $\gamma_{tR}(G_p + e) \leq 4$ for any edge $e \in E\left(\overline{G}_p\right)$. Hence for each p, G_p is a γ_{tR}-edge-critical graph containing a dead vertex. ■

We can also construct a connected γ_{tR}-edge-critical graph H with $\gamma_{tR}(H) = 10$ containing two dead vertices, as illustrated in the Figure 4.3, but we will omit the details. So, we demonstrated the existence of connected γ_{tR}-edge-critical graphs containing dead vertices. However, in the next section, it will be shown that a γ_{tR}-edge-supercritical graph cannot have this property, that is, no connected γ_{tR}-edge-supercritical graph contains a dead vertex.

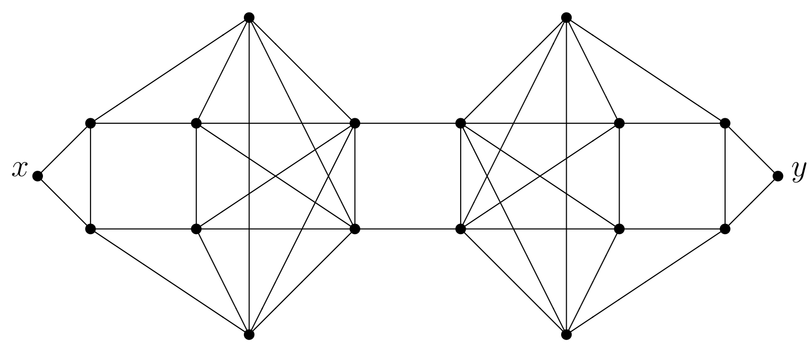

Figure 4.3: The graph H, where x and y are dead vertices.

4.1.2 Proof of conjectures

First we mention a result proved in [55].

Proposition 4.1.2 ([55]) *For a graph G with no isolated vertices, if $uv \in E\left(\overline{G}\right)$ is a critical edge, then there exists a $\gamma_{tR}(G+uv)$-function f such that $\{f(u), f(v)\} \in \{\{2,2\}, \{2,1\}, \{2,0\}, \{1,1\}\}$.*

Theorem 4.1.1 ([64]) *Let G be a γ_{tR}-edge-supercritical graph with no isolated vertices. Then, for every vertex $u \in V(G)$, there exists a $\gamma_{tR}(G)$-function f such that $u \in V_f^+$.*

Proof. Let G be a γ_{tR}-edge-supercritical graph of order n, and let $u \in V(G)$. Then, for any edge $uv \in E\left(\overline{G}\right)$, $\gamma_{tR}(G+uv) = \gamma_{tR}(G) - 2$. Suppose for a contradiction that there is a vertex $x \in V(G)$ such that $f(x) = 0$ for every $\gamma_{tR}(G)$-function f. Note that, since G is connected x is adjacent to some vertex, say w, in G. If x is adjacent to all other vertices of G, then clearly the function h, such that $h(x) = 2$, $h(w) = 1$ and $h(z) = 0$ for all other $z \in V(G)$, is a $\gamma_{tR}(G)$-function, which contradicts our assumption. Assume now that there exists a vertex y such that $xy \in E\left(\overline{G}\right)$. By Proposition 4.1.2, there exists a $\gamma_{tR}(G+xy)$-function $g = (V_0, V_1, V_2)$ such that $\{g(x), g(y)\} \in \{\{2,2\}, \{2,1\}, \{2,0\}, \{1,1\}\}$. We distinguish between three cases.

Case 1. $\{g(x), g(y)\} \in \{\{2,2\}, \{2,1\}, \{1,1\}\}$. Then we have three possibilities. If $N_G(x) \cap V_g^+ \neq \emptyset$ and $N_G(y) \cap V_g^+ \neq \emptyset$, then the function g is a TRDF on G, which contradicts the minimality of f. If, without loss of generality, $N_G(x) \cap V_g^+ = \emptyset$ and $N_G(y) \cap V_g^+ \neq \emptyset$, then x has a neighbor x' in V_0, since G is without isolated vertices. Then the function $f' : V \longrightarrow \{0,1,2\}$, such that $f'(x') = 1$ and $f'(z) = g(z)$ for all other $z \in V(G)$, is a TRDF on G, a contradiction too. Assume now that $N_G(x) \cap V_g^+ = \emptyset$ and $N_G(y) \cap V_g^+ = \emptyset$. Then x and y have neighbors in V_0. If $N_G(x) \cap N_G(y) \neq \emptyset$, say $x' \in N_G(x) \cap N_G(y)$, then the function $f' : V \longrightarrow \{0,1,2\}$ such that $f'(x') = 1$ and $f'(z) = g(z)$ for all other $z \in V(G)$, is a TRDF on G, again we have a contradiction with the minimality of f. If $N_G(x) \cap N_G(y) = \emptyset$, then the function $f' : V \longrightarrow \{0,1,2\}$, such that $f'(x') = f'(y') = 1$, where $x' \in N_G(x)$ and $y' \in N_G(y)$, and $f'(z) = g(z)$ for all other $z \in V(G)$ is a TRDF on G, with $w(f') = \gamma_{tR}(G)$ and $f'(x) > 0$, which contradicts our assumption.

Case 2. $g(x) = 2$ and $g(y) = 0$. If $N_G(y) \cap V_g^+ \neq \emptyset$, then the function $f' : V \longrightarrow \{0,1,2\}$ defined on G, as follows: $f'(y) = 1$ and $f'(z) = g(z)$ for all other $z \in V(G)$ is a TRDF on G, we have a contradiction with the minimality of f. If $N_G(y) \cap V_g^+ = \emptyset$, then y has a neighbor y' in V_0. Define $f' : V \longrightarrow \{0,1,2\}$ on G, as follows: $f'(y) = f'(y') = 1$ and $f'(z) = g(z)$ for all other $z \in V(G)$. Thus f' is a TRDF on G, with $w(f') = \gamma_{tR}(G)$ and $f'(x) = 2$, which contradict our assumption.

Case 3. $g(x) = 0$ and $g(y) = 2$. If $N_G(x) \cap V_g^+ \neq \emptyset$, then the function $f' : V \longrightarrow \{0,1,2\}$ defined on G, as follows: $f'(x) = 1$ and $f'(z) = g(z)$ for all other $z \in V(G)$ is a TRDF on G, we have a contradiction with the minimality of f'. If $N_G(x) \cap V_g^+ = \emptyset$, then x has a neighbor x' in V_0. Define $f' : V \longrightarrow \{0,1,2\}$ on G, as follows: $f'(x) = f'(x') = 1$ and $f'(z) = g(z)$ for all other $z \in V(G)$.

Thus f' is a TRDF on G, with $w(f') = \gamma_{tR}(G)$ and $f'(x) = 1$, contradicting our assumption. ■

The authors noted in [61] that Conjecture 4.1.2 would be a direct result of Conjecture 4.1.1. Consequently, Conjecture 4.1.2 has also been proven. So, we obtain the following corollary:

Corollary 4.1.1 ([64]) *If G is a k-γ_{tR}-edge-supercritical graph, then $G \cup K_n$ is $(k+3)$-γ_{tR}-edge-critical, for $n \geq 3$.*

Remark 4.1.1 *Mynhardt et al. [61] provided an excellent graph in response to Question 4.1.1, but they left another related question unanswered:*

Question 4.1.3 ([61]) *Do there exist connected 6-γ_{tR}-edge-supercritical graphs with diameter 2?*

4.2 Double Roman domination edge critical graphs

It is shown in [9] that the addition of an edge to a graph can decrease the double Roman domination number by at most two.

Theorem 4.2.1 ([9]) *Let G be a graph and e be an edge in $\overline{G}$. Then $\gamma_{dR}(G) - 2 \leq \gamma_{dR}(G+e) \leq \gamma_{dR}(G)$.*

A graph G is said to be *double Roman domination edge critical*, or just γ_{dR}*-edge critical*, if $\gamma_{dR}(G+e) < \gamma_{dR}(G)$ for any $e \in E(\overline{G})$, that is; for any edge $e \in E(\overline{G})$, $\gamma_{dR}(G)-2 \leq \gamma_{dR}(G+e) \leq \gamma_{dR}(G)-1$. Double Roman domination edge critical graphs are studied in [63].

In this section, we continue our study of the critical concept for double Roman domination in graphs, providing a characterization of double Roman domination edge-critical trees. This work answers a problem posed by Nazari-Moghaddam et al. in [63].

Conjecture 4.2.1 ([63]) *A tree T is γ_{dR}-edge critical if and only if $T = P_4$.*

4.2.1 Preliminary results

We begin by recalling some important results, given by Beeler, Haynes and Hedetniemi [13], Ahangar, Chellali and Sheikholeslami [4] and Anu [8], that will be useful in our investigations.

Proposition 4.2.1 ([13]) *In a double Roman dominating function of weight $\gamma_{dR}(G)$, no vertex needs to be assigned the value* 1.

Using Proposition 4.2.1, we have the following straightforward observation.

Observation 4.2.1 *Let v be a support vertex in a graph G. Then any $\gamma_{dR}(G)$-function $f = (V_0, \emptyset, V_2, V_3)$ assigns 0 or 3 to v.*

Proposition 4.2.2 ([8]) $\gamma_{dR}\left(cor(P_n)\right) = \gamma_{dR}\left(cor(C_n)\right) = 2n + \left\lceil \frac{n}{3} \right\rceil.$

Proposition 4.2.3 ([63]) *Let G be a γ_{dR}-edge critical graph and a, b two non-adjacent vertices. Then for any $\gamma_{dR}(G + ab)$-function $f = (V_0, \emptyset, V_2, V_3)$ we have $f(a) = 0$ and $f(b) \geq 2$, or $f(b) = 0$ and $f(a) \geq 2$.*

Proposition 4.2.4 ([63]) *Any support vertex in a γ_{dR}-edge critical graph is adjacent to exactly one leaf.*

4.2.2 Double Roman domination edge critical trees

In [63], Nazari-Moghaddam and Volkmann gave the following result for trees.

Theorem 4.2.2 ([63]) *Let T be a tree of order $n \geq 5$ and $diam(T) \neq 5$. Then T is not γ_{dR}-edge critical.*

The necessary condition in 4.2.1 is not true, as can be seen by the trees T_1 and T_2, where T_1 is obtained from two copies of path P_5, and T_2 is obtained from a copy of path P_5 and a copy of path P_4, by joining their center vertices, respectively (Refer Figure 4.4).

Observe that every DRDF of T_1 or T_2 assigns a weight of at least 6 and 5 to the copies P_5 and P_4, respectively. Hence, $\gamma_{dR}(T_1) \geq 12$ and $\gamma_{dR}(T_2) \geq 11$. We can also define two DRDFs on T_1 and T_2, with weights 12 and 11, respectively. Hence, $\gamma_{dR}(T_1) = 12$ and $\gamma_{dR}(T_2) = 11$. Now, by a simple calculation we see that $\gamma_{dR}(T_1 + e) = 11$ for any edge $e \in E(\overline{T_1})$, and $\gamma_{dR}(T_2 + e) \in \{9, 10\}$ for any edge $e \in E(\overline{T_2})$ (Refer to Figures 4.5 and 4.6).

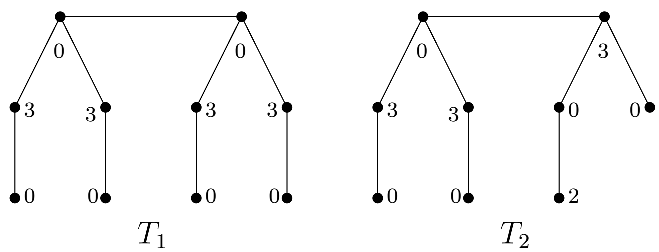

Figure 4.4: Two γ_{dR}-edge critical trees

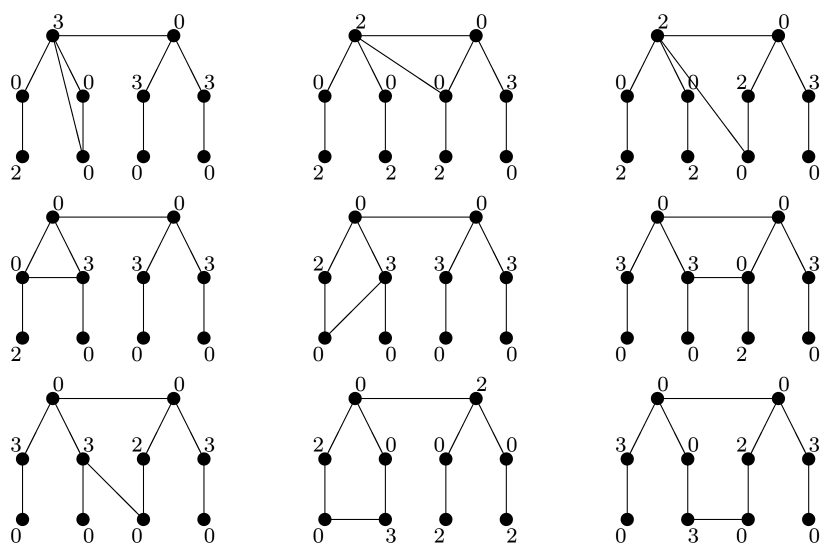

Figure 4.5: All possibilities of graphs $T_1 + e$, where $\gamma_{dR}(T_1 + e) = 11$.

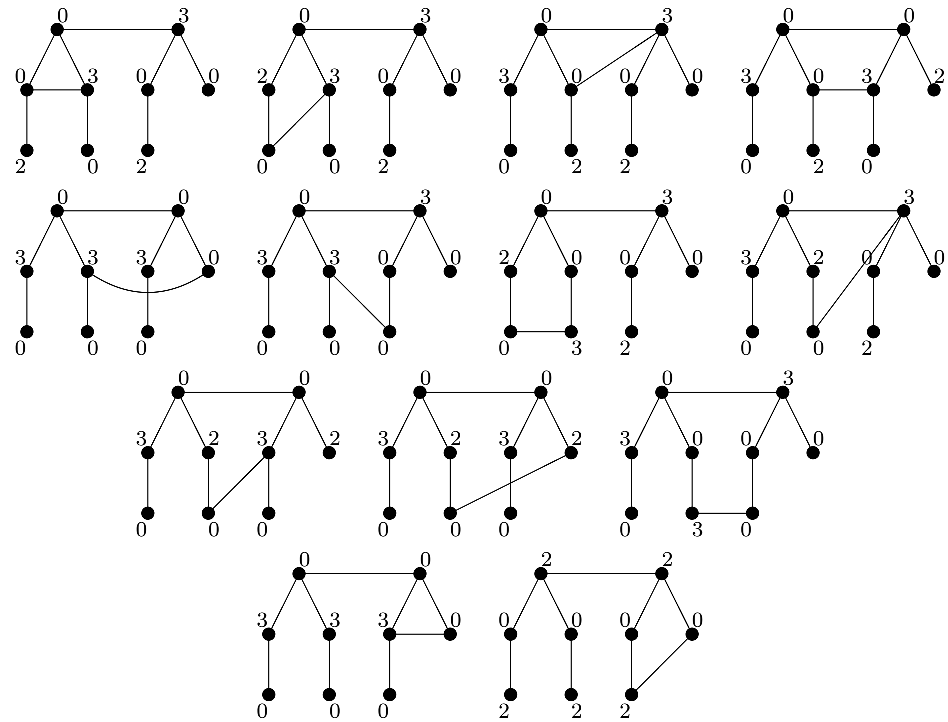

Figure 4.6: All possibilities of graphs $T_2 + e$, where $\gamma_{dR}(T_2 + e) \in \{9, 10\}$..

In the following we show that the only γ_{dR}-edge critical trees are P_4, T_1 and T_2.

Theorem 4.2.3 ([67]) *A tree T of order $n \geq 3$ is γ_{dR}-edge critical if and only if $T \in \{P_4, T_1, T_2\}$.*

Proof. Let T be a γ_{dR}-edge critical tree. Considering the contrapositive of Theorem 4.2.2, we may assume that T is a tree of order $n \leq 4$ or $diam(T) = 5$. If $n \leq 4$, then $T \in \{P_3, P_4, K_{1,3}\}$, and so by Proposition 4.2.4, P_3 and $K_{1,3}$ are excluded. Hence $T = P_4$. So in the following we may assume that $diam(T) = 5$. Let $v_0v_1...v_5$ be a diametrical path in T. Note that by Proposition 4.2.4, we may assume that $\deg(v_1) = \deg(v_4) = 2$. We proceed according to the value of $\deg_T(v_2)$:

Case 1. $\deg_T(v_2) = 2$. Suppose that f is a $\gamma_{dR}(T + v_0v_2)$-function. By the definition of f we must have $f(v_0) + f(v_1) + f(v_2) = 3$, since $\{v_0, v_1, v_2\}$ induces a pendant complete graph. Note that, if $f(v_3) = 0$, then we have $f(v_4) + f(v_5) = 3$. However, defining g on $V(T)$ by $g(v_1) = g(v_4) = 3$, $g(v_0) = g(v_2) = g(v_5) = 0$ and $g(v) = f(v)$ for $v \notin \{v_0, v_1, v_2, v_4, v_5\}$ produces a DRDF for T with weight $\gamma_{dR}(T + v_0v_2)$, so $\gamma_{dR}(T) \leq \gamma_{dR}(T + v_0v_2)$, which contradicts the fact that T is γ_{dR}-edge critical. Now assume that $f(v_3) \geq 2$. Then we define g on $V(T)$ by $g(v_1) = 3$, $g(v_0) = g(v_2) = 0$ and $g(v) = f(v)$ for $v \notin \{v_0, v_1, v_2\}$ produces a DRDF for T with weight $\gamma_{dR}(T + v_0v_2)$, a contradiction.

Case 2. $\deg(v_2) \geq 4$. Recall that v_2 has at most one leaf (see Proposition 4.2.4). Suppose that f is a $\gamma_{dR}(T+v_1v_1')$-function, where $v_1' \in N(v_2) - \{v_1, v_3\}$ is a support vertex adjacent to a single leaf v_0' in T. By Proposition 4.2.3, Observation 4.2.1, and without loss of generality, we may assume that $f(v_1) = 0$ and $f(v_1') = 3$. By the definition of f we must have $f(v_0) = 2$ and $f(v_0') = 0$. If $f(v_2) \geq 2$, then the function h defined on $V(T + v_1v_1')$ by $h(v_1') = 0$, $h(v_0') = 2$, and $h(v) = f(v)$ for $v \notin \{v_0', v_1'\}$, produces a DRDF for $T + v_1v_1'$ with weight less than f, a contradiction. Hence, we may assume that $f(v_2) = 0$. If v_2 has exactly one leaf neighbor, say v_2', then this leaf would be assigned a 2 under f. But the function g defined by $g(v_2) = 3$, $g(v_1') = g(v_2') = 0$, $g(v_0') = 2$ and $g(v) = f(v)$ for $v \notin \{v_2, v_0', v_1', v_2'\}$ produces a DRDF for T with weight $\gamma_{dR}(T + v_1v_1')$, a contradiction. Finally assume that v_2 has no leaf. Note that, since $\deg(v_2) \geq 4$, there exists a support vertex $w \in N(v_2) - \{v_1, v_3, v_1'\}$ adjacent to a single leaf t such that $f(w) + f(t) = 3$. But the function g defined by $g(v_2) = 2$, $g(v_1') = g(w) = 0$, $g(v_0') = g(t) = 2$ and $g(x) = f(x)$ otherwise, produces a DRDF on T with weight $\gamma_{dR}(T + v_1v_1')$, a contradiction too.

Case 3. $\deg(v_2) = 3$. Then by above cases and by symmetry, we must have that $\deg(v_3) = 3$. We claim that v_2 or v_3 is not support vertex. Suppose to the contrary that v_2 and v_3 are support vertices. Then $T = Cor(P_4)$ and $T + v_1v_4 = Cor(C_4)$. From Proposition 4.2.2, we have $\gamma_{dR}(Cor(P_4)) = \gamma_{dR}(Cor(C_4)) = 10$, again a contradiction. Hence at least one of v_2 and v_3 is not support vertex, and thus $T = T_1$ or T_2.

The converse part is obvious. This completes the proof. ■

We recall some results before going further. A set D of vertices in a graph G is a 2-*dominating set* of G if every vertex in $V - D$ has at least two neighbors in D. The 2-*domination number* of a graph G, denoted by $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set of G.

Proposition 4.2.5 ([13]) *For any graph G, $2\gamma(G) \leq \gamma_{dR}(G) \leq 3\gamma(G)$.*

Proposition 4.2.6 ([13]) *For any graph G, $2\gamma(G) = \gamma_{dR}(G)$ if and only if $\gamma(G) = \gamma_2(G)$.*

The *independence number* $\alpha(G)$ of a graph G is the cardinality of a maximum independent set of vertices. The following result, due to Balbuena and Hansberg [11], establishes a connection between 3-γ-edge-critical graphs, the independence number, and the 2-domination number.

Theorem 4.2.4 ([11]) *If G is a connected 3-γ-edge-critical graph with independence number $\alpha(G) \geq 4$, then $\gamma_2(G) \leq 5$.*

In addition, the authors [11] mentioned the following: "Until now, we have not found a single example of a 3-edge critical graph with $\gamma_2(G) = 5$. Moreover, there are many examples where $\gamma_2(G)$ is 3 or 4."

Thus, this remark can be translated into the following problem.

Conjecture 4.2.2 ([11]) *If G is a connected 3-γ-edge-critical graph with independence number $\alpha(G) \geq 4$, then $\gamma_2(G) \leq 4$.*

4.2.3 k-γ_{dR}-edge supercritical graphs

A graph G is said to be *double Roman domination edge supercritical*, or just γ_{dR}-*edge supercritical*, if $\gamma_{dR}(G+e) = \gamma_{dR}(G) - 2$ for any edge $e \in E(\overline{G})$. A double Roman domination edge supercritical graph G with $\gamma_{dR}(G) = k$ is called k-γ_{dR}-*edge supercritical.* The concept of edge supercriticality was studied, for the first time, by Haynes, Mynhardt and van der Merwe [46] for the total domination number.

The next result follows immediately from Theorem 4.2.3.

Corollary 4.2.1 ([67]) *There is no γ_{dR}-edge-supercritical tree.*

In the following, we study k-γ_{dR}-edge supercritical graphs where $k \in \{5, 6, 7, 8\}$.

Theorem 4.2.5 ([67]) *A graph G is 5-γ_{dR}-edge supercritical if and only if $\overline{G}$ is a disjoint union of stars, each of order at least 3. Figure 4.7 shows the smallest example of such a graph.*

Proof. We first prove the necessity. Let G be 5-γ_{dR}-edge supercritical graph. Then for any edge $e \in E\left(\overline{G}\right)$, we have $\gamma_{dR}\left(G+e\right) = 3$, and thus Proposition 2.4.5 implies that the addition of any edge to G creates a universal vertex, say u. Therefore, u is isolated in $\overline{G} - uv$, where $uv \in E\left(\overline{G}\right)$. Hence, we have shown that every edge of $\overline{G}$ is incident with a leaf of $\overline{G}$. So, the components of $\overline{G}$

are nontrivial stars. Moreover, each star must be of order at least 3, otherwise $\gamma_{dR}(G) \in \{3,4\}$ (see Proposition 2.4.5).

Now, we consider the sufficiency. Suppose $\overline{G}$ is the disjoint union of stars, each of order at least 3. Then G has no universal vertices and $G \neq \overline{K_2} \vee H$, where H is a graph with $\Delta(H) \leq |V(H)| - 2$. Thus, by Proposition 2.4.5, $\gamma_{dR}(G) \geq 5$. Let u be a leaf in $\overline{G}$, with v its support vertex and define $f : V(G) \longrightarrow \{0,1,2,3\}$ by $f(u) = 3$, $f(v) = 2$ and $f(x) = 0$ for all $x \in V(G) - \{u,v\}$. Clearly f is a DRDF on G, and hence $\gamma_{dR}(G) = 5$. Since deleting any edge in $\overline{G}$ produces an isolated vertex, the addition of any edge to G creates a universal vertex. Hence we obtain that $\gamma_{dR}(G+e) = 3$ for all $e \in E(\overline{G})$, and so G is 5-γ_{dR}-edge supercritical. ■

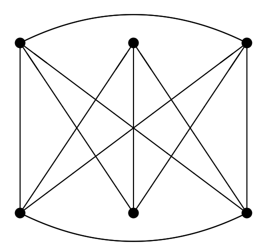

Figure 4.7: The smallest 5-γ_{dR}-edge supercritical graph.

Theorem 4.2.6 ([67]) *There are no k-γ_{dR}-edge supercritical graphs for $k \in \{6,7\}$.*

Proof. Consider $k = 6$. Suppose for a contradiction that G is a 6-γ_{dR}-edge supercritical graph. Then $\gamma_{dR}(G+uv) = 4$ for any $uv \in E(\overline{G})$. By Proposition 2.4.5 there exist two non adjacent vertices x and y each of which is adjacent to all other vertices in $G+uv$. It is clear that $\{x,y\} \neq \{u,v\}$. If $\{x,y\}$ and $\{u,v\}$ are disjoint, then $\gamma_{dR}(G) = 4$, contradicting G being 6-γ_{dR}-edge supercritical. Without loss of generality, assume that $u = x$. In this case assign 2 to x and 3 to y in G to obtain a DRDF of weight 5, a contradiction too.

Consider $k = 7$. Suppose for a contradiction that G is a 7-γ_{dR}-edge supercritical graph. Then $\gamma_{dR}(G+uv) = 5$ for any $uv \in E(\overline{G})$. Again by Proposition 2.4.5, we have $\Delta(G+uv) = n-2$ and $G+uv \neq \overline{K_2} \vee H$ for any graph H of order $n-2$. Thus in $G+uv$, there exist two non adjacent vertices, say x and y, such that $\deg_{G+uv}(x) = n-2$. It is clear that $\{x,y\} \neq \{u,v\}$. If $\{x,y\}$ and $\{u,v\}$ are disjoint, then u and v are in $N_G(x)$, and thus $\gamma_{dR}(G) = 5$, contradicting

G being 7-γ_{dR}-edge supercritical. If $y \in \{u, v\}$, then u or v is in $N_G(x)$, and thus $\gamma_{dR}(G) = 5$, a contradiction. Assume now that $x \in \{u, v\}$, without loss of generality, let $x = u$. We consider three cases:

Case 1. Suppose that $v \in N_{G+uv}(y)$. In this case assign 3 to x and y, and 0 to the remaining vertices to obtain a DRDF on G of weight 6, a contradiction.

Case 2. Suppose that $N_{G+uv}(x) - N_{G+uv}(y) = \{v\}$. In this case assign 2 to x, v and y, and 0 to the remaining vertices to obtain a DRDF on G of weight 6, a contradiction.

Case 3. Suppose that $|N_{G+uv}(x) - N_{G+uv}(y)| \geq 2$ and $v \in N_{G+uv}(x) - N_{G+uv}(y)$. Obviously, we have $\deg_G(x) = n - 3$ and $\deg_G(y) \leq n - 4$. Now, we will show that $\deg_G(v) \leq n - 4$, and for any w in $V - \{x, y, v\}$, $\deg_G(w) \leq n - 3$. Firstly, suppose that $\deg_G(v) \geq n - 3$. Then $N_G(v) = V - \{x, y\}$, and thus $g = (V - \{x, y, v\}, \emptyset, \{x, y, v\}, \emptyset)$ is a DRDF on G with $w(g) = 6$, a contradiction. Secondly, suppose that there is a vertex w in $V - \{x, y, v\}$ such that $\deg_G(w) \geq n-2$. Then by Proposition 2.4.5, we have $\gamma_{dR}(G) \leq 5$, a contradiction too. Hence $\Delta(G + vy) \leq n - 3$, and so by Proposition 2.4.5, we have $\gamma_{dR}(G + vy) \geq 6$, contradicting the supercriticality of G. This completes the proof. ■

For γ_{dR}-edge supercritical graphs, the analogous result to Proposition 4.2.3 is more restrictive, as we now show.

Corollary 4.2.2 ([67]) *Let G be a γ_{dR}-edge supercritical graph and a, b two non-adjacent vertices. Then for any $\gamma_{dR}(G + ab)$-function $f = (V_0, \emptyset, V_2, V_3)$, we have $\{f(a), f(b)\} = \{0, 3\}$.*

Proof. Let G be a γ_{dR}-edge supercritical graph. Then for any $ab \in E\left(\overline{G}\right)$, we have $\gamma_{dR}(G + ab) = \gamma_{dR}(G) - 2$. Let $f = (V_0, \emptyset, V_2, V_3)$ be a $\gamma_{dR}(G + ab)$-function. By Proposition 4.2.3, and without loss of generality, we may assume that $f(a) \geq 2$ and $f(b) = 0$. Suppose to the contrary that $f(a) = 2$ and $f(b) = 0$. Then there exists a vertex, say w, in $N_G(b)$ such that $f(w) \in \{2, 3\}$. If $f(w) = 3$, then $\gamma_{dR}(G) \leq \gamma_{dR}(G + ab)$, a contradiction. Assume now that $f(w) = 2$, and define the function g by $g(w) = 3$ and $g(x) = f(x)$ otherwise. Clearly, g is a DRDF on G of weight $\gamma_{dR}(G + ab) + 1$, a contradiction too. ■

In the following proposition, we show that there is no leaf in connected 8-γ_{dR}-edge supercritical graphs.

Proposition 4.2.7 ([67]) *There is no connected* 8-γ_{dR}*-edge supercritical graph having a leaf.*

Proof. Suppose there exists a connected 8-γ_{dR}-edge supercritical graph G with a leaf y adjacent to a vertex x in G. Clearly $N(x) - \{y\} \neq \emptyset$. We claim that $N(x) - \{y\}$ induces a complete graph. Suppose for a contradiction that there exist two non adjacent vertices u and v in $N(x) - \{y\}$, and consider a γ_{dR}-function $f = (V_0, \emptyset, V_2, V_3)$ on $G+uv$. By Corollary 4.2.2, $\{f(u), f(v)\} = \{0, 3\}$. We can assume $f(x) = 3$, since y is a leaf and $\gamma_{dR}(G + uv) = 6$. In this case, f is also a DRDF on G, contradicting $\gamma_{dR}(G) = 8$. Therefore $G[N(x) - \{y\}]$ is complete. Now, let $w \in N(x) - \{y\}$. It is a simple matter to see that $\deg(w) \leq n - 4$ (for otherwise $\gamma_{dR}(G) \leq 7$). So, there exist two vertices a and b in $V - N[x]$ that are not adjacent to w, and consider a γ_{dR}-function $g = (V_0, \emptyset, V_2, V_3)$ on $G + aw$. Again, by Corollary 4.2.2, we have $\{g(a), g(w)\} = \{0, 3\}$. We can assume, without loss of generality, that $g(x) = 3$. If $g(a) = 3$ and $g(w) = 0$, then g is a DRDF on G, a contradiction. Now, assume that $g(a) = 0$ and $g(w) = 3$, but in this case b is not double Roman dominated, a contradiction too. Hence there is no leaf in G. ■

Remark 4.2.1 *If G is* 8-γ_{dR}*-edge supercritical graph, then G is easily seen to be connected.*

Now, we consider connected 8-γ_{dR}-edge supercritical graphs and give results concerning 3-γ-edge critical graphs, and the diameter of such graphs. We need the following result for 3-γ-edge critical graphs.

Theorem 4.2.7 ([78]) *The diameter of a* 3-γ*-edge critical graph is at most* 3.

Proposition 4.2.8 ([67]) *If G is a connected* 8-γ_{dR}*-edge supercritical graph, then G is* 3-γ*-edge critical.*

Proof. Let G be a connected 8-γ_{dR}-edge supercritical graph, and let e be any edge of $E(\overline{G})$. First we show that $2 \leq \gamma(G + e) \leq 3$. The upper bound of Proposition 4.2.5, leads to $\gamma(G) \geq$

$\left\lceil \frac{\gamma_{dR}(G)}{3} \right\rceil = \left\lceil \frac{8}{3} \right\rceil = 3$, and since $\gamma(G+e) \geq \gamma(G) - 1$ for any edge $e \in E\left(\overline{G}\right)$, $\gamma\left(G+e\right) \geq 2$. On the other hand, the lower bound of Proposition 4.2.5, leads to $\gamma(G+e) \leq \frac{\gamma_{dR}(G+e)}{2} = \frac{6}{2} = 3$. Now we show that $\gamma(G+e) \neq 3$ for any edge $e \in E\left(\overline{G}\right)$. Suppose, to the contrary, that $\gamma\left(G+ab\right) = 3$ for some edge $ab \in E\left(\overline{G}\right)$. Since G is an 8-γ_{dR}-edge supercritical, $\gamma_{dR}\left(G+ab\right) = 2\gamma\left(G+ab\right)$. Thus by Proposition 4.2.6, we have $\gamma(G+ab) = \gamma_2(G+ab)$. Let D be a γ_2-set of $G+ab$. Note that $|D| = 3$. If $\{a,b\} \subset V(G) - D$ or $\{a,b\} \subset D$, then assigning a 2 to every vertex of D and a 0 to every vertex not in D provides a DRDF of G with weight $2|D| = 6$, contradicting $\gamma_{dR}(G) = 8$. Now, without loss of generality, assume that $a \in D$. In this case assign 2 to every vertex of D, 1 to b and 0 to every vertex not in $D \cup \{b\}$ to obtain a DRDF of G with weight $2|D| + 1 = 7$, a contradiction too. Hence, $\gamma\left(G+e\right) = 2$ for any edge $e \in E\left(\overline{G}\right)$, that is G is 3-γ-edge critical. ■

Remark 4.2.2 *Balbuena and Hansberg [11] characterized a special family of* 3*-*γ*-edge critical graphs with minimum degree one and presented a figure (Figure 4.8) illustrating all such graphs of order at most* 8*. So, by Proposition 4.2.7, the converse of Proposition 4.2.8 is not true.*

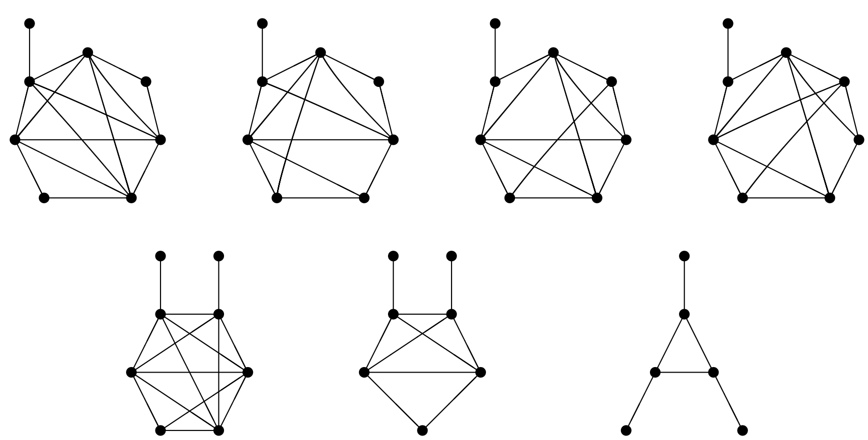

Figure 4.8: All 3-γ-edge-critical graphs with minimum degree one and order at most 8.

Theorem 4.2.8 ([67]) *If* G *is a connected* 8*-*γ_{dR}*-edge supercritical graph, then* $diam(G) \in \{2,3\}$*. Moreover, there exist connected* 8*-*γ_{dR}*-edge supercritical graphs* F *and* H *with* $diam(F) = 2$ *and* $diam(H) = 3$*, as illustrated in Figure 4.9.*

Proof. Obviously $diam(G) \geq 2$. By Proposition 4.2.8 and Theorem 4.2.7, we have $diam(G) \leq 3$. Now we show that the graphs F and H in Figure 4.9 are 8-γ_{dR}-edge supercritical:

Consider the graph F. The function f that assigns 3 to each of a and a', 2 to d' and 0 to all other vertices is a DRDF of F with weight $w(f) = 8$. It can be verified that there are no DRDFs of smaller weights of F. Hence $\gamma_{dR}(F) = 8$. It is simple matter to check that $\{a, d'\}$, $\{a, c'\}$, $\{a, a'\}$ and $\{a', d\}$ are dominating sets of $F + ab'$, $F + ae'$, $F + ad'$ and $F + a'c'$, respectively, and thus $\gamma_{dR}(F + e) \leq 6$ for any $e \in \{ab', ac', ae', a'c'\}$. Since all possible edges of $\overline{F}$ have been considered, we must have $\gamma_{dR}(F + e) \leq 6$ for any $e \in E\left(\overline{F}\right)$. In either case, we have $\gamma_{dR}(F + e) = 6$ for any $e \in E\left(\overline{F}\right)$. Moreover, it is clear that $ecc(u) = 2$ for any $u \in V(F)$, and thus $diam(F) = 2$. It follows that F is 8-γ_{dR}-edge supercritical with $diam(F) = 2$.

Consider the graph H. The function h that assigns 3 to each of x_8 and x_9, 2 to x_4 and 0 to all other vertices is a DRDF of H with weight $w(h) = 8$. It can be verified that there are no DRDFs of smaller weights of H. Hence $\gamma_{dR}(H) = 8$. As shown in [77], the graph H is 3-γ-edge critical. Then $\gamma(H + e) = 2$ for any $e \in E\left(\overline{H}\right)$. By Proposition 4.2.5, we must have $\gamma_{dR}(H + e) = 6$ for any $e \in E\left(\overline{H}\right)$. Now, by inspection, $ecc(x_3) = ecc(x_5) = ecc(x_8) = 2$ and $ecc(u) = 3$ for any $u \in V(H) - \{x_3, x_5, x_8\}$, and thus $diam(H) = 3$. Hence H is 8-γ_{dR}-edge supercritical with $diam(H) = 3$. ■

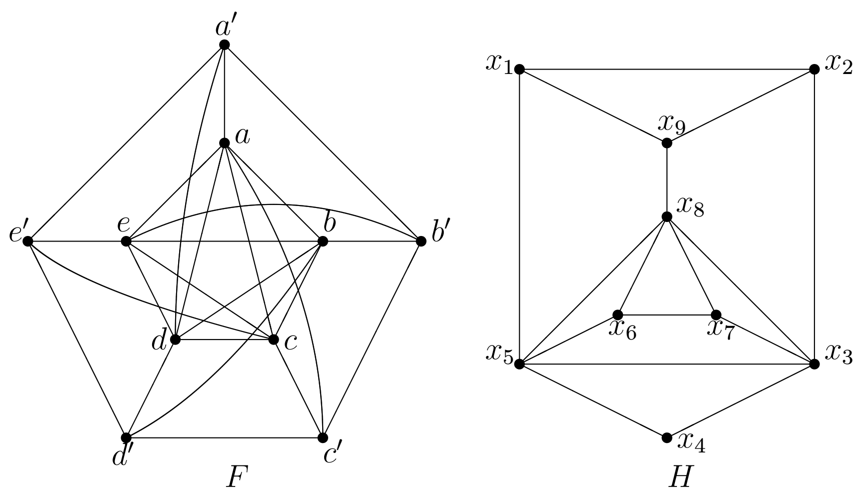

Figure 4.9: Examples of 8-γ_{dR}-edge-supercritical graphs

It is shown in [29] that there exists an infinite class $\mathcal{F}$ of 3-γ-edge critical graphs as follows: Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, ..., b_n\}$ for $n \geq 3$, and $C = \{c_1, c_2, c_3\}$. Set $V(G) = A \cup B \cup C \cup \{v\}$. Form complete graphs on A, B and C. Join v to each vertex of A, join each vertex in C to exactly two vertices in A such that each vertex of A is adjacent to exactly two vertices of C. Form a perfect matching between three vertices in B and vertices in A, and join the other vertices in B to each vertex of A (see Figure 4.10 for $n = 5$). Next, we show that $\mathcal{F}$ is an infinite families of 8-γ_{dR}-edge supercritical graphs.

Let $G \in \mathcal{F}$. It is not difficult to show that $\gamma_{dR}(G) = 8$. Now, since G is 3-γ-edge critical, then $\gamma(G+e) = 2$ for any $e \in E(\overline{G})$. By Proposition 4.2.5, we must have $\gamma_{dR}(G+e) \leq 6$ for any $e \in E(\overline{G})$. Hence G is 8-γ_{dR}-edge supercritical.

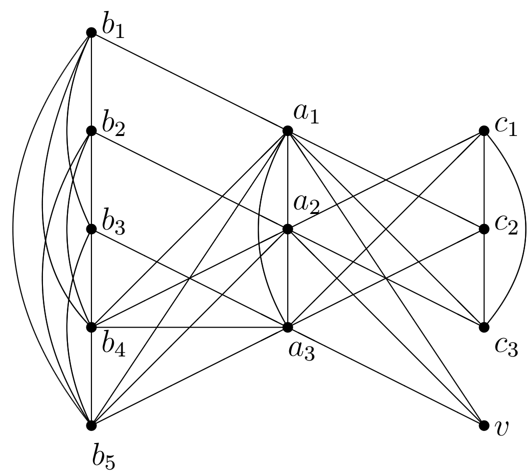

Figure 4.10: Example of 8-γ_{dR}-edge-supercritical graph with diameter equal to 3

In 2011, Jafari Rad et al. [50] showed that the only γ-edge critical cactus graphs are P_2, C_3, C_4 and $cor(C_3)$. However, these graphs have a double Roman domination number of at most 7. Therefore, the following corollary can be directly deduced from Proposition 4.2.8.

Corollary 4.2.3 ([67]) *There is no connected* 8-γ_{dR}-*edge supercritical cactus.*

In [39], Goddard and Henning proved that every planar graph G with diameter 2 has $\gamma(G) \leq 2$, except for the graph F depicted in Figure 4.11, which has $\gamma(F) = 3$.

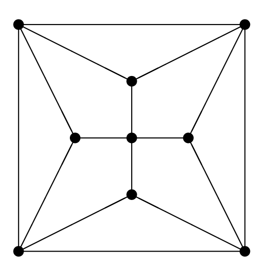

Figure 4.11: The planar graph F

Therefore, we conclude that there is no 8-γ_{dR}-edge supercritical planar graph of diameter 2, since $\gamma_{dR}(G) \leq 3\gamma(G) \leq 6$, and $\gamma_{dR}(F) = 6$. On the other hand, Furuya and Matsumoto [36] provided a well-organized and inventive proof showing that the order of a connected 3-γ-edge critical planar graph is at most 23. Thus the order of a connected 8-γ_{dR}-edge supercritical planar graph is at most 23. Note that the graph H illustrated above is planar of order 9.

CONCLUSION

In this thesis, we characterized the graphs G achieving the upper bound in the inequality $\gamma_{dR}(G) + \gamma_{dR}(G) \leq 2n$, a type of Nordhaus-Gaddum inequality, and determined the graphs G satisfying $\gamma_{dR}(G) = 2\gamma_R(G) - 1$, improving upon previous studies. Additionally, we extended the concept of supercriticality to double Roman domination for the first time, yielding significant results that enhance the understanding of criticality in graph theory and build upon classical domination concepts. Furthermore, we solved some open problems.

Based on these contributions, it is evident that Roman domination functions remain an attractive research area in graph theory. There have been many achievements on this topic, but still some open problems remain that have not been completely solved. In closing, we recall a few notable examples:

Conjecture 4.2.3 ([15]) *If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma_R(G) + \gamma(G) \leq n$.*

Conjecture 4.2.4 ([21]) *Let G be a graph with no isolated vertex. Then $\gamma_{tR}(G) = 3\gamma(G)$ if and only if $\gamma_{tR}(G) = \gamma_R(G) + \gamma(G)$.*

Question 4.2.1 ([67]) *Can you establish structural properties of 8-γ_{dR}-edge supercritical graphs?*

Question 4.2.2 ([67]) *Can you find some classes of 8-γ_{dR}-edge supercritical graphs?*

Question 4.2.3 ([22]) *Is it the case that $\gamma_{oiR}(G) = n - i(G) + \gamma(G)$ if and only if G is a complete graph?*

Question 4.2.4 ([61]) *Do there exist connected 6-γ_{tR}-edge-supercritical graphs with diameter 2?*

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