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A Study of certain fractional-order boundary value problems on non-regular domains

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بسم الله الرحمن الرحيم

قَالُواْ سُبْحَانَكَ لاَ عِلْمَ لَنَا إِلاَّ مَا عَلَّمْتَنَا إِنَّكَ
أَنتَ الْعَلِيمُ الْحَكِيمُ

سورة البقرة , الآية 32

Dedication

This modest work is dedicated to:

*My dearest mother **Aicha,***

*My beloved brother **Mohamed,***

All my family and friends.

Appreciation

First and foremost, I express my profound gratitude to the almighty God for granting me the will, courage, and patience throughout these arduous years of study.

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Abstract

The objective of this thesis is to investigate fractional-order boundary value problems in non-regular domains by examining the existence and uniqueness of solutions for various types of abstract differential equations involving fractional operators. The study begins with an analysis of three-dimensional fourth-order differential equations incorporating fractional powers of the negative Laplace operator under Cauchy-Dirichlet boundary conditions in cuspidal domains. The investigation techniques are based on transforming the main problem, through a natural change of variables, into a complete abstract fourth-order differential equation involving fractional powers of linear operators, which allows us to provide results on well-posedness. Furthermore, we explore periodic-type solutions for fractional neutral evolution equations involving Caputo and ψ-Hilfer derivatives, utilizing classical fixed point theorems as a preliminary step toward further investigation of fractional-order boundary value problems in non-smooth domains.

Keywords: Fractional-order boundary value problems, non-regular domains, existence and uniqueness, abstract differential equations, well-posedness, periodic-type solutions.

Résumé

L'objectif de cette thèse est d'étudier les problèmes aux limites d'ordre fractionnaire dans des domaines non réguliers, en examinant l'existence et l'unicité des solutions pour différents types d'équations différentielles abstraites impliquant des opérateurs fractionnaires. L'étude commence par l'analyse d'équations différentielles du quatrième ordre en dimension trois, incorporant des puissances fractionnaires de l'opérateur de Laplace négatif, sous conditions de Cauchy-Dirichlet sur la frontière, dans des domaines contenant des points de rebroussement. La méthode d'investigation s'appuie sur la transformation du problème principal, via un changement naturel de variables, en une équation différentielle abstraite complète du quatrième ordre comportant des puissances fractionnaires d'opérateurs linéaires, ce qui permet d'obtenir des résultats concernant le problème bien posé. Par ailleurs, on explore des solutions de type périodique pour des équations d'évolution neutres fractionnaires impliquant les dérivées de Caputo et les dérivées ψ-Hilfer, en utilisant les théorèmes classiques du point fixe comme étape préliminaire vers une étude approfondie des problèmes aux limites d'ordre fractionnaire dans des domaines non lisses.

Mots clés: Problèmes aux limites d'ordre fractionnaire, domaines non réguliers, existence et unicité, équations différentielles abstraites, bien-posé, solutions de type périodique.

الملخص

تهدف هذه الأطروحة إلى دراسة مسائل القيم الحدية ذات الرتبة الكسرية في مجالات غير منتظمة، من خلال فحص وجود الحل ووحدانيته لأنماط مختلفة من المعادلات التفاضلية المجردة التي تتضمن مؤثرات كسرية. تبدأ الدراسة بتحليل معادلات تفاضلية من الرتبة الرابعة في الفضاء الثلاثي الأبعاد، والتي تشمل قوى كسرية للمؤثر سالب لابلاس، تحت شروط كوشي- ديريخليه على الحافة، وذلك في مجالات تحتوي على نقاط قرنة (رجوع). وتعتمد منهجية البحث على تحويل المسألة الأصلية، من خلال تغيير طبيعي للمتغيرات، إلى معادلة تفاضلية مجردة تامة من الرتبة الرابعة تتضمن قوى كسرية لمؤثرات خطية، مما يتيح التوصل إلى نتائج تتعلق بمسألة مصوغة جيدًا. كما يتم أيضًا استكشاف الحلول ذات النمط الدوري لمعادلات التطور الحيادية الكسرية التي تتضمن مشتقات كابوتو ومشتقات ψ-هيلفر، وذلك باستخدام مبرهنات النقطة الثابتة الكلاسيكية كخطوة أولية نحو تعميق البحث في مسائل القيم الحدية ذات الرتبة الكسرية في مجالات غير ملساء.

الكلمات المفتاحية: مسائل القيم الحدية ذات الرتبة الكسرية، مجالات غير منتظمة، وجود الحل ووحدانيته، معادلات تفاضلية مجردة ،مسألة مصوغة جيدًا، حلول ذات نمط دوري.

Publications and communications

International publications

- **N. Chegloufa**, B. Chaouchi, F. Boutaous, and M. Kostić, S-asymptotically Bloch type periodic solutions for abstract fractional equations involving ψ-Hilfer derivatives, Journal of Applied Nonlinear Dynamics. **14** (2025), no. 2, 343–354.

- **N. Chegloufa**, B. Chaouchi, M. Kostić, and F. Boutaous, On a class of abstract fourth-order differential equations set on cusp domains, Journal of Prime Research in Mathematics. **21** (2025), no. 1, 22–39.

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International communications

- **N. Chegloufa**. Existence of pseudo S-asymptotically periodic α-mild solution of class p for neutral fractional derivative delayed evolution equations. ICCMA, 26-27th November 2023, Mila, Algeria.

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National communications

- **N. Chegloufa**. On a class of neutral fractional delayed evolution equations. MMS, 26-27th October 2023, M'sila- Bousaâda, Algeria.

Contents

Notations

- $\mathbb{R}^n$: n-Dimensional Euclidean space, where n is a non-zero natural number.
- Ω: Arbitrary set in $\mathbb{R}^n$.
- $\partial\Omega$: Boundary of Ω.
- $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$: Set of natural numbers, real numbers, and complex numbers, respectively.
- $\mathbb{R}^+$: Set of positive real numbers.
- I: Arbitrary interval in $\mathbb{R}$.
- $Re(\lambda)$: Real part of complex number λ.
- $n!$: Factorial of n.
- $[\alpha]$: Integer part of real number α.
- $\hat{u}$: Fourier transform of function u.
- $\partial_{x_i} = \partial/\partial_{x_i}$: Partial derivative with respect to x_i.
- $\nabla = \left(\partial_{x_1}, ..., \partial_{x_n}\right)$: Gradient with respect to x.
- $\dfrac{d^m u}{dt^m} = u^{(m)}$ and $D^\alpha u$: Derivatives in the sense of distributions of u.
- $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$: Banach spaces over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.
- H: Complex separable Hilbert space.
- $< \cdot, \cdot >$: Scalar product on a Hilbert space H.
- X': Dual space of X with the strong dual topology.
- $\overline{X}$: Closure of X.
- I_X : Identity operator on X.
- $\mathcal{L}(X, Y)$: Space of linear bounded operators defined from the whole space X into Y. To simplify notation, we write $\mathcal{L}(X)$ instead $\mathcal{L}(X, X)$.
- $(T(t))_{t\geq 0}$: Family of bounded linear operators on X.
- A: Linear operator on X.
- $D(A)$ and $R(A)$: Domain and range of the operator A.
- $\sigma(A)$: Spectrum of the operator A.
- $\rho(A)$ and $R(\lambda, A)$: Resolvent set and resolvent operator of A, respectively.
- $\Gamma(\cdot)$: Gamma function.
- I_t^α: Fractional integral of order $\alpha > 0$.

- $I_t^{\alpha,\psi}$: Fractional integral of order $\alpha > 0$, with respect to function ψ.
- ${}^{RL}D_t^\alpha$ and ${}^{C}D_t^\alpha$: Riemann-Liouville and Caputo fractional derivatives of order $\alpha > 0$.
- ${}^{H}D_{a+}^{\alpha,\beta}$: Hilfer fractional derivative of order α and type $0 \leq \beta \leq 1$.
- $D_t^{\alpha,\psi}$ and ${}^{C}D_t^{\alpha,\psi}$: ψ-Riemann-Liouville and ψ-Caputo fractional derivatives of order $\alpha > 0$.
- ${}^{H}D_{a+}^{\alpha,\beta,\psi}$: ψ-Hilfer fractional derivative of order α and type $0 \leq \beta \leq 1$.
- $u_t(\cdot) = u(\cdot + t)$: Classical history function.
- $\mathcal{C}$: Space of all continuous functions from $[-r, 0]$ into X, $r > 0$.
- $\mathcal{B}$: Phase space.
- $C^n(\Omega)$: Space of n-times continuously differentiable functions on Ω.
- $C_0^\infty(\Omega)$: The set of all smooth functions with compact support in Ω, having (continuous in Ω) partial derivatives of arbitrary order.
- $\mathcal{S}$: Space of rapidly decreasing functions at infinity (Frechet space).
- $\mathcal{S}'$: Space of Schwartz's tempered distributions.
- $L^1_{loc}(\Omega)$: Space of all locally integrable functions on Ω.
- $L^p(\Omega)$: Lebesgue space consisting of all p-integrable measurable functions on Ω.
- $W^{m,p}(\Omega)$: Sobolev space constructed on $L^p(\Omega)$.
- $H^m(\Omega) = W^{m,2}(\Omega)$.
- $H^s(\mathbb{R}^n)$ and $H^s(\Omega)$: Fractional order Sobolev spaces (Bessel potentials spaces).
- $L^p(I, X)$: Lebesgue space consisting of all p-integrable measurable functions from I into X.
- $L^\infty(I, X)$: Lebesgue space consisting of all measurable essentially bounded functions from I into X.
- $(X, Y)_{\theta,p}$ and $(X, Y)_\theta$: Interpolation spaces between X and Y.
- $C(I, X)$: Space of all continuous functions from I into X.
- $C_b(I, X)$: Space of bounded continuous functions from I into X.
- $C_b(I \times Y, X)$: Space of all continuous functions from $I \times Y$ into X .
- $SAP_\omega(X)$: Space of S-asymptotically ω-periodic functions from $\mathbb{R}^+$ into X.
- $SAAP_\omega(X)$: Space of S-asymptotically ω-anti-periodic functions from $\mathbb{R}^+$ into X.
- $SABP_{\omega,k}(X)$: Space of S-asymptotically Bloch type periodic functions from $\mathbb{R}^+$ into X.
- $PSAP_\omega(X)$: Space of pseudo S-asymptotically ω-periodic functions from $\mathbb{R}^+$ into X.
- $PSAP_{\omega,p}(X)$: Space of pseudo S-asymptotically ω-periodic functions of class p from $\mathbb{R}^+$ into X.
- $PSAP_{\omega,p}(\mathbb{R}^+ \times Y, X)$: Space of uniformly (Y, X) pseudo S-asymptotically ω-periodic functions of class p from $\mathbb{R}^+ \times Y$ into X.

Introduction

The field of fractional calculus, which is mainly based on the study of integrals and derivatives of arbitrary real or complex orders, has become a rapidly growing area of applied mathematics, providing a powerful framework for modeling complex phenomena. The concept dates back to the late 17th century when L'Hôpital posed a question to Leibniz about the meaning of $d^n y/dx^n$ for $n = 1/2$. Initially regarded as a purely theoretical construct, fractional calculus has evolved significantly through the contributions of many mathematicians (see [57], [65] and references therein). In recent years, it has played a crucial role in various branches of science and engineering [18, 48, 69, 70, 93]. Its applications extend to fields such as theoretical physics, fluid mechanics, biology, and image processing [13, 15, 19, 20, 53, 62, 63, 83, 92, 102, 103].

The study of fractional boundary value problems (FBVPs) is one of the most important areas of fractional calculus. These problems have attracted considerable interest from researchers due to their ability to include memory effects, allowing fractional derivatives and integrals to provide a more realistic representation of physical phenomena compared to classical approaches. Significant research has focused on investigating the existence, uniqueness, and stability of solutions for different types of FBVPs, using various forms of fractional derivatives; see, for example, [1, 2, 3, 5, 8, 10, 22, 23, 25, 29, 41, 42, 43, 71, 89, 90, 100, 106, 111].

A comprehensive theory has been established for FBVPs in domains with smooth boundaries, where sufficiently smooth coefficients, boundary operators, and domain boundaries result in solutions with corresponding smoothness. However, the situation becomes considerably more complex when the domain contains non-regular or non-smooth boundary points, and we recall here that a point x in the boundary of a domain $\Pi \subset \mathbb{R}^n$ (i.e., $x \in \partial\Pi$) is called non-regular if, for every neighborhood U around x, there is no smooth, non-degenerate map $U \to \mathbb{R}^n$ that carries $\partial\Pi \cap U$ into an $(n-1)$-dimensional sphere; see [66] for more details. The study of classical boundary value problem (BVPs) in domains with non-regular boundaries has roots in early research efforts, including T. Carleman's Ph.D. dissertation (1916) [28]. Subsequent surveys by researchers such as V. A. Kondrat'ev and

O. A. Oleinik (1983) [66] extended this analysis to fundamental equations in mathematical physics, including elasticity theory, the Navier-Stokes equations, and the biharmonic equation.

In the contemporary theory of boundary-value problems, correctly formulating BVPs in non-smooth domains requires considering solutions, the right-hand sides of equations, and boundary conditions in appropriately chosen function spaces. Often, it is convenient to use function spaces with a weighted norm, where the weight is a power of the distance to the set of non-regular boundary points. This approach allows for a precise description of the singularities in the solution and its derivatives near these points, see [66] for more details. Since the 1970s, various classical methods have been adapted to study the complexities of BVPs in non-cylindrical and non-smooth domains. Notable methods include:

- Domain decomposition method: By approximating the non-smooth domain with a sequence of sub-domains that can be transformed into smooth ones, researchers have obtained significant results [17, 44, 64, 95, 96].
- Layer potential method: S. Hofmann and J. L. Lewis (2005) [59] utilized this method for the solvability of the heat equation in non-cylindrical domains with Lipschitz-type conditions.
- Rothe's method: Initially introduced in the 1930s by E. Rothe [94] for second-order linear parabolic equations, this method has since been extended to handle linear parabolic BVPs in non-cylindrical domains [46, 68].
- Sum of operators method: Developed by P. Grisvard and Da. Prato (1975) [91], this powerful method involves representing the solution through a Dunford integral containing resolvents of the operators involved. This method has been successfully applied to solve parabolic problems in non-cylindrical domains, yielding results that demonstrate maximal regularity [72, 73, 74].

Many important applied problems reduce to studying BVPs in domains with non-smooth boundaries, numerous studies have focused on equations in specific domains with particular types of boundary conditions. Nevertheless, only a few results are dedicated to the study of FBVPs in non-smooth domains. For instance, B. Chaouchi et al. (2023) [34] investigated the solvability of a time-conformable fractional equation given by

$$\mathcal{D}_t^{\alpha} u(t,x) + \sum_{i=1}^{N} D_{x_i}^{2m} u(t,x) = h(t,x),\ \alpha \in (0,1],\ m \in \mathbb{N}^*, \tag{1}$$

associated with the following initial and boundary conditions

$$\begin{aligned} u\big|_{\{0\}\times\Omega} = 0,\ u\big|_{\{1\}\times\Omega} = 0, \\ u\big|_{[0,1]\times\partial\Omega} = 0, \end{aligned} \tag{2}$$

set in a singular cylindrical domain

$$\Pi = [0,1] \times \Omega(t),$$

$$\Omega = \left\{ (x_1, x_2, ..., x_n) \in \mathbb{R}^N \Big/ \sqrt{x_1^2 + x_2^2 + ... + x_n^2} \leq \varphi(t) \right\},$$

where, φ represents a parametrization function satisfying $\varphi(0) = 0$ and $\varphi(t) > 0,\ t \in]0,1]$, while $\mathcal{D}_t^{\alpha}$ is the standard conformable time fractional derivative of order α in the sense stated in [4]. The investigation techniques are based on transforming the problem (1)–(2) through a natural change of variables into an abstract differential problem

$$w^{'}(t) + A(t)w(t) = g(t),\ t \in [0,1],$$

with

$$w(0) = w(1) = 0.$$

In line with this objective, this thesis is devoted to the study of fractional-order boundary value problems in non-regular domains by examining the existence and uniqueness of solutions for various types of abstract differential equations involving fractional operators. The analysis employs a variety of functional analysis tools, including semigroup theory, fractional powers of closed operators, interpolation theory, and some classical fixed point theorems. This approach has been utilized in numerous works; see [33, 35, 36, 37, 49, 50, 87].

The organization and main ideas of the thesis are summarized as follows. The first chapter provides essential definitions and results related to Sobolev spaces, fractional integrals and derivatives, semigroup theory, and significant findings regarding the fractional power of closed operators. Additionally, it introduces definitions and properties of interpolation spaces and the trace theorem, concluding with several classical fixed point theorems that are foundational for the subsequent analysis.

Chapter 2 explores the existence and uniqueness of solutions for three-dimensional fourth-order differential equations involving fractional powers of the negative Laplace operator with Cauchy-Dirichlet boundary conditions and initial conditions

$$\frac{d^4}{dt^4} u(t,x) + (1 + \rho_4(x))(-\Delta)^{1/2} u(t,x) + \sum_{j=1}^{3} \left(\rho_j(x) (-\Delta)^{j/8} \right) \frac{d^{4-j}}{dt^{4-j}} u(t,x) = f(t,x), \tag{3}$$

$$u|_{\mathbb{R}^+ \times \partial \Pi} = 0, \tag{4}$$

and

$$\left. \frac{du}{dt} \right|_{\{0\} \times \Pi} = 0,\ \left. \frac{d^3 u}{dt^3} + b(-\Delta)^{3/8} u \right|_{\{0\} \times \Pi} = 0, \tag{5}$$

on the cusp domain $\mathbb{R}^+ \times \Pi$,

$$\Pi := \left\{ x \in \mathbb{R}^3 \Big/ 0 < x_3 < 1,\ \left(\frac{x_1}{(x_3)^{\alpha}}, \frac{x_2}{(x_3)^{\alpha}} \right) \in \Omega \right\}, \quad \alpha > 1,$$

where $\Omega \subseteq \mathbb{R}^2$ is a bounded smooth, $\rho_j(\cdot)$, $j = 1,2,3,4$, are continuous real functions defined on Π, and $f(t,\cdot) \in L^2(\Pi)$.
The principal strategy for solving problem (3)-(4)-(5) involves transforming the equation (3), posed in the non-cylindrical domain Π, into a variable-coefficient equation in a cylindrical domain Q given by

$$Q = \Omega \times \left]\frac{1}{\alpha - 1}, +\infty\right[.$$

Section 2.2 provides sufficient conditions for the well-posedness and regular solvability of a class of complete abstract fourth-order differential equations

$$\frac{d^4 w(t)}{dt^4} + A^{4\theta} w(t) + \sum_{j=1}^{4} A_j \frac{d^{4-j} w(t)}{dt^{4-j}} = h(t), \ t \in \mathbb{R}^+,$$

endowed with the initial conditions

$$\frac{dw(0)}{dt} = \varphi_1 \in H_{5\theta/2}, \quad \frac{d^3 w(0)}{dt^3} + Kw(0) = \varphi_2 \in H_{\theta/2},$$

where $\theta \in]0,1]$, A is a self-adjoint positive definite operator in a separable Hilbert space H, A_j, $j \in \{1,2,3,4\}$ are linear operators acting on H, $K \in \mathcal{L}(H_{7\theta/2}, H_{\theta/2})$, with H_θ denoting the Hilbert scale spaces generated by A^θ for $\theta \geq 0$, and $h \in L^2(\mathbb{R}^+; H)$. Lastly, after preparing some intermediate results that directly follow from the findings in Section 2.3, we revisit the original problem by applying the inverse change of variables.

Chapter 3 examines the existence and uniqueness of pseudo S-asymptotically periodic mild solutions for a class of neutral evolution equations involving the Caputo fractional operator with finite delay. This study applies classical fixed point theorems, including the Banach contraction principle and Krasnoselskii's fixed point theorem. Section 3.2 compiles essential definitions and preliminary results needed to justify the main findings, particularly the class of pseudo S-asymptotically periodic functions. Section 3.3 establishes sufficient conditions for the existence of such solutions, while Section 3.4 illustrates an example of a delayed partial differential equation.

Chapter 4 investigates the existence and uniqueness of S-asymptotically Bloch periodic mild solutions for a class of neutral evolution equations governed by the ψ-Hilfer fractional operator with infinite delay. The analysis employs classical fixed point theorems, specifically the Banach contraction principle and Schauder's fixed point theorem. Section 4.2 presents the fundamental definitions and preliminary results required to establish the main findings, including the class of S-asymptotically Bloch type periodic functions and the associated phase space. Section 4.3 derives sufficient conditions ensuring the existence of the desired solution. Finally, Section 4.4 provides an illustrative example of a fractional partial differential equation to demonstrate the applicability of the theoretical results.

Chapter 1

Preliminaries

In this chapter, we review some standard definitions and properties that we will need throughout this work. The chapter is intended to make the work as self-contained as possible. For a deeper discussion of the theory discussed here, we refer the reader to [7, 12, 14, 45, 65, 79, 80, 82, 87, 99, 105].

1.1 Sobolev spaces

In this section, we provide some definitions and properties of Sobolev spaces which will be used later. The primary references for further detailed information are [7, 45, 80].

Definition 1.1.1. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary set, and let $x = (x_1, \ldots, x_n)$ be an element of Ω with $dx = dx_1 \ \ldots \ dx_n$. For a real number p where $1 \leq p < +\infty$, the space $L^p(\Omega)$ is defined by*

$$L^p(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{K} \Big/ \ u \text{ measurable, and } \int_\Omega |u(x)|^p \, dx < +\infty \right\},$$

equipped with the norm

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p}. \tag{1.1}$$

If $p = 2$, it is a classical result that $L^2(\Omega)$ is a Hilbert space for the scalar product

$$< u, v >_{L^2(\Omega)} = \int_\Omega u(x) v(x) dx,$$

associated to the norm (1.1).

Definition 1.1.2. *Let m be a positive integer and $1 \leq p < +\infty$.*

The Sobolev space $W^{m,p}(\Omega)$ of order m on Ω is defined by

$$W^{m,p}(\Omega) = \{ u \in L^p(\Omega) / \ D^\alpha u \in L^p(\Omega) \ \text{for} \ \ 0 \leq |\alpha| \leq m \},$$

where

$$\alpha = \{\alpha_1, \ldots, \alpha_n\}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n, \quad D^\alpha u = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}. \tag{1.2}$$

The derivatives $D^\alpha u$ are taken in the sense of distributions on Ω, i.e., $D^\alpha u = v_\alpha$ in the weak sense provided $v_\alpha \in L^1_{loc}(\Omega)$ satisfies

$$\int_\Omega u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega v_\alpha(x) \varphi(x) dx, \tag{1.3}$$

for every $\varphi \in C_0^\infty(\Omega)$.

In particular case if $p = 2$, we set

$$H^m(\Omega) = W^{m,2}(\Omega).$$

Remark 1.1.1. *For m_1 and m_2, two integers such that $m_1 < m_2$, we observe strict inclusions*

$$H^{m_2}(\Omega) \subset H^{m_1}(\Omega) \subset L^2(\Omega) = H^0(\Omega).$$

Theorem 1.1.1. *Let m be a positive integer and $1 \leq p < +\infty$.*

(i) The space $W^{m,p}(\Omega)$ equipped with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}, \tag{1.4}$$

is a Banach space.

(ii) $H^m(\Omega)$ is a Hilbert space for the scalar product

$$< u, v >_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \langle D^\alpha u, D^\alpha v \rangle_{L^2(\Omega)},$$

associated to the norm

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}. \tag{1.5}$$

Definition 1.1.3. *For m being a positive integer and $1 \leq p < +\infty$.*

(i) $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$, i.e.,

$$W_0^{m,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{m,p}(\Omega)}.$$

(ii) $H_0^m(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the space $H^m(\Omega)$, i.e.,

$$H_0^m(\Omega) = \overline{C_0^\infty(\Omega)}^{H^m(\Omega)}.$$

Proposition 1.1.1. ***(Poincaré's inequality).*** *Let Ω be a bounded open set and $1 \leq p < +\infty$. Then, there exists a constant $C > 0$ such that*

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega).$$

In other words, on $W_0^{1,p}(\Omega)$, the quantity $\|\nabla u\|_{L^p(\Omega)}$ is a norm equivalent to the $W^{1,p}(\Omega)$ norm.

Definition 1.1.4. ***(Fractional order Sobolev spaces).*** *Let s be a real number, we define*

$$H^s(\mathbb{R}^n) = \left\{u \Big/ u \in \mathcal{S}'(\mathbb{R}^n),\ (1+|\xi|^2)^{\frac{s}{2}}\hat{u} \in L^2(\mathbb{R}^n)\right\}, \tag{1.6}$$

where $|\xi|^2 = \xi_1^2 + \cdots + \xi_n^2$ and $\mathcal{S}'$ is dual space of

$$\mathcal{S} = \left\{u \Big/ x^\alpha D^\beta u \in L^2(\mathbb{R}^n)\ \forall\alpha\ \forall\beta\right\},$$

with $x^\alpha = x_1^{\alpha_1} \ldots\ x_n^{\alpha_n}$.

Theorem 1.1.2. *The space $H^s(\mathbb{R}^n)$ equipped with the norm*

$$\|u\|_{H^s(\mathbb{R}^n)} = \left\|(1+|\xi|^2)^{\frac{s}{2}}\hat{u}\right\|_{L^2(\mathbb{R}^n)},$$

is a Hilbert space.

Definition 1.1.5. *For any real number s and arbitrary domain $\Omega \subset \mathbb{R}^n$.*

(i) $H^s(\Omega)$ consists of restrictions $u|_\Omega$ of elements $u \in H^s(\mathbb{R}^n)$ and is normed by

$$\|f\|_{H^s(\Omega)} = \inf\left\{\ \|u\|_{H^s(\mathbb{R}^n)}\ /\ u|_\Omega = f,\ u \in H^s(\mathbb{R}^n)\right\}.$$

(ii) $H_0^s(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the space $H^s(\Omega)$, i.e.,

$$H_0^s(\Omega) = \overline{C_0^\infty(\Omega)}^{H^s(\Omega)}.$$

1.2 Fractional derivation

In this section, we present the definitions and properties of fractional integrals and fractional derivatives of a function f with respect to another function ψ. Some of these definitions and results were provided in [12, 65, 99].

1.2.1 Gamma function

The Gamma function is a fundamental element of fractional calculus, playing an essential role in the theory. More detailed information may be found in [65, Section 1.1.5].

Definition 1.2.1. *The Gamma function, denoted by $\Gamma(x)$, is defined for any complex number z such that $Re(z) > 0$ by*

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt,$$

where $t^{z-1} = e^{(z-1)\ln(t)}$.

Proposition 1.2.1. *For all $z \in \mathbb{C}$, $Re(z) > 0$ we have*

$$\Gamma(z+1) = z\Gamma(z).$$

1.2.2 Fractional integrals and fractional derivatives

Definition 1.2.2. *[12, 99] Let (a,b) be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Let $\psi(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi^{'}(x)$ on (a,b). The fractional integrals of a function f with respect to another function ψ on $[a,b]$ are defined by*

$$I_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi^{'}(t)(\psi(x)-\psi(t))^{\alpha-1} f(t)dt,\ x \in (a,b), \tag{1.7}$$

where f is an integrable function defined on $[a,b]$.

Lemma 1.2.1. *Let $\alpha > 0$ and $\beta > 0$. Then, we have*

$$I_{a+}^{\alpha,\psi} I_{a+}^{\beta,\psi} f(x) = I_{a+}^{\alpha+\beta,\psi} f(x),\ x \in (a,b).$$

Definition 1.2.3. *Let $\psi'(x) \neq 0$ $(-\infty \leq a < x < b \leq +\infty)$ and $\alpha > 0$, $n \in \mathbb{N}$. The Riemann-Liouville derivative of a function f with respect to ψ of order α correspondent to the Riemann-Liouville, is defined by*

$$\begin{aligned} D_{a+}^{\alpha,\psi} f(x) &= \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{a+}^{n-\alpha,\psi} f(x) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \int_a^x \psi^{'}(t)(\psi(x)-\psi(t))^{n-\alpha-1} f(t)dt,\ \ x \in (a,b). \end{aligned}$$

1.2.3 Caputo-type and Hilfer-type fractional derivatives

Definition 1.2.4. *[12] Let $\alpha > 0$, $n \in \mathbb{N}$, I is the interval $-\infty \leq a < b \leq +\infty$, f, $\psi \in C^n(I)$ two functions such that ψ is increasing and $\psi' \neq 0$ on I. The ψ-Caputo fractional derivative of order α of a function f is given by*

$${}^C D_{a+}^{\alpha,\psi} f(x) = I_{a+}^{n-\alpha,\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n f(x),\ x \in (a,b),$$

where,

$$n = [\alpha]+1 \ \ for \ \ \alpha \notin \mathbb{N}, \quad n = \alpha \ \ for \ \ \alpha \in \mathbb{N}.$$

Remark 1.2.1. *To simplify notation, we will use the abbreviated symbol*

$$f_{\psi}^{[n]}(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n f(x),\ x \in (a,b);$$

it is clear that, given $\alpha = n \in \mathbb{N}$

$${}^C D_{a+}^{\alpha,\psi} f(x) = f_{\psi}^{[n]}(x),$$

and, if $\alpha \notin \mathbb{N}$, then

$${}^C D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{n-\alpha-1} f_{\psi}^{[n]}(t)dt.$$

In particular, when $\alpha \in (0,1)$, we have

$$^{C}D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (\psi(x)-\psi(t))^{-\alpha} f'(t)dt.$$

Definition 1.2.5. *[99] Let $n-1 < \alpha < n$ with $n \in \mathbb{N}$, $I = [a,b]$ be the interval such that $-\infty \leq a < b \leq +\infty$ and $f, \psi \in C^n(I)$ two functions such that ψ is increasing and $\psi' \neq 0$ on I. The ψ-Hilfer fractional derivative of order α and type $0 \leq \beta \leq 1$ of a function f is defined by*

$$^{H}D_{a+}^{\alpha,\beta,\psi} f(x) = I_{a+}^{\beta(n-\alpha),\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n I_{a+}^{(1-\beta)(n-\alpha),\psi} f(x), \ x \in (a,b). \tag{1.8}$$

1.2.4 Some properties of fractional derivation

In what follows, we present relationships between different types of fractional derivatives and fractional integrals. For further details, we refer to [12, 99].

Theorem 1.2.1. *If $f \in C^n(I)$ and $\alpha > 0$, then*

$$^{C}D_{a+}^{\alpha,\psi} f(x) = D_{a+}^{\alpha,\psi} \left[f(x) - \sum_{k=0}^{n-1} \frac{(\psi(x)-\psi(a))^k}{k!} f_{\psi}^{[k]}(a) \right],$$

and

$$I_{a+}^{\alpha,\psi}\ {}^{C}D_{a+}^{\alpha,\psi} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(\psi(x)-\psi(a))^k}{k!} f_{\psi}^{[k]}(a).$$

Theorem 1.2.2. *If $f \in C^n(I)$, $n-1 < \alpha < n$ and $0 \leq \beta \leq 1$, then*

$$^{H}D_{a+}^{\alpha,\beta,\psi} f(x) = D_{a+}^{n-\beta(n-\alpha),\psi} \left[I_{a+}^{(1-\beta)(n-\alpha),\psi} f(x) - \sum_{k=0}^{n-1} \frac{(\psi(x)-\psi(a))^k}{k!} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^k I_{a+}^{(1-\beta)(k-\alpha),\psi} f(a) \right],$$

and

$$I_{a+}^{\alpha,\psi}\ {}^{H}D_{a+}^{\alpha,\beta,\psi} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(\psi(x)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\psi}^{[n-k]} I_{a+}^{(1-\beta)(n-\alpha),\psi} f(a),$$

where $\gamma = \alpha + \beta(n-\alpha)$.

Theorem 1.2.3. *Let $f \in C^1(I)$, $\alpha > 0$ and $0 \leq \beta \leq 1$, we have*

$$^{C}D_{a+}^{\alpha,\psi} I_{a+}^{\alpha,\psi} f(x) = f(x) \ \text{ and } \ ^{H}D_{a+}^{\alpha,\beta,\psi} I_{a+}^{\alpha,\psi} f(x) = f(x).$$

Remark 1.2.2. *If $\psi(x) = x$, then all the definitions mentioned above coincide with the definition of the classical fractional derivative and integral; see [65, 97]. Therefore, we can write:*

$$I_{a+}^{\alpha,x} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt = I_{a+}^{\alpha} f(x),$$

$$D_{a+}^{\alpha,x} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_a^x (x-t)^{n-\alpha-1} f(t)dt = {}^{RL}D_{a+}^{\alpha} f(x),$$

$$ {}^{C}D_{a+}^{\alpha,x} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} \left(\frac{d}{dt}\right)^n f(t)dt = {}^{C}D_{a+}^{\alpha} f(x), $$

$$ {}^{H}D_{a+}^{\alpha,\beta,x} f(x) = I_{a+}^{\beta(n-\alpha)} \left(\frac{d}{dx}\right)^n I_{a+}^{(1-\beta)(n-\alpha)} f(x) = {}^{H}D_{a+}^{\alpha,\beta} f(x). $$

1.3 Semigroups of bounded linear operators

For detailed proofs in this section, we refer the reader to [87, Chapter 1].

Definition 1.3.1. *A one-parameter family $(T(t))_{t\geq 0} \in \mathcal{L}(X)$ of bounded linear operators is a semigroup of bounded operators on X if*

(i) $T(0) = I_X$,

(ii) $T(t+s) = T(t)T(s)$ *for all* $t, s \geq 0$.

Definition 1.3.2. *A one-parameter family $(T(t))_{t\geq 0} \in \mathcal{L}(X)$ of bounded operators on X is a uniformly continuous semigroup if*

$$ \lim_{t \longrightarrow 0^+} \|T(t) - I_X\| = 0. $$

Definition 1.3.3. *The linear operator $A : D(A) \subset X \longrightarrow X$ is defined by*

$$ D(A) = \left\{ x \in X \middle/ \lim_{t \longrightarrow 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}, $$

and

$$ Ax = \lim_{t \longrightarrow 0^+} \frac{T(t)x - x}{t}, \text{ for all } x \in X, $$

is the infinitesimal generator of the semigroup $(T(t))_{t\geq 0}$.

Theorem 1.3.1. *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

Definition 1.3.4. *A semigroup $(T(t))_{t\geq 0} \in \mathcal{L}(X)$ of bounded operators on X is a strongly continuous semigroup (or C_0- semigroup) if*

$$ \lim_{t \longrightarrow 0^+} T(t)x - x = 0, \text{ for all } x \in X. $$

Lemma 1.3.1. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X. Then*

(i) There exist constants $\nu \geq 0$ and $M \geq 1$ such that

$$ \|T(t)\| \leq Me^{\nu t}, \text{ for all } t \geq 0. $$

(ii) For every $x \in X$, $t \mapsto T(t)x$ is a continuous function from $[0, +\infty)$ into X.

Theorem 1.3.2. *Assume that $(T(t))_{t\geq 0}$ is a C_0-semigroup on X and let $A : D(A) \subset X \longrightarrow X$ be its infinitesimal generator. Then*

(i) $T(t)x \in D(A)$, *for* $x \in D(A)$ *and* $t \geq 0$. *Moreover, for* $x \in D(A)$ *the function* $[0,+\infty) \ni t \mapsto T(t)x$ *is differentiable and*
$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

(ii) For $x \in D(A)$ *and* $0 \leq s \leq t < +\infty$,
$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau.$$

(iii) $\cap_{n\geq 1} D(A^n)$ *is dense in* X.

(vi) If $\|T(t)\| \leq Me^{\nu t}$, $t \geq 0$, *for some* $M \geq 1$ *and* $\nu \in \mathbb{R}$, *then for all* $x \in X$ *and* $\lambda \in \mathbb{C}$ *with* $Re(\lambda) > \nu$ *we have*
$$R(\lambda, A)x := (\lambda I_X - A)^{-1} x = \int_0^{+\infty} e^{-\lambda t}T(t)xdt.$$

Theorem 1.3.3. *Let* $A : D(A) \subset X \longrightarrow X$ *be the infinitesimal generator of two* C_0*-semigroups* $(T(t))_{t\geq 0}$ *and* $(S(t))_{t\geq 0}$. *Then*
$$T(t) = S(t), \text{ for all } t \geq 0.$$

Definition 1.3.5. *[84] A* C_0*-semigroup* $(T(t))_{t\geq 0}$ *is said to be a uniformly exponentially stable if there exist constants* $M > 0$ *and* $\nu > 0$ *such that*
$$\|T(t)\| \leq Me^{-\nu t}, \text{ for all } t \geq 0. \tag{1.9}$$

Moreover, we define
$$\nu_0 = \inf\Big\{ \nu \in \mathbb{R} \Big/ \exists M > 0 \text{ such that } \|T(t)\| \leq Me^{\nu t}, \forall t \geq 0\Big\}.$$

1.3.1 Compact semigroup

Definition 1.3.6. *A* C_0*-semigroup* $(T(t))_{t\geq 0}$ *is called compact for* $t > t_0$ *if for every* $t > t_0$, $T(t)$ *is a compact operator.* $(T(t))_{t\geq 0}$ *is called compact if it is compact for* $t > 0$.

We need to clarify the relationship between the compactness of the semigroup $(T(t))_{t\geq 0}$ and its continuity, which is expressed via the following lemma. The proof is detailed in [87].

Lemma 1.3.2. *Let* $(T(t))_{t\geq 0}$ *be a* C_0 *semigroup and let* A *be its infinitesimal generator.* $(T(t))_{t\geq 0}$ *is a compact semigroup if and only if*

(i) $T(t)$ *is continuous in the uniform operator topology for* $t > 0$, *and*

(ii) $R(\lambda, A)$ *is compact for* $\lambda \in \rho(A)$.

Corollary 1.3.1. *Let* $(T(t))_{t\geq 0}$ *be a uniformly continuous semigroup.* $(T(t))_{t\geq 0}$ *is a compact semigroup if and only if* $R(\lambda, A)$ *is compact for every* $\lambda \in \rho(A)$.

1.3.2 Existence of semigroups

Theorem 1.3.4. ***(Hille-Yosida).*** *If $A : D(A) \subset X \longrightarrow X$ is a linear operator, then the following conditions are equivalent:*

(i) A is the infinitesimal generator of a C_0-semigroup of contractions, i.e., A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ such that,

$$\|T(t)\| \leq 1 \text{ for all } t \geq 0.$$

(ii) (a) A is closed and $\overline{D(A)} = X$,

(b) the resolvent set $\rho(A)$ of A contains $(0, +\infty)$ and for every $\lambda > 0$

$$\left\|(\lambda I_X - A)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}.$$

(iii) (a) A is closed and $\overline{D(A)} = X$,

(b) the resolvent set $\rho(A)$ of A contains the half plane $\{\lambda \in \mathbb{C} / Re(\lambda) > 0\}$ and for such λ

$$\left\|(\lambda I_X - A)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{1}{Re(\lambda)}.$$

Theorem 1.3.5. ***(Feller-Miyadera-Phillips).*** *If $A : D(A) \subset X \longrightarrow X$ is a linear operator and $M \geq 1, \nu \in \mathbb{R}$ are constants, then the following conditions are equivalent:*

(i) A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ such that,

$$\|T(t)\| \leq M e^{\nu t} \text{ for all } t \geq 0.$$

(ii) (a) A is closed and $\overline{D(A)} = X$,

(b) the resolvent set $\rho(A)$ of A contains $(\nu, +\infty)$ and and for every $\lambda > \nu$ and $n \in \mathbb{N}$

$$\|(\lambda I_X - A)^{-n}\|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda - \nu)^n}.$$

(iii) (a) A is closed and $\overline{D(A)} = X$,

(b) the resolvent set $\rho(A)$ of A contains the half plane $\{\lambda \in \mathbb{C} \ / \ Re(\lambda) > \nu\}$ and for such λ and $n \in \mathbb{N}$

$$\|(\lambda I_X - A)^{-n}\|_{\mathcal{L}(X)} \leq \frac{M}{(Re(\lambda) - \nu)^n}.$$

1.3.3 Analytic semigroups

For the results in this section we refer the reader to [87, Chapter 2].

Definition 1.3.7. *A semigroup $(T(t))_{t\geq 0}$ is called an analytic if there exist a sector on the complex plane*

$$\Delta_\delta = \{z \in \mathbb{C} / \ \delta_1 < arg(z) < \delta_2, \ \delta_1 < 0 < \delta_2\},$$

and a family of bounded linear operators $(T(z))_{z\in\Delta_\delta}$ which coincide with $T(t)$ for $t \geq 0$, such that

(i) the mapping $z \longmapsto T(z)$ is analytic in Δ_δ,

(ii) $T(0) = I_X$ and $\lim_{z\longrightarrow 0, z\in\Delta_\delta} T(z)x = x$ for all $x \in X$,

(iii) $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Delta_\delta$.

Definition 1.3.8. *[82] Let $0 < \delta < \frac{\pi}{2}$, $M \geq 1$ and $a \in \mathbb{R}$. We say that an operator $A : D(A) \subset X \longrightarrow X$ is sectorial if*

(i) A is a densely defined closed operator,

(ii) the resolvent set $\rho(A)$ contains the sector

$$S_{a,\delta} = \{\lambda \ / \ \delta \leq |arg(\lambda - a)| \leq \pi, \lambda \neq a\},$$

and the estimate

$$\left\|(\lambda I_X - A)^{-1}\right\| \leq \frac{M}{|\lambda - a|}$$

holds for all $\lambda \in S_{a,\delta}$.

Theorem 1.3.6. *Let $A : D(A) \subset X \longrightarrow X$ be a linear operator. Then the following conditions are equivalent:*

(i) A is the infinitesimal generator of an analytic semigroup.

(ii) $-A$ is a sectorial operator in X.

Theorem 1.3.7. *Let A be the infinitesimal generator of an analytic semigroup. If B is a bounded linear operator then $A + B$ is the infinitesimal generator of an analytic semigroup.*

1.4 Fractional powers of closed operators

To characterize the fractional powers of linear operators, we use the following assumption:

$$(H) : \begin{cases} \text{Let } A \text{ be a densely defined closed linear operator for which:} \\ (i) \ \ \rho(A) \supset \sum^+ = \{\lambda \in \mathbb{C} / \ 0 < \omega < |arg(\lambda)| \leq \pi\} \cup V, \text{ where } V \text{ is a neighborhood of zero,} \\ (ii) \ \ \|R(\lambda, A)\| \leq \frac{M}{1+|\lambda|}, \ \ \text{for } \lambda \in \sum^+. \end{cases}$$

If $M = 1$ and $\omega = \frac{\pi}{2}$, then $-A$ generates a C_0-semigroup. For $\omega < \frac{\pi}{2}$, $-A$ generates an analytic semigroup; see [87, Theorem 2.5.2].

1.4.1 Negative fractional powers of linear operators

Let A be an operator satisfying assumption (H), and let $\theta > 0$. If $\omega < \frac{\pi}{2}$, i.e., $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$. The negative fractional powers of A are given by

$$A^{-\theta} := \frac{1}{\Gamma(\theta)} \int_0^\infty t^{\theta-1} T(t)\, dt, \tag{1.10}$$

where the integral converges in the uniform operator topology for every $\theta > 0$. For alternative representations of $A^{-\theta}$ involving the Dunford integral or real line resolvent integrals, we refer the reader to [14, 87].

Remark 1.4.1. *In the subsequent discussion, if $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$, we adopt Equation (1.10) as the definition of $A^{-\theta}$ for $\theta > 0$, while setting*

$$A^0 := I_X.$$

Lemma 1.4.1. *Suppose A satisfies Assumption (H) with $\omega < \frac{\pi}{2}$. Then, we have*

(i) For $\theta_1, \theta_2 \geq 0$

$$A^{-(\theta_1+\theta_2)} = A^{-\theta_1} \cdot A^{-\theta_2}.$$

(ii) There exists a constant C such that

$$\left\| A^{-\theta} \right\| \leq C, \quad \textit{for } 0 \leq \theta \leq 1.$$

(iii) $A^{-\theta}$ is one-to-one.

1.4.2 Positive fractional powers of linear operators

Definition 1.4.1. *Let A satisfies Assumption (H) with $\omega < \frac{\pi}{2}$. For every $\theta > 0$ we define*

$$A^\theta = \begin{cases} \left(A^{-\theta}\right)^{-1}, & \theta > 0, \\ I_X, & \theta = 0. \end{cases}$$

Here are some properties of these operators.

Theorem 1.4.1. *Let A^θ be defined by Definition 1.4.1. Then,*

(i) A^θ is a closed operator with domain $D(A^\theta) = R(A^{-\theta})$.

(ii) For $\theta_1 \geq \theta_2 > 0$ implies $D(A^{\theta_1}) \subset D(A^{\theta_2})$.

(iii) If θ_1, θ_2 are real then

$$A^{\theta_1+\theta_2} = A^{\theta_1} \cdot A^{\theta_2},$$

for every $x \in D(A^\theta)$ where $\theta = max(\theta_1, \theta_2, \theta_1 + \theta_2)$.

For $x \in D(A) \subset D(A^\theta)$ and $0 < \theta < 1$, we can explicitly define the operators $A^\theta x$.

Lemma 1.4.2. ***(Balakrishnan's formula).*** *Let $0 < \theta < 1$. If $x \in D(A) \subset D(A^\theta)$ then*

$$A^\theta x = \frac{\sin(\pi\theta)}{\pi} \int_0^{+\infty} t^{\theta-1} A(tI_X + A)^{-1} x dt. \tag{1.11}$$

Theorem 1.4.2. *Assume that $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$, and that $\|T(t)\| \leq Me^{-\delta t}$ for all $t \geq 0$ and some $\delta > 0$. If $0 \in \rho(A)$, then one has*

(i) $T(t) : X \to D(A^\theta)$ *for every $t > 0$ and $\theta \geq 0$.*

(ii) For every $t \geq 0$ and $x \in D(A^\theta)$, we have

$$T(t)A^\theta x = A^\theta T(t)x.$$

(iii) For every $t > 0$, the operator $A^\theta T(t)$ is bounded and

$$\|A^\theta T(t)\| \leq M_\theta t^{-\theta} e^{-\delta t}. \tag{1.12}$$

(vi) Let $0 < \theta < 1$ and $x \in D(A^\theta)$, then

$$\|T(t)x - x\| \leq C_\theta t^\theta \|A^\theta x\|.$$

The following result, as cited in [98, p. 15], will be needed in Chapter 2.

Theorem 1.4.3. *If $A : D(A) \subset H \longrightarrow H$ is a positive definite self-adjoint operator in a Hilbert space H, then the operator $A^\theta : D(A^\theta) \subset H \longrightarrow H$ is positive definite self-adjoint for each $\theta > 0$.*

1.5 Interpolation spaces

For further details on this section, we refer the reader to [79, 80, 82, 105].

Definition 1.5.1. ***(Intermediate and interpolation spaces).*** *Let X, Y, Z be Banach spaces. The space Z is called an* intermediate space *between X and Y if*

$$Y \subset Z \subset X,$$

with continuous embeddings. Furthermore, Z is called an interpolation space *between X and Y if, for every linear operator $T \in \mathcal{L}(X)$ such that the restriction $T|_Y \in \mathcal{L}(Y)$, it follows that $T|_Z \in \mathcal{L}(Z)$.*

Now, we provide specific characterizations of interpolation spaces.

Definition 1.5.2. *Let X and Y be two Banach spaces with $Y \subset X$, and let $C > 0$ be such that*

$$\|x\|_X \leq C\,\|x\|_Y, \text{ for all } x \in Y.$$

Let $0 < \theta \leq 1$ and $1 \leq p \leq +\infty$. We define:

(i)
$$(X,Y)_{\theta,p} = \left\{ x \in X \;\Big/ t \mapsto t^{-\theta-1/p} K(t,x,X,Y) \in L^p(0,+\infty) \right\}.$$

(ii) The norm on $(X,Y)_{\theta,p}$ is given by
$$\|x\|_{(X,Y)_{\theta,p}} = \left\| t^{-\theta-1/p} K(t,x,X,Y) \right\|_{L^p(0,+\infty)}.$$

(iii) The space
$$(X,Y)_{\theta} = \left\{ x \in X \;\Big/ \lim_{t \to 0} t^{-\theta} K(t,x,X,Y) = 0 \right\}.$$

Here, for every $x \in X$ and $t > 0$, the function $K(t,x,X,Y)$ is defined by

$$K(t,x,X,Y) = \inf\{\, \|a\|_X + t\,\|b\|_Y \,/\, x = a + b,\ a \in X,\ b \in Y \}.$$

For the proofs of the following proposition, refer to A. Lunardi (1995) [82, Chapter 1].

Proposition 1.5.1. *Let $\theta, \theta_1, \theta_2, p_1$ and p_2 be real numbers.*

(i) If $0 < \theta < 1$, $1 \leq p_1 \leq p_2 \leq +\infty$ then
$$(X,Y)_{\theta,p_1} \subset (X,Y)_{\theta,p_2} \subset (X,Y)_{\theta} \subset (X,Y)_{\theta,+\infty}.$$

(ii) If $0 < \theta_1 < \theta_2 \leq 1$, then $(X,Y)_{\theta_2,+\infty} \subset (X,Y)_{\theta_1,1}$.

Regarding fractional power operators, here is another definition that will be essential.

Definition 1.5.3. *Let $0 < \theta < 1$, $1 \leq p \leq +\infty$, and A be a closed linear operator with its domain $D(A) \subset X$, we define the intermediate space between $D(A)$ and X by*

$$D_A(\theta,p) = (D(A),X)_{1-\theta,p}.$$

Following [40], when the operator A satisfies certain additional assumptions, it is then possible to provide explicit characterizations of $D_A(\theta,p)$. To accomplish this, it is necessary to utilize the space $L^p(\mathbb{R}^+,X)$ as defined by

$$L^p(\mathbb{R}^+,X) = \left\{ u : \mathbb{R}^+ \longrightarrow X \Big/ \; \|u\|_{L^p(\mathbb{R}^+,X)} = \left(\int_0^{+\infty} \|u(t)\|^p\, dt \right)^{\frac{1}{p}} < +\infty \right\},$$

with the usual modification for $p = +\infty$; that is

$$L^\infty(\mathbb{R}^+,X) = \left\{ u : \mathbb{R}^+ \longrightarrow X \Big/ \; \|u\|_{L^\infty(\mathbb{R}^+,X)} = \sup_{t \geq 0} \|u(t)\| < +\infty \right\}.$$

Theorem 1.5.1. *Let $0<\theta<1$, $1\le p\le+\infty$. Assume that $\rho(A)\supset\mathbb{R}^+$ and that there exists a constant $C>0$ such that*

$$\left\|\left(A-\lambda I_X\right)^{-1}\right\|_{\mathcal{L}(X)}\le\frac{C}{\lambda},\ \text{for all } \lambda>0,$$

then

$$D_A(\theta,p)=\left\{x\in X\Big/\ t^{\theta-1/p}A\left(A-tI_X\right)^{-1}x\in L^p\left(\mathbb{R}^+,X\right)\right\},$$

and

$$D_A(\theta,+\infty)=\left\{x\in X\Bigg/\sup_{t>0}\left\|t^{\theta}A\left(A-tI_X\right)^{-1}x\right\|<+\infty\right\},$$

equipped with the norm

$$\|x\|_{D_A(\theta,+\infty)}=\|x\|+\sup_{t>0}\left\|t^{\theta}A\left(A-tI_X\right)^{-1}x\right\|.$$

Lemma 1.5.1. *Let $A:D(A)\subset H\longrightarrow H$ be a positive definite self-adjoint operator in a Hilbert space H. Let α and β be two positive constants. Then, for $0<\theta<1$,*

$$\left[D\left(A^{\alpha}\right),D\left(A^{\beta}\right)\right]_{\theta}:=\left(D\left(A^{\alpha}\right),D\left(A^{\beta}\right)\right)_{\theta,2}=D\left(A^{\alpha(1-\theta)+\theta\beta}\right).$$

In particular case where $\beta=0$, we have

$$D_A(\theta,2)=\left(D\left(A^{\alpha}\right),X\right)_{\theta,2}=D\left(A^{\alpha(1-\theta)}\right).$$

Proof. See H. Triebel (1995) [105, Page 142]. □

Remark 1.5.1. *According to Remark 1.1.1, for $s\ge0$, the space $H^s(\Omega)$ may also be defined as the interpolation space between $H^m(\Omega)$ and $L^2(\Omega)$, given by*

$$H^s(\Omega)=\left[H^m(\Omega),L^2(\Omega)\right]_{\theta},\ \ (1-\theta)m=s,\ m\text{ integer},\ 0<\theta<1.$$

For a more detailed discussion, we refer the reader to [80].

1.6 Trace theorem

The proofs in this section can be found in [80, Chapter 1].

Definition 1.6.1. *Let X and Y be two separable Hilbert spaces such that $Y\subset X$ and Y dense in X with continuous injection. For an integer $m\ge1$, we denote by $W\left(\mathbb{R}^+,Y,X\right)$ the classes of functions u such that*

$$W\left(\mathbb{R}^+,Y,X\right)=\left\{u\Bigg/\ u\in L^2\left(\mathbb{R}^+,Y\right),\ \frac{d^mu}{dt^m}=u^{(m)}\in L^2\left(\mathbb{R}^+,X\right)\right\},$$

where $u^{(m)}$ is taken in the sense of distributions, and the space is equipped with the norm

$$\|u\|_{W(\mathbb{R}^+,Y,X)}=\left(\|u\|^2_{L^2(\mathbb{R}^+,Y)}+\left\|u^{(m)}\right\|^2_{L^2(\mathbb{R}^+,X)}\right)^{\frac{1}{2}}.$$

Theorem 1.6.1. *For $u \in W^m(\mathbb{R}^+, Y, X)$, we have*

$$u^{(j)} \in C_b\left(\mathbb{R}^+, [Y, X]_{(j+1/2)/m}\right), \quad 0 \leq j \leq m-1,$$

and $u \mapsto u^{(j)}$ being a continuous and linear mapping of

$$W^m(\mathbb{R}^+, Y, X) \longrightarrow C_b\left(\mathbb{R}^+, [Y, X]_{(j+1/2)/m}\right).$$

Theorem 1.6.2. *Let $u \in W\left(\mathbb{R}^+, Y, X\right)$, we have*

$$u^{(j)}(0) \in [Y, X]_{(j+1/2)/m}, \quad 0 \leq j \leq m-1.$$

Moreover, the mapping

$$u \longmapsto \left\{u^{(j)}(0) \ / \ 0 \leq j \leq m-1\right\} \quad \text{of} \quad W\left(\mathbb{R}^+, Y, X\right) \longrightarrow \Pi_{j=0}^{m-1} [Y, X]_{(j+1/2)/m},$$

is surjective.

1.7 Fixed point theorems

For convenience, we recall essential fixed point theorems related to our study.

Definition 1.7.1. *[52] Let X, Y be topological spaces. A map $f : X \longrightarrow Y$ is called compact if $f(X)$ is contained in a compact subset of Y.*

Theorem 1.7.1. ***(Banach contraction principle [26, 52]).*** *Let (E, d) be a complete metric space and $f : E \to E$ be contractive. Then f has a unique fixed point u, and $\lim\limits_{n \to +\infty} f^n(y) \to u$, for each $y \in E$.*

Theorem 1.7.2. ***(Schauder's fixed point theorem [26, 52]).*** *Let Ω be a closed convex subset of a normed linear space and let $f : \Omega \to \Omega$ be a compact map. Then f has a fixed point.*

Theorem 1.7.3. ***(Krasnoselskii's fixed point theorem [27]).*** *Let Ω be a closed convex nonempty subset of a Banach space $(X, \|\cdot\|)$. Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ map Ω into X such that*

- *$\mathcal{A}_1 x + \mathcal{A}_2 y \in \Omega$ for every pair $x, y \in \Omega$,*
- *$\mathcal{A}_1$ is continuous and $\mathcal{A}_1(\Omega)$ is contained in a compact set,*
- *$\mathcal{A}_2$ is a contraction.*

Then, there exists $y \in \Omega$ with $\mathcal{A}_1 y + \mathcal{A}_2 y = y$.

Chapter 2

On a class of abstract fourth-order differential equations set on cusp domains

In this chapter, we concentrate on a boundary value problem set on a singular domain involving a cuspidial point. In our analysis, we obtain some existence results. We also study the boundary value problems for a class of the complete abstract fourth-order differential equations involving fractional powers of unbounded linear operators.

2.1 Introduction and motivation

In this section, we assume that $x = (x_1, x_2, x_3)$ is a generic point of $\mathbb{R}^3$. Let $\Pi \subseteq \mathbb{R}^3$ be a cusp domain defined by

$$\Pi := \left\{ x \in \mathbb{R}^3 \middle/ 0 < x_3 < 1,\ \left(\frac{x_1}{(x_3)^\alpha}, \frac{x_2}{(x_3)^\alpha} \right) \in \Omega \right\},$$

where $\Omega \subseteq \mathbb{R}^2$ is a bounded smooth domain and $\alpha > 1$.

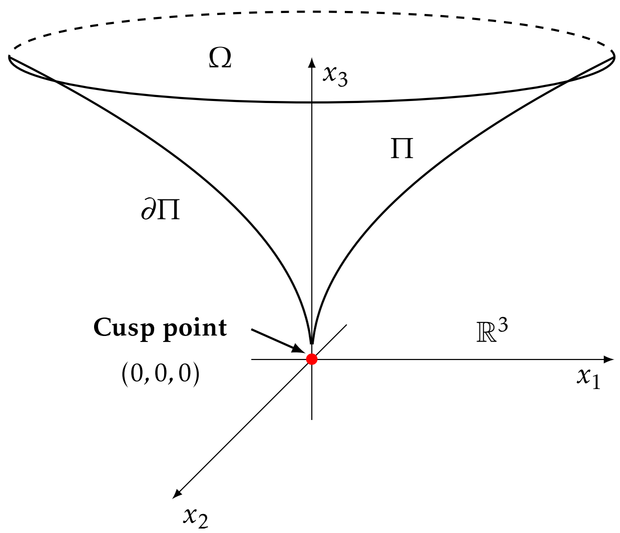

Fig. 1: The cusp domain Π, where Ω is the unit disk in $\mathbb{R}^2$ and $\alpha = 2$.

In the cusp domain $\mathbb{R}^+ \times \Pi$, we consider the following problem

$$\frac{d^4}{dt^4}u(t,x) + (1+\rho_4(x))(-\Delta)^{4\theta}u(t,x) + \sum_{j=1}^{3} \rho_j(x)(-\Delta)^{j\theta}\frac{d^{4-j}}{dt^{4-j}}u(t,x) = f(t,x), \tag{2.1}$$

where u is a function from $\mathbb{R}^+ \times \Pi$ into the complex plane $\mathbb{C}$, $\theta \in]0,1]$, and Δ is the classical Laplace operator on $\mathbb{R}^3$ defined by $\Delta = \sum_{i=1}^{3} \partial_{x_i}^2$. The functions $\rho_j(\cdot)$, $j = 1,2,3,4$, are continuous real functions defined on Π, such that

$$\lim_{x_3 \to 0^+} \left(\frac{1}{x_3}\right)^{\frac{3\alpha}{8}(4-j)} \rho_j(x) < +\infty, \quad j = 1,2,3,4. \tag{2.2}$$

The right hand side of equation (2.1) is assumed to belong to the Hilbert space $L^2(\mathbb{R}^+ \times \Pi) = L^2(\mathbb{R}^+, L^2(\Pi))$. We will also accompany to (2.1) some boundary conditions and initial conditions involving Laplace operator. More precisely, we look for a solution $u(\cdot)$ satisfying

$$u|_{\mathbb{R}^+ \times \partial\Pi} = 0, \tag{2.3}$$

$$\left.\frac{du}{dt}\right|_{\{0\}\times\Pi} = 0, \quad \left.\frac{d^3u}{dt^3} + b(-\Delta)^{3\theta}u\right|_{\{0\}\times\Pi} = 0, \quad b \in \mathbb{C}. \tag{2.4}$$

The first step is to transform the cusp domain $\mathbb{R}^+ \times \Pi$ into a cylindrical one. To do this, we consider the following change of variables

$$\begin{aligned} \Psi: \quad & \mathbb{R}^+ \times \Pi \to \mathbb{R}^+ \times Q \\ & (t,x) \mapsto (t,\xi), \end{aligned}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ is also a new generic point of $\mathbb{R}^3$ such that

$$\xi_1 = \frac{x_1}{(x_3)^\alpha}, \qquad \xi_2 = \frac{x_2}{(x_3)^\alpha}, \quad \text{and} \quad \xi_3 = \frac{(x_3)^{1-\alpha}}{\alpha - 1}. \tag{2.5}$$

Here,

$$Q = \Omega \times]\xi_{3,0}, +\infty[,$$

with $\xi_{3,0} = \frac{1}{\alpha-1} > 0$.

In this study, we confine ourselves to the neighborhood of the origin $0_{\mathbb{R}^3}$; this means that we consider the case in which $\xi_3 > \xi_{3,0}$ is large enough. At this level, let us introduce the following change of functions

$$v(t,\xi) = u(t,x), \qquad g(t,\xi) = f(t,x).$$

According to (2.5), it is easy to check that

$$f \in L^2(\mathbb{R}^+ \times \Pi) \quad \text{if and only if} \quad \left(\frac{\gamma}{\xi_3}\right)^{\frac{-3\alpha}{2\beta}} g \in L^2(\mathbb{R}^+ \times Q)), \tag{2.6}$$

where
$$\beta = 1/\gamma = \alpha - 1.$$
To avoid the use of weighted L^2-spaces, we opt for the use of a new change of functions given by
$$w = \left(\frac{\gamma}{\xi_3}\right)^{-s} v, \quad h = \left(\frac{\gamma}{\xi_3}\right)^{\frac{-3\alpha}{2\beta}} g,$$
with
$$s = \frac{\alpha}{\beta}\left(\frac{3}{8\theta} + 2\right).$$
As a direct consequence, the problem (2.1)-(2.3)-(2.4) is written as follows
$$\mathcal{P}_1(\xi_3)\frac{d^4}{dt^4}w(t,\xi) + (1+\sigma_4(\xi))(\mathcal{L})^{4\theta}w(t,\xi) + \sum_{j=1}^{3}\left(\sigma_j(\xi)(\mathcal{L})^{j\theta}\right)\frac{d^{4-j}}{dt^{4-j}}w(t,\xi) = h(t,\xi), \tag{2.7}$$
$$w|_{\mathbb{R}^+\times\partial Q} = 0, \tag{2.8}$$
and
$$\left.\frac{dw}{dt}\right|_{\{0\}\times Q} = 0, \qquad \left.\mathcal{P}_2(\xi_3)\frac{d^3w}{dt^3} + b(\mathcal{L})^{3\theta}w\right|_{\{0\}\times Q} = 0. \tag{2.9}$$
Here
$$\mathcal{P}_1(\xi_3) = \left(\frac{\gamma}{\xi_3}\right)^{\frac{\alpha}{\beta}\left(\frac{3}{8\theta}+\frac{1}{2}\right)}, \quad \mathcal{P}_2(\xi_3) = \left(\frac{\gamma}{\xi_3}\right)^{\frac{\alpha}{\beta}\left(\frac{3}{8\theta}+\frac{7}{8}\right)},$$
and
$$\mathcal{L} = -\Delta + \frac{1}{\xi_3}\mathcal{M}, \ \xi_3 > \xi_{3,0} > 0,$$
where $\mathcal{M}$ is the second-order differential operator with smooth coefficients given by
$$\begin{aligned}(\mathcal{M}w)(\xi) = {}& \frac{(\alpha\gamma)^2}{\xi_3}\left\{\xi_1^2\partial_{\xi_1}^2 w + \xi_2^2\partial_{\xi_2}^2 w + 2\xi_1\xi_2\partial_{\xi_1\xi_2}^2 w\right\} + 2\alpha\gamma\left\{\xi_1\partial_{\xi_1\xi_3}^2 w + \xi_2\partial_{\xi_2\xi_3}^2 w\right\} \\ & + (\alpha\gamma - 2s)\partial_{\xi_3} w + \frac{\alpha\gamma}{\xi_3}((\alpha+1)\gamma - 2s)\left\{\xi_1\partial_{\xi_1} w + \xi_2\partial_{\xi_2} w\right\} - \frac{s}{\xi_3}\{s + 1 + \alpha\gamma\} w.\end{aligned}$$
Note also that the family of functions $\sigma_j(\xi)$, $j \in \{1,2,3,4\}$ are defined as follows
$$\left(\frac{\gamma}{\xi_3}\right)^{\frac{3\alpha}{8\beta}(4-j)}\sigma_j(\xi) = \rho_j(x), \ j \in \{1,2,3,4\}.$$
Due to the change of variables Ψ defined by (2.5) and the condition (2.2), these functions are bounded on Q.

Observe that the study of (2.7)-(2.8)-(2.9) needs the investigation of the following abstract problem
$$\frac{d^4w(t)}{dt^4} + A^{4\theta}w(t) + \sum_{j=1}^{4} A_j\frac{d^{4-j}w(t)}{dt^{4-j}} = h(t), \ t \in \mathbb{R}^+, \tag{2.10}$$

endowed with the initial conditions

$$\frac{dw(0)}{dt} = 0, \quad \frac{d^3w(0)}{dt^3} + Kw(0) = 0. \tag{2.11}$$

where the vector-valued functions w and h are defined by

$$\begin{aligned} w : \mathbb{R}^+ \to H \,;\; t \to w(t) \,; \quad w(t)(\xi) = w(t,\xi), \\ h : \mathbb{R}^+ \to H \,;\; t \to h(t) \,; \quad h(t)(\xi) = h(t,\xi), \end{aligned}$$

with $H = L^2(Q)$. Here,

$$\begin{cases} (A\phi)(\xi) & := -\Delta\phi(\xi), \\ D(A) & := \left\{ \phi \in L^2(Q) \middle/ A\phi \in L^2(Q), \phi|_{\partial Q} = 0 \right\}, \end{cases} \tag{2.12}$$

and

$$\begin{cases} \left(A_j\psi\right)(\xi) & := \left[\sigma_j(\xi)(-\Delta)^{j\theta}\right]\phi(\xi), \quad j = 1,2,3,4, \\ D\left(A_j\right) & := \left\{ \phi \in L^2(Q) \middle/ A_j\phi \in L^2(Q), \phi|_{\partial Q} = 0 \right\}. \end{cases} \tag{2.13}$$

We define the operator K by

$$\begin{cases} (K\phi)(\xi) & := b\left[(-\Delta)^{3\theta}\right]\phi(\xi), \\ D(K) & := \left\{ \phi \in L^2(Q) \middle/ K\phi \in L^2(Q) \right\}. \end{cases} \tag{2.14}$$

Following [24] and [51], the fractional power of the operator (2.12) is well defined. Furthermore, we have the following practical characterization of $D(A^\theta)$ through the classical Sobolev spaces. For the reader convenience, we recall that

$$D(A^\theta) = \begin{cases} H^{2\theta}(Q), & 0 < \theta < 1/4, \\ H_{00}^{1/2}(Q), & \theta = 1/4, \\ H_0^{2\theta}(Q), & 1/4 < \theta \le 1/2, \\ H^{2\theta}(Q) \cap H_0^1(Q), & 1/2 < \theta \le 1; \end{cases} \tag{2.15}$$

here, $H_{00}^{1/2}(Q)$ is the interpolation space defined in [80, Chapter 1, p. 66].

2.2 Statement of the abstract problem

In this section, a particular attention is given to the study of a general class of the abstract fourth-order differential equations with operator coefficients posed in Hilbert spaces.

2.2.1 Preliminaries

We consider a complex separable Hilbert space H and a self-adjoint positive-definite operator A on H. By H_θ, $\theta \geq 0$ we denote the scale of Hilbert spaces generated by the operator A^θ, i.e.,

$$H_\theta := D\left(A^\theta\right); \left\langle x,y\right\rangle_\theta := \left\langle A^\theta x, A^\theta y\right\rangle,\ x,\ y \in D\left(A^\theta\right).$$

According to Theorem 1.4.3, it is well established that A^θ is a self-adjoint positive definite operator for $\theta > 0$. This allows us to define the Sobolev space $W^{4,\theta}(\mathbb{R}^+, H)$ as follows

$$W^{4,\theta}(\mathbb{R}^+, H) := \left\{w \Big/ w^{(4)} \in L^2(\mathbb{R}^+, H),\ A^{4\theta} w \in L^2(\mathbb{R}^+, H)\right\}, \tag{2.16}$$

endowed with the norm

$$\|w\|_{W^{4,\theta}(\mathbb{R}^+,H)} := \left(\left\|w^{(4)}\right\|^2_{L^2(\mathbb{R}^+,H)} + \left\|A^{4\theta} w\right\|^2_{L^2(\mathbb{R}^+,H)}\right)^{1/2}.$$

For more details about these spaces, see [80, Chapter 1].

Now, let us consider the following abstract differential equation

$$w^{(4)}(t) + A^{4\theta} w(t) + \sum_{j=1}^{4} A_j w^{(4-j)}(t) = h(t),\quad t \in \mathbb{R}^+, \tag{2.17}$$

where $\theta \in\ ]0,1]$, $h \in L^2(\mathbb{R}^+, H)$ and A_j, $j = 1,2,3,4$, are linear operators acting on H. We also assume that Eq. (2.17) is accompanied with the following nonhomogeneous abstract boundary conditions given by

$$w'(0) = \varphi_1,\ w'''(0) + K w(0) = \varphi_2, \tag{2.18}$$

with K being an element of $\mathcal{L}(H_{7\theta/2}, H_{\theta/2})$, $\varphi_1 \in H_{5\theta/2}$ and $\varphi_2 \in H_{\theta/2}$.

First of all, we seek for a regular solution for (2.17), i.e., a vectorial function $w \in W^{4,\theta}(\mathbb{R}^+; H)$ satisfying (2.17)-(2.18) a.e. in $\mathbb{R}^+$. Next, we provide some necessary conditions ensuring the regular solvability of our problem (2.17)-(2.18). For the reader's convenience, we recall from Definition 2.1 in [9] that the problem (2.17)-(2.18) is said to be regularly solvable if and only if it admits a regular solution w which satisfies the following conditions

$$\begin{cases} \lim\limits_{t\to 0^+} \|w'(t) - \varphi_1\|_{H_{5\theta/2}} = 0, \\ \\ \lim\limits_{t\to 0^+} \|w'''(t) + K w(t) - \varphi_2\|_{H_{\theta/2}} = 0, \end{cases}$$

and for any $h \in L^2(\mathbb{R}^+; H)$, there exists $C > 0$ such that

$$\|w\|_{W^{4,\theta}(\mathbb{R}^+,H)} \leq C\left(\|\varphi_1\|_{H_{5\theta/2}} + \|\varphi_2\|_{H_{\theta/2}} + \|h\|_{L^2(\mathbb{R}^+,H)}\right).$$

In the current literature, we find many works considering various classes of the fourth-order operator-differential equations. For example, in [11], some optimal results about the existence and uniqueness of regular solutions have been established for the problem

$$\begin{aligned} &\frac{d^4w(t)}{dt^4}+A^4w(t)+\sum_{j=1}^{4}A_j\frac{d^{4-j}w(t)}{dt^{4-j}}=h(t),\ t\in\mathbb{R}, \\ &\frac{d^3w(0)}{dt^3}=0,\ \frac{d^2w(0)}{dt^2}-K\frac{dw(0)}{dt}=0, \end{aligned} \tag{2.19}$$

where

- $h\in L^2(\mathbb{R}^+,H)$,
- $(A,D(A))$ is a self-adjoint positive definite operator in a Hilbert space H,
- A_j, $j\in\{1,2,3,4\}$ are, in general, linear unbounded operators,
- $K\in\mathcal{L}(H_{5/2},H_{3/2})$.

In [9], many interesting regularity results are established for the problem

$$\begin{aligned} &\frac{d^4w(t)}{dt^4}+A^4w(t)+\sum_{j=1}^{4}A_j\frac{d^{4-j}w(t)}{dt^{4-j}}=h(t),\ t\in\mathbb{R}^+, \\ &w(0)=\varphi\in H_{7/2},\ \frac{d^2w(0)}{dt^2}-K\frac{dw(0)}{dt}=\psi\in H_{3/2}, \end{aligned} \tag{2.20}$$

with the same assumptions as above.

In the same direction, in [60] we find a complete study concerning the problem

$$\begin{aligned} &\frac{d^4w(t)}{dt^4}+\rho(t)A^4w(t)+\sum_{j=1}^{4}A_j\frac{d^{4-j}w(t)}{dt^{4-j}}=h(t),\ t\in\mathbb{R}^+, \\ &w(0)=\varphi,\ \frac{dw(0)}{dt}=\psi, \end{aligned}$$

with ρ being a scalar measurable function in $\mathbb{R}^+$.

2.2.2 Existence of regular solution

In the sequel, the abbreviation $W_K^{4,\theta}(\mathbb{R}^+,H)$ stands for the space defined by

$$W_K^{4,\theta}(\mathbb{R}^+,H)=\left\{w\Big/\ w\in W^{4,\theta}(\mathbb{R}^+,H),\ w'(0)=0,\ w'''(0)=-Kw(0)\right\},$$

where $K\in\mathcal{L}(H_{7\theta/2},H_{\theta/2})$.

Remark 2.2.1. *As a direct consequence of the well known Lions-Peetre interpolation, the traces*

$$w'(0),\ w'''(0) \text{ and } w(0)$$

are well defined; see Section 1.6. Furthermore, for $w \in W^{4,\theta}(\mathbb{R}^+, H)$, one has

$$w^j(0) \in D\Big(A^{\theta(7/2-j)}\Big), \quad j = 0, 1, 2, 3,$$

and the mapping

$$\begin{aligned} &W^{4,\theta}(\mathbb{R}^+, H) \to \textstyle\prod_{j=0}^{3} D\Big(A^{\theta(7/2-j)}\Big), \\ &w \mapsto \Big\{w^{(j)}(0)\Big\},\ 0 \le j \le 3, \end{aligned}$$

is surjective; see also Theorem 1.6.2.

The first step of our strategy is based on the study of the principal part of Eq. (2.17), that is

$$w^{(4)}(t) + A^{4\theta} w(t) = h(t),\ t \in \mathbb{R}^+, \tag{2.21}$$

equipped with the homogeneous initial conditions

$$w'(0) = 0,\ w'''(0) + Kw(0) = 0. \tag{2.22}$$

Towards this end, let us denote by P_0 the operator defined as follows

$$\begin{array}{lll} P_0: & W_K^{4,\theta}(\mathbb{R}^+, H) & \to L^2(\mathbb{R}^+, H) \\ & w & \mapsto P_0 w(t) = w^{(4)}(t) + A^{4\theta} w(t). \end{array} \tag{2.23}$$

Lemma 2.2.1. *Let B be the operator defined by*

$$B := A^{\theta/2} K A^{-7\theta/2}. \tag{2.24}$$

Assume that $-\sqrt{2} \notin \sigma(B)$. Then the equation

$$P_0 w(t) = 0$$

has only a zero solution in the space $W_K^{4,\theta}(\mathbb{R}^+, H)$.

Proof. As in [11], we look for a solution of equation

$$P_0 w(t) = 0,$$

set on the space $W^{4,\theta}(\mathbb{R}^+, H)$. This solution has the following standard form

$$w_0(t) = e^{\eta_1 t A^\theta}\phi_1 + e^{\eta_2 t A^\theta}\phi_2, \quad t \in \mathbb{R}^+,$$

where $(e^{\eta_1 t A^\theta})_{t\ge 0}$ and $(e^{\eta_2 t A^\theta})_{t\ge 0}$ are the C_0-semigroups generated by $\eta_1 A^\theta$ and $\eta_2 A^\theta$, respectively, with

$$\eta_1 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i,\ \eta_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i,$$

and $\phi_1,\ \phi_2 \in H_{7\theta/2}$. Taking into account conditions (2.22), we obtain

$$\begin{cases} \eta_1 A^\theta \phi_1 + \eta_2 A^\theta \phi_2 = 0, \\ A^{3\theta}(\eta_1^3 \phi_1 + \eta_2^3 \phi_2) = -K(\phi_1 + \phi_2). \end{cases} \tag{2.25}$$

A direct computation implies that

$$\phi_2 = -\frac{\eta_1}{\eta_2}\phi_1, \tag{2.26}$$

and

$$\left(\sqrt{2}I_H + B\right)A^{7\theta/2}\phi_1 = 0. \tag{2.27}$$

Keeping in mind that

$$-\sqrt{2} \notin \sigma(B),$$

this leads to $\phi_1 = 0$ and from (2.26) it results that $\phi_2 = 0$. Therefore, $w_0(t) = 0$. □

Now, we are able to state our main result concerning the solvability of problem (2.21)-(2.22).

Theorem 2.2.1. *Let the assumptions of Lemma 2.2.1 hold. Then, the problem (2.21)-(2.22) has a unique regular solution* $w \in W_K^{4,\theta}(\mathbb{R}^+, H)$.

Proof. **Step 1.** Thanks to Lemma 2.2.1, we know that the problem

$$w^{(4)}(t) + A^{4\theta}w(t) = 0,\ t \in \mathbb{R}^+, \tag{2.28}$$

$$w'(0) = 0,\ w'''(0) = -Kw(0), \tag{2.29}$$

has only zero solution in $W_K^{4,\theta}(\mathbb{R}^+, H)$. Let us show that the equation

$$P_0 w(t) = h(t)$$

has a solution $w \in W_K^{4,\theta}(\mathbb{R}^+, H)$ for every $h \in L^2(\mathbb{R}^+, H)$.

First, set

$$H(t) := \begin{cases} h(t), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Let $\hat{H}(\xi)$ be the Fourier transform of $H(t)$, i.e.,

$$\hat{H}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} H(t)e^{-i\xi t}\,dt,\ \xi \in \mathbb{R}.$$

Then, performing the direct and inverse Fourier transforms, it is clear that the vector-valued function

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\xi^4 I_H + A^{4\theta})^{-1}\left(\int_0^{+\infty} h(s)e^{-i\xi s}ds\right)e^{i\xi t}d\xi,\ t \in \mathbb{R}, \tag{2.30}$$

satisfies the equation

$$v^{(4)}(t) + A^{4\theta}v(t) = H(t) \quad \text{a.e. in } \mathbb{R}.$$

Now, we prove that $v(\cdot)$ defined by the formula (2.30) belongs to the space $W^{4,\theta}(\mathbb{R}^+, H)$. By Plancherel's theorem, we have

$$\begin{aligned} \|v\|^2_{W^{4,\theta}(\mathbb{R}^+,H)} &= \left\|v^{(4)}\right\|^2_{L^2(\mathbb{R}^+,H)} + \left\|A^{4\theta}v\right\|^2_{L^2(\mathbb{R}^+,H)} \\ &= \left\|\xi^4\hat{v}\right\|^2_{L^2(\mathbb{R}^+,H)} + \left\|A^{4\theta}\hat{v}\right\|^2_{L^2(\mathbb{R}^+,H)}; \end{aligned}$$

hence,

$$\begin{aligned}&\|v\|^2_{W^{4,\theta}(\mathbb{R}^+,H)}\\ = \;& \left\|\xi^4(\xi^4 I_H + A^{4\theta})^{-1}\hat{H}(\xi)\right\|^2_{L^2(\mathbb{R}^+,H)} + \left\|A^{4\theta}(\xi^4 I_H + A^{4\theta})^{-1}\hat{H}(\xi)\right\|^2_{L^2(\mathbb{R}^+,H)}.\end{aligned}$$

Then

$$\begin{aligned}&\|v\|^2_{W^{4,\theta}(\mathbb{R}^+,H)}\\ \leq \;& \left(\sup_{\xi\in\mathbb{R}}\left\|\xi^4(\xi^4 I_H + A^{4\theta})^{-1}\right\|_{\mathcal{L}(H)} + \sup_{\xi\in\mathbb{R}}\left\|A^{4\theta}(\xi^4 I_H + A^{4\theta})^{-1}\right\|_{\mathcal{L}(H)}\right)\|H\|^2_{L^2(\mathbb{R}^+,H)}.\end{aligned}$$

According to the classical spectral theory of self-adjoint operators, we obtain

$$\left\|\xi^4(\xi^4 I_H + A^{4\theta})^{-1}\right\|_{\mathcal{L}(H)} \leq \sup_{\lambda\in\sigma(A^\theta)}\left|\xi^4(\xi^4+\lambda^4)^{-1}\right| \leq 1,$$

and

$$\left\|A^{4\theta}(\xi^4 I_H + A^{4\theta})^{-1}\right\|_{\mathcal{L}(H)} \leq \sup_{\lambda\in\sigma(A^\theta)}\left|\lambda^4(\xi^4+\lambda^4)^{-1}\right| \leq 1;$$

hence $v \in W^{4,\theta}(\mathbb{R}^+, H)$.

Step 2 Put

$$w_1(t) := v(t)|_{\mathbb{R}^+}.$$

Then $w_1 \in W^{4,\theta}(\mathbb{R}^+, H)$ and satisfies the equation (2.21) almost everywhere in $\mathbb{R}^+$. On the other hand, Theorem 1.6.2 yields that

$$w_1^{(j)}(0) \in H_{(7/2-j)\theta},\ j = 0,1,2,3.$$

Similarly, as in the previous step, the solution of problem (2.21)-(2.22) can be written in the following form

$$w(t) = w_1(t) + e^{\eta_1 t A^\theta}\phi_1 + e^{\eta_2 t A^\theta}\phi_2,$$

where

$$\eta_1 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \quad\text{and}\quad \eta_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,$$

$\phi_1,\ \phi_2 \in H_{7\theta/2}$; see also (2.22). Consequently, we obtain the following system

$$\begin{cases} w_1'(0) + \eta_1 A^\theta\phi_1 + \eta_2 A^\theta\phi_2 = 0, \\ w_1'''(0) + \eta_1^3 A^{3\theta}\phi_1 + \eta_2^3 A^{3\theta}\phi_2 = -K(w_1(0)+\phi_1+\phi_2). \end{cases} \tag{2.31}$$

Taking into account that

$$\phi_2 = -\frac{\eta_1}{\eta_2}\phi_1 - \frac{1}{\eta_2}A^{-\theta}w_1'(0),$$

and keeping in mind the condition

$$-\sqrt{2} \notin \sigma(B),$$

we uniquely deduce that

$$\phi_1 = A^{-7\theta/2}(\sqrt{2}I + B)^{-1}A^{7\theta/2}\eta \in H_{7\theta/2},$$

where

$$\eta = -\frac{1-i}{2}A^{-3\theta}\left(w_1'''(0) + iA^{2\theta}w'(0) + Kw_1(0) - \eta_1 KA^{-\theta}w_1'(0)\right) \in H_{7\theta/2}.$$

Thus, w belongs to the space $W^{4,\theta}(\mathbb{R}^+, H)$ and it is a solution for the problem (2.21)-(2.22). Moreover, the operator

$$P_0 : W_K^{4,\theta}(\mathbb{R}^+, H) \to L^2(\mathbb{R}^+, H)$$

is bounded. In fact, we have

$$\|P_0 w\|^2_{L^2(\mathbb{R}^+,H)} = \left\|w^{(4)} + A^{4\theta}w\right\|^2_{L^2(\mathbb{R}^+,H)} \leq 2\|w\|^2_{W^{4,\theta}(\mathbb{R}^+,H)}.$$

Therefore, by the Banach inverse operator theorem, we deduce that operator P_0 is invertible and

$$P_0^{-1} : L^2(\mathbb{R}^+, H) \to W_K^{4,\theta}(\mathbb{R}^+, H).$$

Furthermore, this operator is bounded and we obtain

$$\|w\|_{W^{4,\theta}(\mathbb{R}^+,H)} \leq C\|h\|_{L^2(\mathbb{R}^+,H)}.$$

□

The following result follows directly from Lemma 2.2.1 and Theorem 2.2.1.

Corollary 2.2.1. *Under the assumptions of Lemma 2.2.1, the operator P_0 defined by (2.23) is an isomorphism.*

Let us prove now the following coercive inequality, which will be used later.

Lemma 2.2.2. *Let B denote the operator defined by (2.24), with $Re(B) \geq 0$. Then, for every $w \in W_K^{4,\theta}(\mathbb{R}^+, H)$, the following inequality holds true*

$$\|P_0 w\|^2_{L^2(\mathbb{R}^+,H)} \geq \|w\|^2_{W^{4,\theta}(\mathbb{R}^+,H)} + 2\left\|A^{2\theta}w''\right\|^2_{L^2(\mathbb{R}^+,H)}. \tag{2.32}$$

Proof. For $w \in W_K^{4,\theta}(\mathbb{R}^+, H)$, we have

$$\begin{aligned} &\|P_0 w\|^2_{L^2(\mathbb{R}^+,H)} \\ = &\left\|w^{(4)}\right\|^2_{L^2(\mathbb{R}^+,H)} + \left\|A^{4\theta}w\right\|^2_{L^2(\mathbb{R}^+,H)} + 2Re\left(< w^{(4)}, A^{4\theta}w >_{L^2(\mathbb{R}^+,H)}\right). \end{aligned} \tag{2.33}$$

On the other hand, integrating by parts, we obtain

$$\begin{aligned} < w^{(4)}, A^{4\theta}w >_{L^2(\mathbb{R}^+,H)} &= \left[< w'''(t), A^{4\theta}w(t) >\right]_0^{+\infty} - \int_0^{+\infty} < w'''(t), A^{4\theta}w'(t) > dt \\ &= < Kw(0), A^{4\theta}w(0) > + \int_0^{+\infty} < A^{2\theta}w''(t), A^{2\theta}w''(t) > dt \\ &= < BA^{7\theta/2}w(0), A^{7\theta/2}w(0) > + \left\|A^{2\theta}w''\right\|^2_{L^2(\mathbb{R}^+,H)}. \end{aligned} \tag{2.34}$$

Taking into account the fact that $Re(B) \geq 0$, the estimate (2.32) is easily deduced from relation (2.34). □

Observe here that the Corollary 2.2.1 implies that the quantity $\|P_0 w\|_{L^2(\mathbb{R}^+,H)}$ is equivalent to $\|w\|_{W^{4,\theta}(\mathbb{R}^+,H)}$ in the space $W_K^{4,\theta}(\mathbb{R}^+,H)$. Moreover, the norms of the intermediate derivative operators

$$A^{j\theta}\frac{d^{4-j}}{dt^{4-j}} : W_K^{4,\theta}(\mathbb{R}^+,H) \longrightarrow L^2(\mathbb{R}^+,H),\ \ j=1,2,3,4,$$

can be estimated with respect to $\|P_0 u\|_{L^2(\mathbb{R}^+,H)}$.

Theorem 2.2.2. *Under the assumptions of Lemma 2.2.2, the following estimates hold true*

$$\left\|A^{j\theta}w^{(4-j)}\right\|_{L^2(\mathbb{R}^+,H)} \leq a_j \|P_0 w\|_{L^2(\mathbb{R}^+,H)},\quad j=1,2,3,4, \tag{2.35}$$

for any $w \in W_K^{4,\theta}(\mathbb{R}^+,H)$ with

$$a_0 = a_1 = a_4 = 1,\quad a_2 = \frac{1}{2},\quad a_3 = \frac{1}{\sqrt{2}}.$$

Proof. Let $w \in W_K^{4,\theta}(\mathbb{R}^+,H)$. From the equality (2.34), we have

$$\begin{aligned} & Re\left(< P_0 w, A^{4\theta}w >_{L^2(\mathbb{R}^+,H)}\right) \\ = \ & \left\|A^{4\theta}w\right\|^2_{L^2(\mathbb{R}^+,H)} + Re\left(< BA^{7\theta/2}w(0), A^{7\theta/2}w(0) >\right) + \left\|A^{2\theta}w''\right\|^2_{L^2(\mathbb{R}^+,H)}. \end{aligned}$$

Then we can see that

$$Re\left(< P_0 w, A^{4\theta}w >_{L^2(\mathbb{R}^+,H)}\right) \geq \left\|A^{4\theta}w\right\|^2_{L^2(\mathbb{R}^+,H)} + \left\|A^{2\theta}w''\right\|^2_{L^2(\mathbb{R}^+,H)}.$$

Applying the well known Cauchy-Schwarz and Young inequalities, we conclude that

$$\begin{aligned} & \left\|A^{4\theta}w\right\|^2_{L^2(\mathbb{R}^+,H)} + \left\|A^{2\theta}w''\right\|^2_{L^2(\mathbb{R}^+,H)} \\ \leq \ & \|P_0 w\|_{L^2(\mathbb{R}^+,H)} \left\|A^{4\theta}w\right\|_{L^2(\mathbb{R}^+,H)}, \end{aligned} \tag{2.36}$$

from which we may deduce that

$$\left\|A^{4\theta}w\right\|^2_{L^2(\mathbb{R}^+,H)} + \left\|A^{2\theta}w''\right\|^2_{L^2(\mathbb{R}^+,H)} \leq \frac{\delta}{2}\|P_0 w\|^2_{L^2(\mathbb{R}^+,H)} + \frac{1}{2\delta}\left\|A^{4\theta}w\right\|^2_{L^2(\mathbb{R}^+,H)}, \tag{2.37}$$

with $\delta > 0$.

Choosing $\delta = \frac{1}{2}$ in (2.37), we get

$$\left\|A^{2\theta}w''\right\|_{L^2(\mathbb{R}^+,H)} \leq \frac{1}{2}\|P_0 w\|_{L^2(\mathbb{R}^+,H)}. \tag{2.38}$$

On the other hand, from (2.36) we have

$$\left\|A^{4\theta}w\right\|^2_{L^2(\mathbb{R}^+,H)} \leq \|P_0 w\|_{L^2(\mathbb{R}^+,H)} \left\|A^{4\theta}w\right\|_{L^2(\mathbb{R}^+,H)},$$

which implies that

$$\left\|A^{4\theta}w\right\|_{L^2(\mathbb{R}^+,H)} \leq \|P_0 w\|_{L^2(\mathbb{R}^+,H)}. \tag{2.39}$$

It follows from inequality (2.32) that

$$\left\|w^{(4)}\right\|_{L^2(\mathbb{R}^+,H)} \leq \|P_0 w\|_{L^2(\mathbb{R}^+,H)}. \tag{2.40}$$

Now let us estimate the norm $\left\|A^{3\theta}w'\right\|_{L^2(\mathbb{R}^+,H)}$.

Taking into account that $w \in W_K^{4,\theta}(\mathbb{R}^+,H)$, the use of the Cauchy-Schwarz inequality combined with inequalities (2.38) and (2.39), allows us to conclude that

$$\begin{aligned} &\left\|A^{3\theta}w'\right\|^2_{L^2(\mathbb{R}^+,H)} \\ = \ &\left[< A^{3\theta}w(t), A^{3\theta}w'(t) >\right]_0^{+\infty} - \int_0^{+\infty} < A^{4\theta}w(t), A^{2\theta}w''(t) > dt, \end{aligned}$$

so

$$\begin{aligned} \left\|A^{3\theta}w'\right\|^2_{L^2(\mathbb{R}^+,H)} &\leq \left\|A^{2\theta}w''\right\|_{L^2(\mathbb{R}^+,H)} \left\|A^{4\theta}w\right\|_{L^2(\mathbb{R}^+,H)} \\ &\leq \frac{1}{2}\|P_0 w\|^2_{L^2(\mathbb{R}^+,H)}. \end{aligned}$$

Consequently,

$$\left\|A^{3\theta}w'\right\|_{L^2(\mathbb{R}^+,H)} \leq \frac{1}{\sqrt{2}}\|P_0 w\|_{L^2(\mathbb{R}^+,H)}. \tag{2.41}$$

Finally, let us estimate the quantity $\left\|A^{\theta}w'''\right\|_{L^2(\mathbb{R}^+,H)}$. We know that, for $w \in W^{4,\theta}(\mathbb{R}^+,H)$, we have

$$\left\|A^{\theta}w'''\right\|^2_{L^2(\mathbb{R}^+,H)} \leq 2\left\|A^{2\theta}w''\right\|_{L^2(\mathbb{R}^+,H)}\left\|w^{(4)}\right\|_{L^2(\mathbb{R}^+,H)}. \tag{2.42}$$

Inserting the inequalities (2.38) and (2.40) in (2.42), we have

$$\left\|A^{\theta}w'''\right\|_{L^2(\mathbb{R}^+,H)} \leq \|P_0 w\|_{L^2(\mathbb{R}^+,H)}, \tag{2.43}$$

which ends the proof of this theorem. □

It is worth noting that the coefficient operator A, in our boundary value problem, was considered with a positive natural power so far. From now on, we will treat our problem in the general case, where the coefficient operators considered will be of the form A^{θ}, $\theta \in (0,1)$. To this end, let us consider the following abstract Cauchy problem with a complete fourth-order differential equation

$$w^{(4)}(t) + A^{4\theta}w(t) + \sum_{j=1}^{4} A_j w^{(4-j)}(t) = h(t), \ t \in \mathbb{R}^+, \tag{2.44}$$

$$w'(0) = 0, \ w'''(0) = -Kw(0). \tag{2.45}$$

Put

$$\begin{aligned} P: \ \ & W_K^{4,\theta}(\mathbb{R}^+,H) && \longrightarrow L^2(\mathbb{R}^+,H) \\ & w && \longmapsto Pw(t) := w^{(4)}(t) + A^{4\theta}w(t) + \sum_{j=1}^{4} A_j w^{(4-j)}(t). \end{aligned} \tag{2.46}$$

The first auxiliary result concerning this operator is formulated as follows.

Lemma 2.2.3. *Assume that $A_jA^{-j\theta} \in \mathcal{L}(H)$, $j = 1,2,3,4$. Then the operator P, defined by (2.46), is bounded.*

Proof. Let $w \in W_K^{4,\theta}(\mathbb{R}^+, H)$. Then, we have

$$\begin{aligned}
\|Pw\|_{L^2(\mathbb{R}^+,H)} &\leq \|P_0 w\|_{L^2(\mathbb{R}^+,H)} + \left\| \sum_{j=1}^{4} A_j w^{(4-j)} \right\|_{L^2(\mathbb{R}^+,H)} \\
&\leq \sqrt{2}\,\|w\|_{W^{4,\theta}(\mathbb{R}^+,H)} + \left\| \sum_{j=1}^{4} A_j w^{(4-j)} \right\|_{L^2(\mathbb{R}^+,H)} \\
&\leq \sqrt{2}\,\|w\|_{W^{4,\theta}(\mathbb{R}^+,H)} + \sum_{j=1}^{4} \left\|A_jA^{-j\theta}\right\|_{\mathcal{L}(H)} \left\|A^{j\theta} w^{(4-j)}\right\|_{L^2(\mathbb{R}^+,H)}.
\end{aligned}$$

Using the theorem for intermediate derivatives in [80], we deduce that

$$\|Pw\|_{L^2(\mathbb{R}^+,H)} \leq C\,\|w\|_{W^{4,\theta}(\mathbb{R}^+,H)}.$$

□

Let us state our essential results concerning the problem (2.44)-(2.45) performed in the space $L^2(\mathbb{R}^+, H)$.

Theorem 2.2.3. *Let $B = A^{\theta/2}KA^{-7\theta/2}$. Assume that*

$$\begin{cases} Re(B) \geq 0, \\ and \\ A_jA^{-j\theta} \in \mathcal{L}(H),\ j = 1,2,3,4, \end{cases}$$

and

$$a = \sum_{j=1}^{4} a_i \left\|A_jA^{-j\theta}\right\|_{\mathcal{L}(H)} < 1,$$

with

$$a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{\sqrt{2}},\ a_4 = 1.$$

Then, for every $h \in L^2(\mathbb{R}^+, H)$, the boundary value problem (2.44)-(2.45) has a unique regular solution.

Proof. First, we write the boundary value problem (2.44)-(2.45) in the form of operator equation

$$P_0 w(t) + (P - P_0)w(t) = h(t),\ t \in \mathbb{R}^+, \tag{2.47}$$

where $h \in L^2(\mathbb{R}^+, H)$ and $w \in W_K^{4,\theta}(\mathbb{R}^+, H)$.

The conditions $Re(B) \geq 0$ ensure that the operator

$$P_0^{-1} : L^2(\mathbb{R}^+, H) \longrightarrow W_K^{4,\theta}(\mathbb{R}^+, H)$$

is well defined.

Set $w(t) := P_0^{-1}v(t)$, with $v \in L^2(\mathbb{R}^+, H)$. Then a direct computation shows that v satisfies the following equation

$$v(t) + (P - P_0)P_0^{-1}v(t) = h(t),\ t \in \mathbb{R}^+.$$

Keeping in mind that $v \in L^2(\mathbb{R}^+, H)$ and taking into account the estimates (2.35), one has

$$\left\|(P - P_0)P_0^{-1}v\right\|_{L^2(\mathbb{R}^+,H)} = \|(P - P_0)w\|_{L^2(\mathbb{R}^+,H)},$$

so

$$\begin{aligned}\left\|(P - P_0)P_0^{-1}v\right\|_{L^2(\mathbb{R}^+,H)} &\le \sum_{j=1}^{4}\left\|A_j A^{-j\theta}\right\|_{\mathcal{L}(H)}\left\|A^{j\theta}w^{(4-j)}\right\|_{L^2(\mathbb{R}^+,H)}\\ &\le \sum_{j=1}^{4} a_j\left\|A_j A^{-j\theta}\right\|_{\mathcal{L}(H)}\|P_0 w\|_{L^2(\mathbb{R}^+,H)}.\end{aligned}$$

Therefore

$$\left\|(P - P_0)P_0^{-1}v\right\|_{L^2(\mathbb{R}^+,H)} = a\|v\|_{L^2(\mathbb{R}^+,H)}.$$

Since $a < 1$, the operator

$$\left(I_H + (P - P_0)P_0^{-1}\right)^{-1}$$

is well defined in the space $L^2(\mathbb{R}^+, H)$. Consequently, the equation (2.47) is uniquely solvable in the space $W_K^{4,\theta}(\mathbb{R}^+, H)$ and

$$w(t) = P_0^{-1}\left(I_H + (P - P_0)P_0^{-1}\right)^{-1}h(t).$$

Moreover,

$$\begin{aligned}&\|w\|_{W^{4,\theta}(\mathbb{R}^+;H)}\\ \le\ &\left\|P_0^{-1}\right\|_{\mathcal{L}\left(L^2(\mathbb{R}^+,H),W^{4,\theta}(\mathbb{R}^+,H)\right)}\left\|(I_H + (P - P_0)P_0^{-1})^{-1}\right\|_{\mathcal{L}\left(L^2(\mathbb{R}^+,H)\right)}\|h\|_{L^2(\mathbb{R}^+,H)}\\ \le\ & C\|h\|_{L^2(\mathbb{R}^+,H)}.\end{aligned}$$

□

Remark 2.2.2. *In Theorem 2.2.3, the condition $Re(B) \ge 0$ with*

$$B = A^{\theta/2}KA^{-7\theta/2},$$

allows us to omit the condition $-\sqrt{2} \notin \sigma(B)$.

Finally, we may get the conditions for the regular solvability of the boundary value problem (2.17)-(2.18) from Theorem 2.2.3.

Theorem 2.2.4. *Assume that all conditions of Theorem 2.2.3 are fulfilled. Then the boundary value problem (2.17)-(2.18) is regularly solvable.*

Proof. In the case $\varphi_1 = \varphi_2 = 0$, the regular solvability of the boundary value problem (2.17)-(2.18) was established.

In the case $A_j = 0$, $j = 1,2,3,4$, and $h = 0$, the problem (2.17)-(2.18) is reduced to the new one given by

$$w^{(4)}(t) + A^{4\theta} w(t) = 0,\ t \in \mathbb{R}^+, \tag{2.48}$$

$$w'(0) = \varphi_1,\ w'''(0) + K w(0) = \varphi_2, \tag{2.49}$$

with $\varphi_1 \in H_{5\theta/2}$ and $\varphi_2 \in H_{\theta/2}$. The solution of problem (2.48)-(2.49) will be written as follows

$$w_0(t) = e^{\eta_1 t A^\theta} \phi_1 + e^{\eta_2 t A^\theta} \phi_2, \tag{2.50}$$

where

$$\eta_1 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \ \text{ and } \ \eta_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i,$$

and ϕ_1, ϕ_2 are the unknown vectors to be determined via the conditions (2.49):

$$\begin{cases} \eta_1 A^\theta \phi_1 + \eta_2 A^\theta \phi_2 = \varphi_1, \\ A^{3\theta}(\eta_1^3 \phi_1 + \eta_2^3 \phi_2) + K(\phi_1 + \phi_2) = \varphi_2. \end{cases} \tag{2.51}$$

System (2.51) yields

$$\phi_2 = \frac{1}{w_2}\left(A^{-\theta}\varphi_1 - \eta_1 \phi_1\right),$$

$$(\sqrt{2} I_H + B) A^{-7\theta/2} \phi_1 = (i-1) A^{\theta/2}(\varphi_2 - \eta_2^2 A^{2\theta} \varphi_1 - \eta_1 A^{-\theta} \varphi_1).$$

Since $-\sqrt{2} \notin \sigma(B)$, then we have

$$\phi_1 = -\frac{(1-i)}{2} A^{7\theta/2} (\sqrt{2} I_H + B) A^{\theta/2} (\varphi_2 - \eta_2^2 A^{2\theta} \varphi_1 - \eta_1 A^{-\theta} \varphi_1).$$

Thus

$$\phi_2 = \eta_1 A^{-\theta} \varphi_1 + (\frac{1-i}{2}) A^{7\theta/2} (\sqrt{2} I_H + B) A^{\theta/2} (\varphi_2 - \eta_2^2 A^{2\theta} \varphi_1 - \eta_1 A^{-\theta} \varphi_1).$$

It is not difficult to show that ϕ_1, $\phi_2 \in H_{7\theta/2}$. From (2.50), we obtain

$$\begin{aligned} \|w_0\|_{W^{4,\theta}(\mathbb{R}^+,H)} &\leq C\left(\left\|\phi_1\right\|_{H_{7\theta/2}} + \left\|\phi_2\right\|_{H_{7\theta/2}}\right) \\ &\leq C\left(\|\varphi_1\|_{H_{5\theta/2}} + \|\varphi_2\|_{H_{\theta/2}}\right). \end{aligned} \tag{2.52}$$

Now, we are able to study the boundary value problem (2.17)-(2.18). We will seek its solutions in the form

$$w(t) = v(t) + w_0(t),$$

where w_0 is a regular solution of the problem (2.48)-(2.49). Then, the function v is the solution to the boundary value problem

$$v^{(4)}(t) + A^{4\theta} v(t) + \sum_{j=1}^{4} A_j v^{(4-j)}(t) = g(t),\ t \in \mathbb{R}^+, \tag{2.53}$$

$$v'(0) = 0,\ v'''(0) + Kv(0) = 0, \tag{2.54}$$

where

$$g(t) = -\sum_{j=1}^{4} A_j w_0^{(4-j)}(t) + h(t).$$

Let us estimate the quantity $\|g\|_{L^2(\mathbb{R}^+,H)}$. One has

$$\begin{aligned}
\|g\|_{L^2(\mathbb{R}^+;H)} &\leq \left\| \sum_{j=1}^{4} A_j w_0^{(4-j)} \right\|_{L^2(\mathbb{R}^+,H)} + \|h\|_{L^2(\mathbb{R}^+,H)} \\
&\leq \sum_{j=1}^{4} \left\| A_j A^{-j\theta} \right\|_{\mathcal{L}(H)} \left\| A^{j\theta} w_0^{(4-j)} \right\|_{L^2(\mathbb{R}^+,H)} + \|h\|_{L^2(\mathbb{R}^+,H)} \\
&\leq C\left(\|\varphi_1\|_{H_{5\theta/2}} + \|\varphi_2\|_{H_{\theta/2}} + \|h\|_{L^2(\mathbb{R}^+,H)} \right).
\end{aligned}$$

Thanks to Theorem 2.2.3 and the estimate (2.52), we have

$$\begin{aligned}
\|w\|_{W^{4,\theta}(\mathbb{R}^+,H)} &\leq \|v\|_{W^{4,\theta}(\mathbb{R}^+,H)} + \|w_0\|_{W^{4,\theta}(\mathbb{R}^+,H)} \\
&\leq \|g\|_{L^2(\mathbb{R}^+,H)} + \|w_0\|_{W^{4,\theta}(\mathbb{R}^+,H)} \\
&\leq C\left(\|\varphi_1\|_{H_{5\theta/2}} + \|\varphi_2\|_{H_{\theta/2}} + \|h\|_{L^2(\mathbb{R}^+,H)} \right).
\end{aligned}$$

□

2.3 Existence of the solution to the main problem

In this section, we return to the original problem. In order to provide a comprehensive study of the problem (2.1)-(2.3)-(2.4), we need some intermediate results which can be viewed as a direct consequence of the results obtained in the previous section.

Remark 2.3.1. *To simplify the computations involving functional spaces and make the study more comprehensible, we consider the case when $\theta = 1/8$. Thus, from (2.15) and (2.16), the space $W^{4,\theta}(\mathbb{R}^+, L^2(Q))$ is defined as follows*

$$W^{4,\theta}(\mathbb{R}^+, L^2(Q)) = \left\{ w \big/ \ w^{(4)} \in L^2\left(\mathbb{R}^+, L^2(Q)\right),\ w \in L^2\left(\mathbb{R}^+, H_0^1(Q)\right) \right\}.$$

Keeping in mind the definition of the operators $(A, D(A))$, $\left(A_j, D\left(A_j\right)\right)$, and $(K, D(K))$, defined respectively by (2.12), (2.13), and (2.14). Our main result for the transformed problem (2.10)-(2.11) is formulated as follows.

Theorem 2.3.1. *Let $h \in L^2(\mathbb{R}^+ \times Q)$. Assume that*

$$\operatorname{Re}(b) \geq 0, \quad \sum_{j=1}^{4} \sup_{\xi \in Q} \left| \sigma_j(\xi) \right| < 1.$$

Then, the problem

$$\frac{d^4 w(t)}{dt^4} + A^{4\theta} w(t) + \sum_{j=1}^{4} A_j \frac{d^{4-j} w(t)}{dt^{4-j}} = h(t),\ t \in \mathbb{R}^+,$$

with

$$\frac{dw(0)}{dt} = 0, \quad \frac{d^3w(0)}{dt^3} + Kw(0) = 0,$$

has a unique regular solution $w \in W^{4,\theta}(\mathbb{R}^+, L^2(Q))$.

By the same argument and using a classical argument of perturbation as in [21, Section 3, p. 49], we conclude the following result.

Theorem 2.3.2. *Let* $h \in L^2(\mathbb{R}^+ \times Q)$. *Assume that*

$$\operatorname{Re}(b) \geq 0, \quad \sum_{j=1}^{4} \sup_{\xi \in \Omega} \left|\sigma_j(\xi)\right| < 1.$$

Then, the problem (2.7)-(2.8)-(2.9) *has a unique regular solution* $w \in W^{4,\theta}(\mathbb{R}^+, L^2(Q))$.

Consider now the inverse change of variables

$$\begin{aligned} \Psi^{-1}: \quad & \mathbb{R}^+ \times Q \to \mathbb{R}^+ \times \Pi \\ & (t, \xi) \mapsto (t, x), \end{aligned}$$

with

$$x_1 = \left(\tfrac{\gamma}{\xi_3}\right)^{\frac{\alpha}{\beta}} \xi_1, \quad x_2 = \left(\tfrac{\gamma}{\xi_3}\right)^{\frac{\alpha}{\beta}} \xi_2, \quad x_3 = \left(\tfrac{\gamma}{\xi_3}\right)^{\frac{1}{\beta}}.$$

We have

$$w = \left(\frac{\gamma}{\xi_3}\right)^{-\frac{\alpha}{\beta}\left(\frac{3\alpha}{8\theta}+2\right)} u\left(\left(\frac{\gamma}{\xi_3}\right)^{\frac{\alpha}{\beta}} \xi_1, \left(\frac{\gamma}{\xi_3}\right)^{\frac{\alpha}{\beta}} \xi_2, \left(\frac{\gamma}{\xi_3}\right)^{\frac{1}{\beta}}\right);$$

this gives

$$w = \left(\frac{\gamma}{\xi_3}\right)^{-\frac{3\alpha}{2\beta}} (x_3)^{-\alpha\left(\frac{3}{2\theta}+\frac{1}{2}\right)} u. \tag{2.55}$$

In an equivalent manner,

$$\partial_{\xi_1} w = \left(\frac{\gamma}{\xi_3}\right)^{-\frac{3\alpha}{2\beta}} (x_3)^{-\alpha\left(\frac{3}{2\theta}-\frac{1}{2}\right)} \partial_{x_1} u, \tag{2.56}$$

and

$$\partial_{\xi_2} w = \left(\frac{\gamma}{\xi_3}\right)^{-\frac{3\alpha}{2\beta}} (x_3)^{-\alpha\left(\frac{3}{2\theta}-\frac{1}{2}\right)} \partial_{x_2} u. \tag{2.57}$$

Due to the fact that w, $\partial_{\xi_1} w$, and $\partial_{\xi_2} w$ are L^2-integrable in Q, (2.6) with (2.55)-(2.56)-(2.57) implies that

$$(x_3)^{-\alpha\left(\frac{3}{2\theta}+\frac{1}{2}\right)} u, \ (x_3)^{-\alpha\left(\frac{3}{2\theta}-\frac{1}{2}\right)} \partial_{x_1} u, (x_3)^{-\alpha\left(\frac{3}{2\theta}-\frac{1}{2}\right)} \partial_{x_2} u \in L^2(\Pi). \tag{2.58}$$

Furthemore, a direct computation shows that

$$
\begin{aligned}
&\partial_{x_2} w \\
= \;& \left(\frac{\gamma}{\xi_3}\right)^{-s}\left[s\xi_3^{-1}u-\frac{\alpha}{\beta}\xi_1\xi_3^{-1}\left(\frac{\gamma}{\xi_3}\right)^{\frac{\alpha}{\beta}}\partial_{x_1}u-\frac{\alpha}{\beta}\xi_2\xi_3^{-1}\left(\frac{\gamma}{\xi_3}\right)^{\frac{\alpha}{\beta}}\partial_{x_2}u\right. \\
& \left.-\frac{1}{\beta}\xi_3^{-1}\left(\frac{\gamma}{\xi_3}\right)^{\frac{1}{\beta}}\partial_{x_3}u\right] \\
= \;& \left(\frac{\gamma}{\xi_3}\right)^{-\frac{3\alpha}{2\beta}}\left[s\xi_3^{-1}x_3^{-\alpha\left(\frac{3}{8\theta}+\frac{1}{2}\right)}u-\frac{\alpha}{\beta}\xi_1\xi_3^{-1}x_3^{-\alpha\left(\frac{3}{8\theta}-\frac{1}{2}\right)}\partial_{x_1}u-\frac{\alpha}{\beta}\xi_2\xi_3^{-1}x_3^{-\alpha\left(\frac{3}{8\theta}-\frac{1}{2}\right)}\partial_{x_2}u\right. \\
& \left.-\frac{1}{\gamma\beta}x_3^{-\alpha\left(\frac{3}{8\theta}-\frac{1}{2}\right)}\partial_{x_3}u\right].
\end{aligned}
$$

Since $\partial_{\xi_3}w$ is L^2-integrable in Q, according to the previous calculations and (2.58), we obtain

$$
x_3^{-\alpha\left(\frac{3}{8\theta}-\frac{1}{2}\right)}\partial_{x_3}u\in L^2(\Pi).
$$

In summary, the following proposition has been established.

Proposition 2.3.1. *The fact that $w\in W^{4,\theta}(\mathbb{R}^+,L^2(Q))$ implies that*

$$
u\in W^{4,\theta}(\mathbb{R}^+,L^2(\Pi)).
$$

This help us to justify our main result set in the cusp domain $\mathbb{R}^+\times\Pi$.

Theorem 2.3.3. *Let $f\in L^2(\mathbb{R}^+\times\Pi)$. Assume that*

$$
\operatorname{Re}(b)\geq 0 \text{ and } \sum_{j=1}^{4}\sup_{x\in\Pi}\left|\rho_j(x)\right|<1.
$$

Then, the problem

$$
\frac{d^4}{dt^4}u(t,x)+(1+\rho_4(x))(-\Delta)^{4\theta}u(t,x)+\sum_{j=1}^{3}\left(\rho_j(x)(-\Delta)^{j\theta}\right)\frac{d^{4-j}}{dt^{4-j}}u(t,x)=f(t,x),
$$

$$
u|_{\mathbb{R}^+\times\partial\Pi}=0,
$$

$$
\left.\frac{du}{dt}\right|_{\{0\}\times\Pi}=0 \text{ and } \left.\frac{d^3u}{dt^3}+b(-\Delta)^{3\theta}u\right|_{\{0\}\times\Pi}=0,
$$

has a unique regular solution $u\in W^{4,\theta}(\mathbb{R}^+,L^2(\Pi))$.

Chapter 3

On the study of pseudo S-asymptotically periodic mild solutions for a class of neutral fractional delayed evolution equations

The goal of this chapter is to investigate the existence and uniqueness of pseudo S-asymptotically periodic mild solutions for a class of neutral fractional evolution equations involving the Caputo fractional operator with finite delay. We essentially use the fractional powers of closed linear operators, the semigroup theory, and some classical fixed point theorems. Furthermore, we provide an example to illustrate the applications of our abstract results.

3.1 Introduction

We consider the following abstract fractional Cauchy problem

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}\Big(u(t)-G(t,u_t)\Big)+Au(t)=F\Big(t,u_t\Big), & t\geq 0,\\ u\,(t)=\varphi\,(t), \quad -r\leq t\leq 0, \end{cases} \tag{3.1}$$

where $\alpha\in(0,1)$, and $(A,D(A))$ is a closed linear operator in a Banach space $(X,\|\cdot\|)$. Here,

$$u:\ [-r,+\infty)\longrightarrow X,$$

and

$$F\ ,G:\ \mathbb{R}^+\times\mathcal{C}\longrightarrow X,\ r>0$$

are two continuous functions, where $\mathcal{C}=C([-r,0],X)$. By u_t we denote the classical history function defined by

$$u_t(s):=u(t+s),\ -r\leq s\leq 0,$$

while the data $\varphi(\cdot)$ belongs to the space $\mathcal{C}$.

The class of pseudo S-asymptotically periodic functions was introduced in [88]. In that paper, the authors have considered the classical version of (3.1) with $\alpha=1$ and established several interesting results concerning the existence and uniqueness of pseudo S-asymptotically periodic mild solutions for such problems. The class of pseudo S-asymptotically periodic functions is a natural generalization of the class of S-asymptotically periodic functions; see [56]. The investigation of existence and uniqueness of pseudo S-asymptotically periodic mild solutions for various classes of the abstract fractional Cauchy problems is an attractive field and was the principal subject of many works. For example, in [55] the authors have examined the existence and uniqueness of pseudo S-asymptotically periodic solutions of the second-order abstract Cauchy problems. Another interesting class of the abstract fractional equations was analyzed in [110], where the authors have considered a fractional integro-differential neutral equations with order $1 < \alpha < 2$. Moreover, the knowledge of the structure of solutions is useful in numerical analysis; see [36, 38, 39].

3.2 Preliminaries

In the rest of this chapter, we always suppose that A is a closed linear operator with $0 \in \rho(A)$ and $-A$ generates a uniformly exponentially stable analytic semigroup $(T(t))_{t\geq 0}$. Moreover, we need to use the notion of fractional powers of closed linear operators. Then, we know that, for every $\theta > 0$, the operator A^θ is well defined; see Definition 1.4.1.

For $\theta \in (0,1)$, we set

$$X_\theta := D(A^\theta).$$

In the particular situation $\theta = 0$, we consider that $A^0 := I_X$ and $X_0 := X$. The fractional power space X_θ is a Banach space when it is endowed with its natural norm

$$\|\cdot\|_\theta = \left\|A^\theta \cdot\right\|.$$

Furthermore, for $0 \leq \theta_1 \leq \theta_2 \leq 1$, one has

$$X_{\theta_2} \hookrightarrow X_{\theta_1},$$

and the embedding $X_{\theta_2} \hookrightarrow X_{\theta_1}$ is compact whenever the resolvent operator of A is compact. In the sequel, we consider the Banach space

$$\mathcal{C}_\theta := C\Big([-r,0], X_\theta\Big),$$

of all continuous vector-valued functions from $[-r,0]$ into X_θ, equipped with the norm

$$\|\varphi\|_{\mathcal{C}_\theta} := \max_{s\in[-r,0]} \|\varphi(s)\|_\theta .$$

Let us define the following families of operators

$$U(t) := \int_0^\infty \zeta_\alpha(\tau) T\Big(t^\alpha \tau\Big) d\tau \quad \text{and} \quad V(t) := \alpha \int_0^\infty \tau \zeta_\alpha(\tau) T\Big(t^\alpha \tau\Big) d\tau, \ t \geq 0, \tag{3.2}$$

where
$$\zeta_\alpha(\tau) := \frac{1}{\alpha\pi}\sum_{n\geq 1}(-\tau)^{n-1}\frac{\Gamma(\alpha n-1)}{n!}\sin(\alpha n\pi), \quad \tau\in\mathbb{R}^+-\{0\}, \tag{3.3}$$
is a probability density function defined on $\mathbb{R}^+-\{0\}$. Let us recall that (see, e.g., [107])
$$\zeta_\alpha(\tau)\geq 0,\ \tau>0 \text{ and } \int_0^\infty \zeta_\alpha(\tau)\,d\tau = 1,$$
and
$$\int_0^\infty \tau^v\zeta_\alpha(\tau)\,d\tau = \frac{\Gamma(1+v)}{\Gamma(1+\alpha v)}, \quad 0\leq v\leq 1. \tag{3.4}$$

The lemma below summarizes the principal properties of these families; see [107].

Lemma 3.2.1. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup. Then, the operator families $(U(t))_{t\geq 0}$ and $(V(t))_{t\geq 0}$ defined by (3.2) have the following properties:*

(i) $(U(t))_{t\geq 0}$ and $(V(t))_{t\geq 0}$ are strongly continuous.

(ii) If $(T(t))_{t\geq 0}$ is uniformly bounded, then $U(t)$ and $V(t)$ are linear bounded operators for any fixed $t\geq 0$.

(iii) If $(T(t))_{t\geq 0}$ is compact, then $U(t)$ and $V(t)$ are compact operators for any $t>0$.

(vi) If $x\in X$, $\theta\in(0,1)$ and $t>0$, then
$$AV(t)x = A^{1-\theta}V(t)A^\theta x,$$
and
$$\left\|A^\theta V(t)\right\| \leq \frac{M_\theta}{t^{\alpha\theta}}\frac{\alpha\Gamma(1-\theta)}{\Gamma(\alpha(1-\theta))}. \tag{3.5}$$

(v) If $t\geq 0$ and $x\in X_\theta$, then
$$\|U(t)x\|_\theta \leq M\|x\|_\theta,$$
and
$$\|V(t)x\|_\theta \leq M\frac{\alpha}{\Gamma(1+\alpha)}\|x\|_\theta.$$

3.2.1 Pseudo S-asymptotically periodic mild solution

To investigate the mild solution for problem (3.1), we must introduce the space of pseudo S-asymptotically periodic functions and some of its properties. Further details about this class of functions can be founded in [88, 109, 110].

Definition 3.2.1. *A function $f\in C_b(\mathbb{R}^+,X)$ is said to be pseudo S-asymptotically periodic if there exists $\omega>0$ such that*
$$\lim_{h\to+\infty}\frac{1}{h}\int_0^h \|f(t+\omega)-f(t)\|\,dt = 0. \tag{3.6}$$
The set of such functions will be denoted by $PSAP_\omega(X)$.

Definition 3.2.2. *Let $p > 0$ and $f \in PSAP_\omega(X)$. Then, we say that $f(\cdot)$ is pseudo S-asymptotically ω-periodic of class p if*

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \sup_{t\in[\xi-p,\xi]} \|f(t+\omega)-f(t)\|\, d\xi = 0.$$

We denote the space of all such functions by $PSAP_{\omega,p}(X)$.

Proposition 3.2.1. *Let $p \geq 0$. Then*

(i) $PSAP_{\omega,p}(X) \subseteq PSAP_\omega(X)$.

(ii) $PSAP_{\omega,p}(X)$ *is a closed subspace of* $C_b(\mathbb{R}^+, X)$.

(iii) Assume that $f \in C_b(\mathbb{R}^+, X)$. Then, $f \in PSAP_{\omega,p}(X)$ if and only if, for every $\varepsilon > 0$, we have

$$\lim_{h\to+\infty} \frac{1}{h} \mu(M_{h,\varepsilon}(f)) = 0,$$

where $\mu(\cdot)$ denotes the classical Lebesgue measure and

$$M_{h,\varepsilon}(f) = \left\{ t \in [p,h] \Big/ \sup_{t\in[\xi-p,\xi]} \|f(t+\omega)-f(t)\| \geq \varepsilon \right\}.$$

At this level, for Banach spaces $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$, we define another functional framework which will be used henceforth.

Definition 3.2.3. *We say that a function $f \in C_b(\mathbb{R}^+ \times Z, W)$ is uniformly (Z,W) pseudo S-asymptotically ω-periodic of class p if*

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \left(\sup_{\|x\|_Z \leq L} \|f(t+\omega,x)-f(t,x)\|_W \right) \right) d\xi = 0,$$

for any $L > 0$. The collection of such functions will be denoted by $PSAP_{\omega,p}(\mathbb{R}^+ \times Z, W)$.

From the previously cited references, we also have the following lemmas.

Lemma 3.2.2. *Let $u \in C_b([-r,+\infty), X)$ and*

$$u|_{\mathbb{R}^+} \in PSAP_{\omega,p}(X).$$

Then, the function $t \mapsto u_t$ belongs to $PSAP_{\omega,p}(\mathcal{C})$.

Lemma 3.2.3. *Let $f \in PSAP_{\omega,p}(\mathbb{R}^+ \times \mathcal{C}, X)$. Assume that*

(1) there exists $L_f \in C_b(\mathbb{R}^+, \mathbb{R}^+)$ such that for all $(t, \phi_i) \in \mathbb{R}^+ \times \mathcal{C}$

$$\|f(t,\phi_1) - f(t,\phi_2)\| \leq L_f(t)\|\phi_1 - \phi_2\|_{\mathcal{C}};$$

(2) $u \in C_b([-r,+\infty), X)$;

(3) $u|_{\mathbb{R}^+} \in PSAP_{\omega,p}(X)$.

Then, the function $t \mapsto f(t,u_t)$ *belongs to* $PSAP_{\omega,p}(X)$.

The following result can be regarded as a generalization of Lemma 3.2.3.

Lemma 3.2.4. *Let* $\theta_1, \theta_2 \in [0,1]$ *and* $f \in PSAP_{\omega,p}(\mathbb{R}^+ \times \mathcal{C}_{\theta_1}, X_{\theta_2})$. *We assume that*

(1) *there exists* $L_f \in C_b(\mathbb{R}^+, \mathbb{R}^+)$ *such that for all* $(t,\phi_i) \in \mathbb{R}^+ \times \mathcal{C}_{\theta_1}$

$$\|f(t,\phi_1) - f(t,\phi_2)\|_{\theta_2} \leq L_f(t)\|\phi_1 - \phi_2\|_{\mathcal{C}_{\theta_1}};$$

(2) $u \in C_b([-r,+\infty), X_{\theta_1})$;

(3) $u|_{\mathbb{R}^+} \in PSAP_{\omega,p}(X_{\theta_1})$.

Then, the function $t \mapsto f(t,u_t)$ *belongs to* $PSAP_{\omega,p}(X_{\theta_2})$.

Proof. First, according to Lemma 3.2.2, we can see that $t \mapsto u_t \in PSAP_{\omega,p}(\mathcal{C}_{\theta_1})$. Now, for $h > 0$, one has

$$\begin{aligned}
&\int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|f(t+\omega, u_{t+\omega}) - f(t,u_t)\|_{\theta_2} \right) d\xi \\
\leq\; &\int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|f(t+\omega, u_{t+\omega}) - f(t,u_{t+\omega})\|_{\theta_2} \right) d\xi + \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|f(t, u_{t+\omega}) - f(t,u_t)\|_{\theta_2} \right) d\xi,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|f(t+\omega, u_{t+\omega}) - f(t,u_t)\|_{\theta_2} \right) d\xi \\
\leq\; &\int_p^h \left(\sup_{t\in[\xi-p,\xi]} \left(\sup_{\|\phi\|_{\mathcal{C}_{\theta_1}} \leq L} \left\|f(t+\omega, \phi) - f(t,\phi)\right\|_{\theta_2} \right) \right) d\xi \\
&+ \left\|L_f\right\|_{C_b(\mathbb{R}^+,\mathbb{R}^+)} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|u_{t+\omega} - u_t\|_{\mathcal{C}_{\theta_1}} \right) d\xi.
\end{aligned}$$

As a result, we get

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|f(t+\omega, u_{t+\omega}) - f(t,u_t)\|_{\theta_2} \right) d\xi = 0.$$

The proof is complete. $\square$

We are now prepared to define the mild solution for problem (3.1).

Definition 3.2.4. *A function $u \in C([-r,+\infty), X_\theta)$ is said to be a θ-mild solution for problem (3.1) if u satisfies problem (3.1) and*

$$u(t) = \varphi(t),$$

for $\varphi \in \mathcal{C}_\theta$ and $t \in [-r,0]$. In this case, u is defined explicitly as follows

$$u(t) = U(t)(\varphi(0) - G(0,\varphi) + G(t,u_t) - \int_0^t \Big((t-s)^{\alpha-1} AV(t-s)G(s,u_s)\Big)\, ds$$
$$+ \int_0^t \Big((t-s)^{\alpha-1} V(t-s)F(s,u_s)\Big)\, ds, \quad t \geq 0.$$

Moreover, if $u|_{\mathbb{R}_+} \in PSAP_{\omega,p}(X_\theta)$, then $u(\cdot)$ is called pseudo S-asymptotically ω-periodic θ-mild solution of class p for problem (3.1).

3.3 Existence and uniqueness of solution

In this section, we discuss some questions related to the existence and uniqueness of pseudo S-asymptotically ω-periodic θ-mild solutions of class p to problem (3.1). Our standing hypotheses are:

(A1) $F \in PSAP_{\omega,p}(\mathbb{R}^+ \times \mathcal{C}_\theta, X)$ and $G \in PSAP_{\omega,p}(\mathbb{R}^+ \times \mathcal{C}_\theta, X_1)$.

(A2) There exists a function $L_G(\cdot) \in C_b(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\left\|AG(t,\phi_1) - AG(t,\phi_2)\right\| \leq L_G(t) \left\|\phi_1 - \phi_2\right\|_{\mathcal{C}_\theta},$$

for all $(t,\phi_i) \in \mathbb{R}^+ \times \mathcal{C}_\theta$.

(A3) There exists a function $L_F(\cdot) \in C_b(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\left\|F(t,\phi_1) - F(t,\phi_2)\right\| \leq L_F(t) \left\|\phi_1 - \phi_2\right\|_{\mathcal{C}_\theta},$$

for all $(t,\phi_i) \in \mathbb{R}^+ \times \mathcal{C}_\theta$.

(A4) Setting $L_G = \sup_{t \in \mathbb{R}^+} L_G(t)$ and $L_F = \sup_{t \in \mathbb{R}^+} L_F(t)$, we so assume that

$$\left(C_{\theta-1} L_G + (L_G + L_F) \frac{M_\theta \Gamma(1-\theta)}{|v_0|^{1-\theta}}\right) < 1,$$

with

$$C_{\theta-1} = \|A^{\theta-1}\|.$$

Theorem 3.3.1. *Suppose that assumptions* (A1)-(A4) *hold and $-A$ be the generator of a uniformly exponentially stable analytic semigroup $T(t)_{(t\geq 0)}$. For $\theta \in [0,1)$, we assume that $\varphi \in \mathcal{C}_\theta$, $F : \mathbb{R}^+ \times \mathcal{C}_\theta \to X$ and $G : \mathbb{R}^+ \times \mathcal{C}_\theta \to X_1$ are continuous functions. Then, problem (3.1) has a unique pseudo S-asymptotic ω-periodic θ-mild solution of class p.*

Proof. We consider the Banach space

$$C_{b,0}(X_\theta) := \Big\{x : [-r,+\infty) \to X_\theta \big/ \; x|_{[-r,0]} = 0 \;\text{ and }\; x|_{\mathbb{R}^+} \in C_b(\mathbb{R}^+, X_\theta)\Big\},$$

endowed with the norm

$$\|x\|_{C_{b,0}} = \|x_0\|_{\mathcal{C}_\theta} + \sup_{t\geq 0} \|x(t)\|_\theta = \sup_{t\geq 0} \|x(t)\|_\theta .$$

According to Proposition 3.2.1, we define the closed subspace of $C_{b,0}(X_\theta)$ as follows

$$PSAP_{\omega,p,0}(X_\theta) := \Big\{x : [-r,+\infty) \to X_\theta \big/ \; x|_{[-r,0]} = 0 \;\text{ and }\; x|_{\mathbb{R}^+} \in PSAP_{\omega,p}(X_\theta)\Big\}.$$

Throughout the proof, y denotes the function defined by

$$y(t) := \begin{cases} 0, & t \geq 0, \\ \varphi(t), & t \in [-r,0]. \end{cases}$$

Let $x \in PSAP_{\omega,p,0}(X_\theta)$. Due to the continuity of $F : \mathbb{R}^+ \times \mathcal{C}_\theta \to X$ and $G : \mathbb{R}^+ \times \mathcal{C}_\theta \to X_1$ and by taking into account assumptions (A1)-(A3) and Lemma 3.2.4, we can conclude that

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \Big\| AG(t+\omega, x_{t+\omega} + y_{t+\omega}) - AG(t, x_t + y_t) \Big\| \right) d\xi = 0, \tag{3.7}$$

and

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \Big\| F(t+\omega, x_{t+\omega} + y_{t+\omega}) - F(t, x_t + y_t) \Big\| \right) d\xi = 0,$$

which means that $F \in PSAP_{\omega,p}(X)$ and $G \in PSAP_{\omega,p}(X_1)$. Further, there exist $M_F > 0$ and $M_G > 0$ such that

$$\|F(t, x_t + y_t)\| \leq M_F \;\text{ and }\; \|AG(t, x_t + y_t)\| \leq M_G \text{ for all } t \geq 0. \tag{3.8}$$

Now, we need to introduce the following operator

$$\begin{aligned} N : \;& PSAP_{\omega,p,0}(X_\theta) \to PSAP_{\omega,p,0}(X_\theta) \\ & x \to Nx, \end{aligned}$$

where

$$\begin{aligned} Nx(t) \;:=\; & U(t)(\varphi(0) - G(0,\varphi)) + G(t, x_t + y_t) - \int_0^t \Big((t-s)^{\alpha-1} A V(t-s) G(s, x_s + y_s) \Big) ds \\ & + \int_0^t \Big((t-s)^{\alpha-1} V(t-s) F(s, x_s + y_s) \Big) ds, \end{aligned}$$

with $t \geq 0$. In what follows, we show that the operator N has a unique fixed point in $PSAP_{w,p,0}(X_\theta)$. Firstly, we check that N is well defined. From the Fubini's theorem and the definition of the operator V given by (3.2), it follows from (1.12),(3.4) and (3.8) that

$$\begin{aligned}
&\int_0^t \left((t-s)^{\alpha-1}\left\|A^\theta V(t-s)\right\|\left\|F(s,x_s+y_s)\right\|\right)ds \\
\leq\ & \alpha M_F M_\theta \int_0^t \left((t-s)^{\alpha(1-\theta)-1}\int_0^\infty \left(\tau^{1-\theta}\zeta_\alpha(\tau)e^{-|\nu_0|(t-s)^\alpha\tau}\right)d\tau\right)ds \\
\leq\ & \frac{M_F M_\theta \Gamma(1-\theta)}{|\nu_0|^{1-\theta}} \\
<\ & +\infty.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^t \left((t-s)^{\alpha-1}\left\|A^\theta V(t-s)\right\|\left\|AG(s,x_s+y_s)\right\|\right)ds \\
\leq\ & \alpha M_G M_\theta \int_0^t \left((t-s)^{\alpha(1-\theta)-1}\int_0^\infty \left(\tau^{1-\theta}\zeta_\alpha(\tau)e^{-|\nu_0|(t-s)^\alpha\tau}\right)d\tau\right)ds \\
\leq\ & \frac{M_G M_\theta \Gamma(1-\theta)}{|\nu_0|^{1-\theta}} \\
<\ & +\infty,
\end{aligned}$$

for every $x \in PSAP_{\omega,p,0}(X_\theta)$. Consequently, we can see that $t \mapsto Nx(t)$ is a bounded function. Then, it remains to show that

$$\lim_{h\to+\infty}\frac{1}{h}\int_p^h \left(\sup_{t\in[\xi-p,\xi]}\|Nx(t+\omega)-Nx(t)\|_\theta\right)d\xi = 0.$$

A direct computation allows us to get

$$Nx(t+\omega)-Nx(t) = \sum_{i=1}^6 J_i(t),$$

where

$$\begin{aligned}
J_1(t) &= (U(t+\omega)-U(t))(\varphi(0)-G(0,\varphi)),\\
J_2(t) &= G(t+\omega,x_{t+\omega}+y_{t+\omega})-G(t,x_t+y_t),\\
J_3(t) &= \int_0^\omega \left((t+\omega-s)^{\alpha-1}V(t+\omega-s)AG(s,x_s+y_s)\right)ds,
\end{aligned}$$

$$J_4(t) = \int_0^t \Big((t-s)^{\alpha-1} V(t-s)(AG(s+\omega, x_{s+\omega} + y_{s+\omega}) - AG(s, x_s + y_s))\Big) ds,$$

$$J_5(t) = \int_0^\omega \Big((t+\omega-s)^{\alpha-1} V(t+\omega-s) F(s, x_s + y_s)\Big) ds,$$

$$J_6(t) = \int_0^t \Big((t-s)^{\alpha-1} V(t-s)(F(s+\omega, x_{s+\omega} + y_{s+\omega}) - F(s, x_s + y_s))\Big) ds.$$

This implies that

$$\int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|Nx(t+\omega) - Nx(t)\|_\theta \right) d\xi \le \sum_{i=1}^{6} \left(\int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_i(t)\|_\theta \right) d\xi \right).$$

Keeping in mind the exponential stability of semigroup $(T(t))_{t\ge0}$ and the definition of the operator U given by (3.2), we deduce that for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$, such that

$$\|U(t)\| \le \frac{\varepsilon}{2}, \text{ for all } t \ge t_\varepsilon.$$

First, let us start with the estimation of the quantity J_1. We have

$$\|J_1(t)\|_\theta \le (\|U(t+\omega)\| + \|U(t)\|) \|\varphi(0) - G(0,\varphi)\|_\theta \ ,$$

so

$$\begin{aligned} &\frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_1(t)\|_\theta \right) d\xi \\ \le\ & \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} (\|U(t+\omega)\| + \|U(t)\|) \|\varphi(0) - G(0,\varphi)\|_\theta \right) d\xi \\ \le\ & \|\varphi(0) - G(0,\varphi)\|_\theta \left(\frac{2Mt_\varepsilon}{h} + \varepsilon \left(1 - \frac{(p+t_\varepsilon)}{h} \right) \right), \end{aligned}$$

this implies that

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_1(t)\|_\theta \right) d\xi = 0.$$

From (3.7) and taking into account that $X_1 \hookrightarrow X_\theta$, we get

$$\begin{aligned} &\frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \left\| G(t+\omega, x_{t+\omega} + y_{t+\omega}) - G(t, x_t + y_t) \right\|_\theta \right) d\xi \\ \le\ & \frac{C_{\theta-1}}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \left\| AG(t+\omega, x_{t+\omega} + y_{t+\omega}) - AG(t, x_t + y_t) \right\| \right) d\xi; \end{aligned}$$

hence,

$$\lim_{h\to+\infty}\frac{1}{h}\int_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\left\|G(t+\omega,x_{t+\omega}+y_{t+\omega})-G(t,x_t+y_t)\right\|_\theta\right)d\xi=0,$$

and

$$t\mapsto G(t,x_t+y_t)\in PSAP_{w,p}(X_\theta).$$

At this level, let us show that

$$\lim_{h\to+\infty}\frac{1}{h}\int_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\|J_i(t)\|_\theta\right)d\xi=0,\ i=3,4,5,6.$$

Taking into account that

$$t+\omega-s\geq\frac{t+\omega}{\omega}(\omega-s),$$

and the estimates (3.5), (3.8), we deduce that

$$\begin{aligned}
&\frac{1}{h}\int_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\|J_3(t)\|_\theta\right)d\xi\\
\leq\ & M_GM_\theta\frac{\omega\Gamma(1-\theta)}{\Gamma(\alpha(1-\theta)+1)}\left(\frac{1}{h}\int_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}(t+\omega)^{\alpha(1-\theta)-1}\right)d\xi\right)\\
\leq\ & M_GM_\theta\frac{\omega\Gamma(1-\theta)}{\Gamma(\alpha(1-\theta)+1)}\left(\frac{1}{h}\int_{p}^{h}(\xi-p+\omega)^{\alpha(1-\theta)-1}\,d\xi\right),
\end{aligned}$$

which implies that

$$\lim_{h\to+\infty}\frac{1}{h}\int_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\|J_3(t)\|_\theta\right)d\xi=0,$$

and

$$\lim_{h\to+\infty}\frac{1}{h}\int_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\|J_5(t)\|_\theta\right)d\xi=0.$$

In fact, it suffices to see that

$$\begin{aligned}
&\frac{1}{h}\int_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\|J_5(t)\|_\theta\right)d\xi\\
\leq\ & M_FM_\theta\frac{\omega\Gamma(1-\theta)}{\Gamma(\alpha(1-\theta)+1)}\left(\frac{1}{h}\int_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}(t+\omega)^{\alpha(1-\theta)-1}\right)d\xi\right)\\
\leq\ & M_FM_\theta\frac{\omega\Gamma(1-\theta)}{\Gamma(\alpha(1-\theta)+1)}\left(\frac{1}{h}\int_{p}^{h}(\xi-p+\omega)^{\alpha(1-\theta)-1}d\xi\right).
\end{aligned}$$

It remains to show that

$$\lim_{h\to+\infty}\frac{1}{h}\int\limits_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\|J_i(t)\|_\theta\right)d\xi=0,\ i=4,6.$$

We examine the term I_4. To make the notation less cluttered, we define the function Q_G as follows

$$Q_G(t)=AG(t+\omega,x_{t+\omega}+y_{t+\omega})-AG(t,x_t+y_t),\ t\geq 0.$$

Thanks to (3.7), we can deduce that, for every $\varepsilon>0$, there exists $h_\varepsilon>0$ such that

$$\frac{1}{h}\int\limits_{p}^{h}\sup_{t\in[\xi-p,\xi]}\|Q_G(t)\|\,d\xi\leq\varepsilon,\ \text{ for all } h\geq h_\varepsilon. \tag{3.9}$$

Consequently,

$$\begin{aligned}
&\int\limits_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\|J_4(t)\|_\theta\right)d\xi\\
\leq\ &\int\limits_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\int\limits_{0}^{\xi-p}\left((t-s)^{\alpha-1}\left\|A^\theta V(t-s)\right\|\|Q_G(s)\|\right)ds\right)d\xi\\
&+\int\limits_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\int\limits_{\xi-p}^{t}\left((t-s)^{\alpha-1}\left\|A^\theta V(t-s)\right\|\|Q_G(s)\|\right)ds\right)d\xi\\
&:=\sum_{i=1}^{2}J_4^i(t).
\end{aligned}$$

Taking into account the definition of the operator V given in (3.2) and the estimates (1.12), we get

$$\begin{aligned}
&J_4^1(t)\\
\leq\ &\alpha M_\theta\int\limits_{p}^{h}\sup_{t\in[\xi-p,\xi]}\int\limits_{0}^{\xi-p}(t-s)^{\alpha(1-\theta)-1}\left(\int_0^\infty\tau^{1-\theta}\zeta_\alpha(\tau)e^{-|\nu_0|(t-s)^\alpha\tau}\,d\tau\right)\|Q_G(s)\|\,dsd\xi\\
\leq\ &\alpha M_\theta\int\limits_{p}^{h}\int\limits_{0}^{\xi-p}(\xi-p-s)^{\alpha(1-\theta)-1}\left(\int_0^\infty\tau^{1-\theta}\zeta_\alpha(\tau)e^{-|\nu_0|(\xi-p-s)^\alpha\tau}d\tau\right)\|Q_G(s)\|\,ds\,d\xi\\
=\ &\alpha M_\theta\int\limits_{p}^{h}\int\limits_{p}^{\xi}(\xi-s)^{\alpha(1-\theta)-1}\left(\int_0^\infty\tau^{1-\theta}\zeta_\alpha(\tau)e^{-|\nu_0|(\xi-s)^\alpha\tau}d\tau\right)\|Q_G(s-p)\|\,ds\,d\xi.
\end{aligned}$$

Using the classical Fubini's theorem, we get

$$\begin{aligned}
&J_4^1(t)\\
\leq\ & \alpha M_\theta \int\limits_p^h \|Q_G(s-p)\| \left(\int\limits_s^h (\xi-s)^{\alpha(1-\theta)-1} \left(\int_0^\infty \tau^{1-\theta}\zeta_\alpha(\tau) e^{-|\nu_0|(\xi-s)^\alpha \tau} d\tau \right) d\xi \right) ds\\
\leq\ & \alpha M_\theta \int\limits_p^h \|Q_G(s-p)\| \left(\int\limits_0^{h-s} (\xi)^{\alpha(1-\theta)-1} \left(\int_0^\infty \tau^{1-\theta}\zeta_\alpha(\tau) e^{-|\nu_0|(\xi)^\alpha \tau} d\tau \right) d\xi \right) ds.
\end{aligned}$$

Similarly as before, we obtain

$$J_4^1(t) \leq M_\theta \frac{\Gamma(1-\theta)}{|\nu_0|^{1-\theta}} \int\limits_p^h \|Q_G(s-p)\|\, ds;$$

it follows from (3.9) and Proposition 3.2.1 that

$$\lim_{h\to+\infty} \frac{1}{h} J_4^1(t) = 0.$$

Assume that $h \geq 2p$, then we get

$$\begin{aligned}
&J_4^2(t)\\
=\ & \int\limits_p^{2p} \left(\sup_{t\in[\xi-p,\xi]} \int\limits_{\xi-p}^t \left((t-s)^{\alpha-1} \left\| A^\theta V(t-s) \right\| \|Q_G(s)\| \right) ds \right) d\xi\\
&+ \int\limits_{2p}^h \left(\sup_{t\in[\xi-p,\xi]} \int\limits_{\xi-p}^t \left((t-s)^{\alpha-1} \left\| A^\theta V(t-s) \right\| \|Q_G(s)\| \right) ds \right) d\xi\\
:=\ & \sum_{i=1}^2 J_4^{2,i}(t).
\end{aligned}$$

Thanks to (3.8) and the estimate (3.5), we obtain

$$\begin{aligned}
&J_4^{2,1}(t)\\
\leq\ & 2M_G M_\theta \frac{\Gamma(1-\theta)}{\Gamma(\alpha(1-\theta))} \int\limits_p^{2p} \left(\sup_{t\in[\xi-p,\xi]} \int\limits_{\xi-p}^t \left((t-s)^{\alpha(1-\theta)-1} \right) ds \right) d\xi\\
=\ & 2M_G M_\theta \frac{\Gamma(1-\theta)}{\Gamma(1+\alpha(1-\theta))} \int\limits_p^{2p} \left(\sup_{t\in[\xi-p,\xi]} (t-\xi+p)^{\alpha(1-\theta)} \right) d\xi\\
\leq\ & 2M_G M_\theta \frac{\Gamma(1-\theta)}{\Gamma(1+\alpha(1-\theta))} \int\limits_p^{2p} p^{\alpha(1-\theta)}\, d\xi,
\end{aligned}$$

therefore
$$\lim_{h\to+\infty}\frac{1}{h}J_4^{2,1}(t)=0.$$
Concerning the term $J_4^{2,2}(t)$, we have
$$\begin{aligned}
& J_4^{2,2}(t)\\
\leq\ & M_\theta\frac{\alpha\Gamma(1-\theta)}{\Gamma(\alpha(1-\theta))}\int_{2p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\int_{\xi-p}^{t}(t-s)^{\alpha(1-\theta)-1}\|Q_G(s)\|\,ds\right)d\xi\\
\leq\ & M_\theta\frac{\alpha\Gamma(1-\theta)}{\Gamma(\alpha(1-\theta))}\int_{2p}^{h}\left(\int_{0}^{p}(s)^{\alpha(1-\theta)-1}\sup_{t\in[\xi-p,\xi]}\|Q_G(t-s)\|\,ds\right)d\xi,
\end{aligned}$$
for $h\geq h_\varepsilon$, the use of Fubini's theorem implies
$$\begin{aligned}
& \frac{1}{h}J_4^{2,2}(t)\\
\leq\ & M_\theta\frac{\Gamma(1-\theta)}{\Gamma(\alpha(1-\theta))}\int_{0}^{p}(s)^{\alpha(1-\theta)-1}\left(\frac{1}{h}\int_{2p}^{h}\sup_{t\in[\xi-p,\xi]}\|Q_G(t-s)\|\,d\xi\right)ds\\
\leq\ & M_\theta\frac{\Gamma(1-\theta)}{\Gamma(\alpha(1-\theta))}\int_{0}^{p}(s)^{\alpha(1-\theta)-1}\left(\frac{1}{h}\int_{p}^{h}\sup_{t\in[\xi-p,\xi]}\|Q_G(t)\|\,d\xi\right)ds\\
\leq\ & \varepsilon M_\theta\frac{\Gamma(1-\theta)}{\Gamma(\alpha(1-\theta))}\int_{0}^{p}s^{\alpha(1-\theta)-1}\,ds\\
\to\ & 0,\ \text{as } \varepsilon\to 0.
\end{aligned}$$
Combining all the previous estimates, we conclude that
$$\lim_{h\to+\infty}\frac{1}{h}\int_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\|J_4(t)\|_\theta\right)d\xi=0.$$
Similarly,
$$\lim_{h\to+\infty}\frac{1}{h}\int_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\|J_6(t)\|_\theta\right)d\xi=0.$$
Summing up, one can deduce that
$$N(PSAP_{\omega,p,0}(X_\theta))\subseteq PSAP_{\omega,p,0}(X_\theta). \tag{3.10}$$
Next, we will show that N is a contraction mapping. Let $x,z\in PSAP_{\omega,p,0}(X_\theta)$, taking into

account the assumptions (A2) and (A3), we get

$$\begin{aligned}
& \|Nx(t)-Nz(t)\|_\theta \\
\leq \;& C_{\theta-1}\left\|AG(t,x_t+y_t)-AG(t,z_t+y_t)\right\|_\theta \\
& +\int_0^t \left((t-s)^{\alpha-1}\left\|A^\theta V(t-s)\right\|\left\|AG(s,x_s+y_s)-AG(s,z_s+y_s)\right\|\right)ds \\
& +\int_0^t \left((t-s)^{\alpha-1}\left\|A^\theta V(t-s)\right\|\left\|F(s,x_s+y_s)-F(s,z_s+y_s)\right\|\right)ds \\
\leq \;& C_{\theta-1}L_G\|x_t-z_t\|_{\mathcal{C}_\theta} \\
& +(L_G+L_F)\int_0^t \left((t-s)^{\alpha-1}\left\|A^\theta V(t-s)\right\|\|x_s-z_s\|_{\mathcal{C}_\theta}\right)ds \\
\leq \;& \left(C_{\theta-1}L_G+(L_G+L_F)\frac{M_\theta\Gamma(1-\theta)}{|v_0|^{1-\theta}}\right)\|x-z\|_{C_{b,0}}.
\end{aligned}$$

As a result, we confirm that

$$\|Nx-Nz\|_{C_{b,0}}\leq\left(C_{\theta-1}L_G+(L_G+L_F)\frac{M_\theta\Gamma(1-\theta)}{|v_0|^{1-\theta}}\right)\|x-z\|_{C_{b,0}}.$$

Hence, taking into account assumption (A4), we conclude that the mapping

$$N: PSAP_{w,p,0}(X_\theta)\to PSAP_{w,p,0}(X_\theta)$$

is a contraction. Then, it follows from the Banach contraction principle that N has a unique fixed point $x\in PSAP_{w,p,0}(X_\theta)$. Set $u(t)=x(t)+y(t)$ for $t\in[-r,+\infty)$, we can confirm that u is a unique pseudo S-asymptotic ω-periodic θ-mild solution of class p of the problem (3.1). □

In the remainder of this section, we prove the existence of the pseudo S-asymptotic ω-periodic θ-mild solution of class p for problem (3.1) without assuming the Lipschitz property of the function F. Our strategy is based on the use of the Krasnoselskii's fixed point theorem. In order to do this, we need the following conditions:

(A5) Let $\psi_i:\ \mathbb{R}^+\to\mathbb{R}^+$, $i=1,2$ be non negative functions that satisfy the following estimate

$$\lim_{h\to+\infty}\frac{1}{h}\int_p^h\left(\sup_{t\in[\xi-p,\xi]}\psi_i(t)\right)d\xi=0,\quad i=1,2,$$

and assume that, for every $t\in\mathbb{R}^+$ and $\phi\in\mathcal{C}_\theta$, there exists $\omega>0$ such that

$$\left\|F(t+\omega,\phi)-F(t,\phi)\right\|\leq\psi_1(t)\quad\text{and}\quad\left\|AG(t+\omega,\phi)-AG(t,\phi)\right\|\leq\psi_2(t).$$

(A6) There exists a function $k: \mathbb{R}^+ \to \mathbb{R}^+$ and a constant $\delta > 0$ such that for all $\phi \in \mathcal{C}_\theta$,

$$\|F(t,\phi)\| \leq k(t), \text{ for all } t \geq 0, \tag{3.11}$$

and k satisfies the following estimate

$$\lim_{h\to+\infty} \frac{1}{h} \int_{\delta}^{h} \left((\xi - p)^{\alpha(1-\theta)} \sup_{s\in[0,\xi]} k(s) \right) d\xi = 0. \tag{3.12}$$

(A7) There exists a positive constant L_G such that

$$\left\| AG(t,\phi_1) - AG(t,\phi_2) \right\| \leq L_G \left\| \phi_1 - \phi_2 \right\|_{\mathcal{C}_\theta},$$

for all $(t,\phi_i) \in \mathbb{R}^+ \times \mathcal{C}_\theta$.

(A8) Assume that

$$\left(C_{\theta-1} L_G + L_G \frac{M_\theta \Gamma(1-\theta)}{|\nu_0|^{1-\theta}} \right) < 1.$$

Theorem 3.3.2. *Assume that* (A5)-(A8) *hold and* $-A$ *generates a compact, uniformly exponentially stable analytic semigroup* $(T(t))_{t\geq 0}$ *on* X. *For* $\theta \in [0,1)$, *we assume that* $\varphi \in \mathcal{C}_\theta$, $F: \mathbb{R}^+ \times \mathcal{C}_\theta \to X_\theta$ *is a bounded continuous function and* $G: \mathbb{R}^+ \times \mathcal{C}_\theta \to X_1$ *is a continuous function that satisfies* $G(t,0) = 0$ *for* $t \geq 0$. *Then, the problem* (3.1) *has at least one pseudo* S*-asymptotic* ω*-periodic* θ*-mild solution of class* p.

Proof. For the sake of convenience, we will conserve the notation adopted in the proof of the Theorem 3.3.1. In the sequel, our aim is to show that

$$N(PSAP_{w,p,0}(X_\theta)) \subseteq PSAP_{w,p,0}(X_\theta),$$

which means that for any $x \in PSAP_{w,p,0}(X_\theta)$,

$$\begin{aligned} Nx \quad : \quad & t \mapsto U(t)(\varphi(0) - G(0,\varphi) + G(t, x_t + y_t) \\ & - \int_0^t \left((t-s)^{\alpha-1} AV(t-s) G(s, x_s + y_s) \right) ds \\ & + \int_0^t \left((t-s)^{\alpha-1} V(t-s) F(s, x_s + y_s) \right) ds, \quad t \geq 0, \end{aligned}$$

belongs to the space $PSAP_{\omega,p}(X_\theta)$. Since the function F is bounded and G is continuous and satisfies the conditions (A5) and (A7), then by the embedding $X_\theta \hookrightarrow X$ and Lemma 3.2.4, there exist two positive constants M'_F and M'_G such that

$$\|F(t, x_t + y_t)\| \leq M'_F \quad \text{and} \quad \|AG(t, x_t + y_t)\| \leq M'_G, \text{ for all } t > 0, \tag{3.13}$$

for any $x \in PSAP_{\omega,p,0}(X_\theta)$. Furthermore, one has

$$t \mapsto G(t, x_t + y_t) \in PSAP_{w,p}(X_1). \tag{3.14}$$

Note that (3.13) guarantees the boundedness of the function Nx. Now, we look to prove that

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|Nx(t+\omega) - Nx(t)\|_\theta \right) d\xi = 0,$$

i.e.,

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_i(t)\|_\theta \right) d\xi = 0, \; i = \{1, 2, ..., 6\}.$$

From Theorem 3.3.1, it is immediate that

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_1(t)\|_\theta \right) d\xi = 0.$$

Exploiting (3.14) and the fact that $X_1 \hookrightarrow X_\theta$, we obtain

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_2(t)\|_\theta \right) d\xi = 0. \tag{3.15}$$

Taking into account (3.13) and Theorem 3.3.1, we confirm that

$$\lim_{h\to\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_i(t)\|_\theta \right) d\xi = 0, \quad i = 3, 5.$$

Our objective now is to show that

$$\lim_{h\to\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_i(t)\|_\theta \right) d\xi = 0, \quad i = 4, 6.$$

First of all, for $t \geq 0$, we set

$$Q_F(t) = F(t+\omega, x_{t+\omega} + y_{t+\omega}) - F(t, x_{t+\omega} + y_{t+\omega}).$$

According to (A5), it is evident to say that the function Q_F satisfies the following estimate

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \sup_{t\in[\xi-p,\xi]} \|Q_F(t)\| \; d\xi = 0. \tag{3.16}$$

On other side, we observe that

$$\int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_6(t)\|_\theta\right) d\xi \leq \sum_{i=1}^{2} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \left\|J_6^i(t)\right\|_\theta\right) d\xi,$$

where

$$J_6^1(t) = \int_0^t \left((t-s)^{\alpha-1} V(t-s) Q_F(t)\right) ds,$$

and

$$J_6^2(t) = \int_0^t \left((t-s)^{\alpha-1} V(t-s) \left(F(s, x_{s+\omega} + y_{s+\omega}) - F(s, x_s + y_s)\right)\right) ds.$$

Due to (3.13), (3.15) and (3.16) with Theorem 3.3.1 that

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_4(t)\|_\theta\right) d\xi = 0,$$

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \left\|J_6^1(t)\right\|_\theta\right) d\xi = 0.$$

It remains to show that

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \left\|J_6^2(t)\right\|_\theta\right) d\xi = 0.$$

Taking into account the exponential stability of the semigroup $(T(t))_{t\geq 0}$ and the definition of the operator V as presented in (3.2), we can deduce that for every $\varepsilon > 0$, there exists

$$t'_\varepsilon = \left(\alpha C_{-\beta} \frac{M_\beta \Gamma(1-\beta)}{\Gamma(\alpha(1-\beta))\varepsilon}\right)^{\frac{1}{\alpha\beta}} > 0\,,$$

such that $\|V(t)\| \leq \varepsilon$, for all $t \geq t'_\varepsilon$ and $\beta \in (0,1)$. In actual fact, it suffices to take into account the estimate (3.5) which allows us to write

$$\|V(t)x\| \leq \left\|A^{-\beta}\right\| \left\|A^\beta V(t)x\right\| \leq \alpha C_{-\beta} \frac{M_\beta \Gamma(1-\beta)}{t^{\alpha\beta}\Gamma(\alpha(1-\beta))} \|x\|,$$

for any $\beta \in (0,1)$ and $x \in X$. Then, we can write

$$\begin{aligned}
&\int_p^h \sup_{t\in[\xi-p,\xi]} J_6^2(t)\, d\xi \\
=& \int_p^{t'_\varepsilon+p} \sup_{t\in[\xi-p,\xi]} \int_0^t \left((t-s)^{\alpha-1} V(t-s) \left(F(s, x_{s+\omega} + y_{s+\omega}) - F(s, x_s + y_s)\right)\right) ds d\xi \\
&+ \int_{t'_\varepsilon+p}^h \sup_{t\in[\xi-p,\xi]} \int_0^t \left((t-s)^{\alpha-1} V(t-s) \left(F(s, x_{s+\omega} + y_{s+\omega}) - F(s, x_s + y_s)\right)\right) ds d\xi.
\end{aligned}$$

Thanks to (3.13) and the estimate (3.5), we conclude that

$$\begin{aligned}
&\frac{1}{h}\int_{p}^{t'_\varepsilon+p}\left(\sup_{t\in[\xi-p,\xi]}\int_0^t\Big((t-s)^{\alpha-1}\left\|A^\theta V(t-s)\right\|\left\|(F(s,x_{s+\omega}+y_{s+\omega})-F(s,x_s+y_s))\right\|\Big)ds\right)d\xi\\
\le\;& \frac{2M'_F M_\theta\Gamma(1-\theta)}{\Gamma(\alpha(1-\theta))}\left(\frac{1}{h}\int_p^{t'_\varepsilon+p}\xi^{\alpha(1-\theta)}d\xi\right)\to 0, \quad \text{as } h\to+\infty.
\end{aligned}$$

Keeping in mind that the function

$$s\mapsto g(s)=(s+(\xi-p))^\alpha-s^\alpha$$

is decreasing for $s\ge 0$, we get $g(0)\ge g(p)$, i.e., $\xi^\alpha-(\xi-p)^\alpha\le p^\alpha$. Hence,

$$\begin{aligned}
&\frac{1}{h}\int_{t'_\varepsilon+p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\int_0^{t-(\xi-p)}\Big((t-s)^{\alpha-1}\|V(t-s)\|\left\|A^\theta F(s,x_{s+\omega}+y_{s+\omega})-A^\theta F(s,x_s+y_s)\right\|\Big)ds\right)d\xi\\
\le\;& \frac{2M'_F\varepsilon}{\alpha C_{-\theta}}\left(\frac{1}{h}\int_{t'_\varepsilon+p}^{h}(\xi^\alpha-(\xi-p)^\alpha)\,d\xi\right)\le\frac{2M'_F\varepsilon}{\alpha C_{-\theta}}p^\alpha \;\to 0, \quad \text{as } \varepsilon\to 0.
\end{aligned}$$

At this level, the use of (3.11) justify the fact that

$$\|F(t,x_s+y_s)\|\le k(t).$$

Therefore, by (3.12), one can find

$$\begin{aligned}
&\frac{1}{h}\int_{t'_\varepsilon+p}^{h}\sup_{t\in[\xi-p,\xi]}\int_{t-(\xi-p)}^{t}\Big((t-s)^{\alpha-1}\left\|A^\theta V(t-s)\right\|\left\|F(s,x_{s+\omega}+y_{s+\omega})-F(s,x_s+y_s)\right\|\Big)dsd\xi\\
\le\;& \frac{2M_\theta\Gamma(1-\theta)}{\Gamma(\alpha(1-\theta))}\left(\frac{1}{h}\int_{t'_\varepsilon+p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\int_{t-(\xi-p)}^{t}\Big((t-s)^{\alpha(1-\theta)-1}k(s)\Big)ds\right)d\xi\right)\\
\le\;& \frac{2M_\theta\Gamma(1-\theta)}{\Gamma(\alpha(1-\theta))}\left(\frac{1}{h}\int_{t'_\varepsilon+p}^{h}\left((\xi-p)^{\alpha(1-\theta)}\sup_{s\in[0,\xi]}k(s)\right)d\xi\right),
\end{aligned}$$

then

$$\begin{aligned}
&\lim_{h\to+\infty}\left(\frac{1}{h}\int_{t'_\varepsilon+p}^{h}\sup_{t\in[\xi-p,\xi]}\int_{t-(\xi-p)}^{t}\Big((t-s)^{\alpha-1}\left\|A^\theta V(t-s)\right\|\left\|F(s,x_{s+\omega}+y_{s+\omega})-F(s,x_s+y_s)\right\|\Big)dsd\xi\right)\\
&=0.
\end{aligned}$$

Thus,

$$\lim_{h\to+\infty}\frac{1}{h}\int_p^h\left(\sup_{t\in[\xi-p,\xi]}\|J_6(t)\|_\theta\right)d\xi=0.$$

Summing up, the above results for J_i, $i\in\{1,2,...,6\}$, we conclude that

$$Nx\in PSAP_{\omega,p}(X_\theta);$$

which justify the following inclusion, that is

$$N(PSAP_{\omega,p,0}(X_\theta))\subseteq PSAP_{\omega,p,0}(X_\theta).$$

We are now in a position to show that the operator N has at least one fixed point $x\in PSAP_{\omega,p,0}(X_\theta)$. For $\rho>0$, we define the closed ball of $PSAP_{\omega,p,0}(X_\theta)$ with center 0 and radius ρ by

$$\Omega_\rho=\left\{x\in PSAP_{\omega,p,0}(X_\theta)\big/\ \|x\|_{C_{b,0}}\leq\rho\right\}.$$

Set $N=N_1+N_2$, with

$$N_1x(t):=U(t)\varphi(0)+\int_0^t\Big((t-s)^{\alpha-1}V(t-s)F(s,x_s+y_s)\Big)ds,$$

$$N_2x(t):=U(t)(G(0,\varphi))+G(t,x_t+y_t)-\int_0^t\Big((t-s)^{\alpha-1}AV(t-s)G(s,x_s+y_s)\Big)ds.$$

We first prove that there exists a positive constant ρ_0 such that $N_1x+N_2z\in\Omega_{\rho_0}$, for every pair $x,z\in\Omega_{\rho_0}$. For this purpose, we assume that for any $\rho>0$, there exist $x,z\in\Omega_\rho$ and $t\geq 0$ such that

$$\begin{aligned}
&\rho\\
\leq\ &\|N_1x(t)+N_2z(t)\|_\theta\\
\leq\ &\|U(t)\|\|\varphi(0)\|_\theta+\int_0^t\Big((t-s)^{\alpha-1}\left\|A^\theta V(t-s)\right\|\left\|F(s,x_s+y_s)\right\|\Big)ds\\
&+\|U(t)\|\|G(0,\varphi)\|_\theta+\left\|G(t,z_s+y_s)\right\|_\theta+\int_0^t\Big((t-s)^{\alpha-1}\left\|A^\theta V(t-s)\right\|\left\|AG(s,z_s+y_s)\right\|\Big)ds,
\end{aligned}$$

so

$$\begin{aligned}
&\rho\\
\leq\ &M\|\varphi\|_{\mathcal{C}_\theta}+M_F'\int_0^t\Big((t-s)^{\alpha-1}\left\|A^\theta V(t-s)\right\|\Big)ds\\
&+MC_{-\theta}L_G\|\varphi\|_{\mathcal{C}_\theta}+C_{\theta-1}L_G\left\|z_s+y_s\right\|_{\mathcal{C}_\theta}+L_G\left\|z_s+y_s\right\|_{\mathcal{C}_\theta}\int_0^t\Big((t-s)^{\alpha-1}\left\|A^\theta V(t-s)\right\|\Big)ds,
\end{aligned}$$

then

$$
\begin{aligned}
& \rho \\
\leq \; & M\|\varphi\|_{\mathcal{C}_\theta} + M_F' M_\theta \frac{\Gamma(1-\theta)}{|\nu_0|^{1-\theta}} \\
& + M C_{-\theta} L_G \|\varphi\|_{\mathcal{C}_\theta} + C_{\theta-1} L_G \left(\rho + \|\varphi\|_{\mathcal{C}_\theta}\right) + L_G \left(\rho + \|\varphi\|_{\mathcal{C}_\theta}\right) \frac{M_\theta \Gamma(1-\theta)}{|\nu_0|^{1-\theta}}.
\end{aligned}
$$

Dividing on both sides by ρ and taking the limit as ρ approaches infinity, we obtain

$$
1 \leq C_{\theta-1} L_G + L_G \frac{M_\theta \Gamma(1-\theta)}{|\nu_0|^{1-\theta}}.
$$

Combining all the above arguments, we can deduce that there exists a positive constant ρ_0, such that for any pair of $x, z \in \Omega_{\rho_0}$, one has $N_1 x + N_2 z \in \Omega_{\rho_0}$.

Now, let us show that the function N_1 is compact and the function N_2 is contraction. To do that, we should do it in several steps as follows.

Step 1: We show that the function N_1 is continuous on Ω_{ρ_0}. In fact, due the continuity of the function F, for any sequence $(x^n) \in \Omega_{\rho_0}$ such that $x^n \to x$ on Ω_{ρ_0}, one can see

$$
\left\| F(s, x_s^n + y_s) - F(s, x_s + y_s) \right\| \to 0, \text{ as } n \to +\infty.
$$

Then, by the dominate convergence theorem, we can conclude that

$$
\begin{aligned}
\|N_1 x^n(t) - N_1 x(t)\|_\theta \; & \leq \; \int_0^t \left((t-s)^{\alpha-1} \left\| A^\theta V(t-s) \right\| \left\| F(s, x_s^n + y_s) - F(s, x_s + y_s) \right\| \right) ds \\
& \to \; 0, \text{ as } n \to +\infty.
\end{aligned}
$$

Step 2: Following [108], for $t \geq 0$, we define

$$
N_1^{\varepsilon,\delta} x(t) := U(t)\varphi(0) + \alpha \int_0^t \left((t-s)^{\alpha-1} \int_\delta^\infty \left(\tau \zeta_\alpha(\tau) T(t^\alpha \tau) F(s, x_s + y_s) \right) d\tau \right) ds.
$$

The compactness of the operator $T(t)$ and Lemma 3.2.1 implies that the set $N_1^{\varepsilon,\delta}\left(\Omega_{\rho_0}\right)(t)$ is relatively compact in X_θ. Moreover, it follows from (1.12) and (3.13) that

$$
\begin{aligned}
& \left\| N_1 x(t) - N_1^{\varepsilon,\delta} x(t) \right\|_\theta \\
\leq \; & \alpha \int_0^t \left((t-s)^{\alpha-1} \int_0^\delta \left(\tau \zeta_\alpha(\tau) \left\| A^\theta T(t^\alpha \tau) \right\| \left\| F(s, x_s + y_s) \right\| \right) d\tau \right) ds \\
& + \alpha \int_{t-\varepsilon}^t \left((t-s)^{\alpha-1} \int_\delta^\infty \left(\tau \zeta_\alpha(\tau) \left\| A^\theta T(t^\alpha \tau) \right\| \left\| F(s, x_s + y_s) \right\| \right) d\tau \right) ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \alpha M_\theta M'_F \Bigg[\int_0^\delta \Bigg(\tau^{1-\theta}\zeta_\alpha(\tau) \int_0^t \Big((t-s)^{\alpha(1-\theta)-1} e^{-|\nu_0|(t-s)^\alpha \tau} \Big) ds \Bigg) d\tau \\
&\quad + \int_{t-\varepsilon}^t \Big((t-s)^{\alpha(1-\theta)-1} \Big) ds \int_0^\infty \Big(\tau^{1-\theta}\zeta_\alpha(\tau) \Big) d\tau \Bigg] \\
&\leq M_\theta M'_F \frac{\Gamma(1-\theta)}{|\nu_0|^{1-\theta}} \left[\int_0^\delta \zeta_\alpha(\tau) d\tau + \frac{1}{\alpha(1-\theta)} \varepsilon^{\alpha(1-\theta)} \right],
\end{aligned}
$$

in other word

$$
\lim_{\varepsilon,\delta\to 0} \left\| N_1 x(t) - N_1^{\varepsilon,\delta} x(t) \right\|_\theta = 0.
$$

Consequently, the set $N_1\left(\Omega_{\rho_0}\right)(t)$ is relatively compact in X_θ.

Step 3: Let $t_1 > t_2 \geq 0$ and $x \in \Omega_{\rho_0}$. Observe that, from Lemma 2.9 in [107] we deduce that

$$
\left\| A^\theta V(t_1 - s) - A^\theta V(t_2 - s) \right\| \to 0, \text{ as } t_1 \to t_2 ,
$$

and

$$
\| U(t_1) - U(t_2) \| \to 0, \text{ as } t_1 \to t_2 .
$$

On other side, one has

$$
\begin{aligned}
&\int_0^{t_2} \left(\frac{(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{(t_2 - s)^{\alpha\theta}} \right) ds \\
&= \int_0^{t_2} \left((t_2 - s)^{\alpha(1-\theta)-1} - (t_1 - s)^{\alpha(1-\theta)-1} \left(\frac{t_1 - s}{t_2 - s} \right)^{\alpha\theta} \right) ds \\
&\leq \int_0^{t_2} \left((t_2 - s)^{\alpha(1-\theta)-1} - (t_1 - s)^{\alpha(1-\theta)-1} \right) ds \\
&\to 0, \text{ as } t_1 \to t_2.
\end{aligned}
$$

This gives

$$
\begin{aligned}
&\| N_1 x(t_1) - N_1 x(t_2) \|_\theta \\
&\leq \| U(t_1) - U(t_2) \| \left\| A^\theta \varphi(0) - A^\theta G(0,\varphi) \right\| \\
&\quad + \int_0^{t_2} \Big((t_1 - s)^{\alpha-1} \left\| A^\theta V(t_1 - s) - A^\theta V(t_2 - s) \right\| \left\| F(s, x_s + y_s) \right\| \Big) ds \\
&\quad + \int_0^{t_2} \Big(\left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{q-1} \right] \left\| A^\theta V(t_2 - s) \right\| \left\| F(s, x_s + y_s) \right\| \Big) ds \\
&\quad + \int_{t_2}^{t_1} \Big((t_1 - s)^{\alpha-1} \left\| A^\theta V(t_1 - s) \right\| \left\| F(s, x_s + y_s) \right\| \Big) ds
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \|U(t_1)-U(t_2)\|\left(\left\|A^{\theta}\varphi(0)\right\|+C_{\theta-1}\|AG(0,\varphi)\|\right)\\
&+M_F'\left[\int_0^{t_2}\left((t_1-s)^{\alpha-1}\left\|A^{\theta}V(t_1-s)-A^{\theta}V(t_2-s)\right\|\right)ds\right.\\
&\left.+M_{\theta}\int_0^{t_2}\left(\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{(t_2-s)^{\alpha\theta}}\right)ds+M_{\theta}\int_{t_2}^{t_1}\left((t_1-s)^{\alpha(1-\theta)-1}\right)ds\right].
\end{aligned}
$$

Then

$$\lim_{t_1\to t_2}\|N_1x(t_1)-N_1x(t_2)\|_{\theta}=0,$$

which means that $N_1\left(\Omega_{\rho_0}\right)$ is equicontinuous. Combining the above steps, the Arzela-Ascoli theorem guarantees that N_1 is a compact operator on Ω_{ρ_0}.

Step 4: What is left is to show that N_2 is contraction. Let $x,z\in\Omega_{\rho_0}$, for $t\geq 0$, one has

$$
\begin{aligned}
& \|N_2x(t)-N_2z(t)\|_{\theta}\\
\leq\ & \left\|G(t,x_t+y_t)-G(t,z_t+y_t)\right\|_{\theta}\\
&+\int_0^{t}\left((t-s)^{\alpha-1}\left\|A^{\theta}V(t-s)\right\|\left\|AG(s,x_s+y_s)-AG(s,z_s+y_s)\right\|\right)ds,
\end{aligned}
$$

then

$$\|N_2x(t)-N_2z(t)\|_{\theta}\leq\left(C_{\theta-1}L_G+L_G\frac{M_{\theta}\Gamma(1-\theta)}{|\nu_0|^{1-\theta}}\right)\|x-z\|_{C_{b,0}}\ ,$$

it follows from (A8) that N_2 is contraction.

Finally, by applying Theorem 1.7.3, we conclude that the operator N has at least one fixed point $x\in\Omega_{\rho_0}\subset PSAP_{\omega,p,0}(X_{\theta})$. Hence, we can affirm that $u=x+y$ is the pseudo S-asymptotically ω-periodic θ-mild solution of class p for problem (3.1). □

3.4 Example

In this section, we present an example to apply our abstract theoretical results. We focus on the following delayed partial differential equation

$$
\begin{cases}
{}^{C}D_{0^+}^{\frac{1}{2}}\left(u(t,\xi)-k_2(t)\int\limits_{t-r}^{t}\left(\int_a^{\xi}b_2(s-t)u(s,\eta)\,d\eta\right)ds\right)-\dfrac{\partial^2}{\partial\xi^2}u(t,\xi)\\
\qquad\qquad =k_1(t)\int_{t-r}^{t}b_1(s-t)u(s,\xi)\,ds,\ \xi\in[0,\pi],\ t\in\mathbb{R}^+,\\
u(t,0)=u(t,\pi)=0,\quad t\in\mathbb{R}^+,\\
u(\tau,\xi)=\varphi(\tau)(\xi),\quad \tau\in[-r,0],\xi\in[0,\pi],
\end{cases}
\tag{3.17}
$$

where r is a positive constant, $\varphi\in C\left([-r,0],L^2([0,\pi])\right)$, $b_1(\cdot),b_2(\cdot)\in C([-r,0],\mathbb{R})$, and $k_1(.),k_2(.)$ are continuous functions on $\mathbb{R}^+$. Let $X:=L^2([0,\pi])$ and $A:D(A)\subseteq X\to X$ is the operator defined by

$$\begin{cases} Au := -u'', \\ D(A) = \{u \in X \,/\, u'', u' \in X, u(0) = u(\pi) = 0\}. \end{cases}$$

Remark 3.4.1. *Most of the useful spectral properties of this operator can be founded in Section 5 in [78] and Example 5.1 in [104]. For the reader's convenience, we recall that*

- *A has a discrete spectrum with eigenvalues n^2, $n \in \mathbb{N}$;*
- *A generates a uniformly exponentially stable analytic semigroup $(T(t))_{(t\geq 0)}$ defined by*

$$T(t)u := \sum_{n=1}^{\infty} e^{-n^2 t}\langle u, e_n\rangle e_n \quad \text{and} \quad \|T(t)\| \leq e^{-t},$$

  where $\{e_n \,/\, n \in \mathbb{N}\}$ is an orthonormal basis of X and $e_n(\xi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(n\xi)$ are the associated normalized eigenvectors;
- *the operator $A^{1/2}$ is well-defined and can be characterized as follows*

$$\begin{cases} (A)^{\frac{1}{2}} u := \sum_{n=1}^{\infty} n\langle u, e_n\rangle e_n, \\ (A)^{-\frac{1}{2}} u := \sum_{n=1}^{\infty} \dfrac{1}{n}\langle u, e_n\rangle e_n, \\ D(A^{1/2}) := \{u \in X \,/\, \sum_{n=1}^{\infty} n\langle u, e_n\rangle e_n \in X\}; \end{cases}$$

- *for $u \in D(A^{1/2})$, we have*

$$\|u\|_{\frac{1}{2}} = \left\|u'\right\|.$$

Let us introduce the following functions $F : \mathbb{R}^+ \times \mathcal{C}_{\frac{1}{2}} \to X$ and $G : \mathbb{R}^+ \times \mathcal{C}_{\frac{1}{2}} \to X_1$ as follows

$$\begin{cases} F(t,\phi)(\xi) = k_1(t)\int_{-r}^{0} b_1(s)\phi(s,\xi)\,ds, \\ \text{and} \\ G(t,\phi)(\xi) = k_2(t)\int_{-r}^{0}\int_{a}^{\xi} b_2(s)\phi(s,\eta)\,d\eta\,ds. \end{cases}$$

According to Theorem 3.3.1, we have the following result.

Proposition 3.4.1. *Suppose that the functions k_1, k_2 belong to $PSAP_{w,p}(\mathbb{R}^+)$ and*

$$(1+\pi)\left(\int_{-r}^{0} |b_1(s)|^2\,ds\right)^{\frac{1}{2}} \|k_1\|_{C_b([0,+\infty),\mathbb{R}^+)} + \pi\left(\int_{-r}^{0} |b_2(s)|^2\,ds\right)^{\frac{1}{2}} \|k_2\|_{C_b([0,+\infty),\mathbb{R}^+)} < r^{-\frac{1}{2}}. \tag{3.18}$$

Then, the problem (3.17) has a unique pseudo S-asymptotic ω-periodic $\frac{1}{2}$-mild solution of class p.

Proof. Note that, for $t \geq 0$ and $\phi \in \mathcal{C}_{\frac{1}{2}}$, one has

$$\begin{aligned} \left|F(t,\phi)(\xi)\right|^2 &\leq |k_1(t)|^2\left(\int_{-r}^{0} |b_1(s)|\left|\phi(s,\xi)\right|ds\right)^2 \\ &\leq |k_1(t)|^2 \int_{-r}^{0} |b_1(s)|^2\,ds \int_{-r}^{0} \left|\phi(s,\xi)\right|^2 ds. \end{aligned}$$

Using the Fubini theorem, we have

$$\begin{aligned}\left\|F(t,\phi)\right\|^2 &\leq |k_1(t)|^2\int_{-r}^{0}|b_1(s)|^2\,ds\int_{-r}^{0}\left\|\phi(s,\cdot)\right\|^2_{L^2([0,\pi])}ds\\ &\leq r\,|k_1(t)|^2\int_{-r}^{0}|b_1(s)|^2\,ds\sup_{s\in[-r,0]}\left\|\phi(s,\cdot)\right\|^2_{L^2([0,\pi])}.\end{aligned}$$

Furthermore,

$$\left\|F(t,\phi)\right\|\leq r^{\frac{1}{2}}\,|k_1(t)|\left(\int_{-r}^{0}|b_1(s)|^2\,ds\right)^{\frac{1}{2}}\left\|\phi\right\|_{\mathcal{C}_{\frac{1}{2}}},$$

and

$$\begin{aligned}&\frac{1}{h}\int_{p}^{h}\sup_{t\in[\xi-p,\xi]}\sup_{\|\phi\|_{\mathcal{C}_{\frac{1}{2}}}\leq L}\left\|F(t+\omega,\phi)-F(t,\phi)\right\|d\xi\\ \leq\;& Lr^{\frac{1}{2}}\left(\int_{-r}^{0}|b_1(s)|^2\,ds\right)^{\frac{1}{2}}\left(\frac{1}{h}\int_{p}^{h}\sup_{t\in[\xi-p,\xi]}|k_1(t+\omega)-k_1(t)|\,d\xi\right),\end{aligned}$$

which implies that

$$\lim_{h\to+\infty}\frac{1}{h}\int_{p}^{h}\sup_{t\in[\xi-p,\xi]}\sup_{\|\phi\|_{\mathcal{C}_{\frac{1}{2}}}\leq L}\left\|F(t+\omega,\phi)-F(t,\phi)\right\|d\xi=0,$$

and

$$F\in PSAP_{\omega,p}(\mathbb{R}^+\times\mathcal{C}_{\frac{1}{2}},X).\tag{3.19}$$

Moreover, we can easily see that

$$\left\|F(t,\phi_1)-F(t,\phi_2)\right\|\leq r^{\frac{1}{2}}\,\|k_1\|_{C_b([0,+\infty),\mathbb{R}^+)}\left(\int_{-r}^{0}|b_1(s)|^2\,ds\right)^{\frac{1}{2}}\left\|\phi_1-\phi_2\right\|_{\mathcal{C}_{\frac{1}{2}}},\tag{3.20}$$

for any $\phi_1,\ \phi_2\in B_{\frac{1}{2}}$. Similarly, one has

$$\begin{aligned}\left|\frac{\partial^2}{\partial\xi^2}G(t,\phi)(\xi)\right|^2 &= |k_2(t)|^2\left|\int_{-r}^{0}b_2(s)\frac{\partial^2}{\partial\xi^2}\int_{a}^{\xi}\phi(s,\eta)\,d\eta\,ds\right|^2\\ &\leq |k_2(t)|^2\left|\int_{-r}^{0}b_2(s)\frac{\partial}{\partial\xi}\phi(s,\xi)ds\right|^2\\ &\leq |k_2(t)|^2\int_{-r}^{0}|b_2(s)|^2\,ds\int_{-r}^{0}\left|\frac{\partial}{\partial\xi}\phi(s,\xi)\right|^2ds.\end{aligned}$$

This yields

$$\begin{aligned}\left\|\frac{\partial^2}{\partial\xi^2}G(t,\phi)\right\| &\leq r^{\frac{1}{2}}\,|k_2(t)|\left(\int_{-r}^{0}|b_2(s)|^2\,ds\right)^{\frac{1}{2}}\sup_{s\in[-r,0]}\left\|\phi'(s,\cdot)\right\|_{L^2([0,\pi])}\\ &= r^{\frac{1}{2}}\,|k_2(t)|\left(\int_{-r}^{0}|b_2(s)|^2\,ds\right)^{\frac{1}{2}}\left\|\phi\right\|_{\mathcal{C}_{\frac{1}{2}}}.\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\frac{1}{h}\int_p^h \sup_{t\in[\xi-p,\xi]}\sup_{\|\phi\|_{\mathcal{C}_{\frac{1}{2}}}\le L}\left\|\frac{\partial^2}{\partial\xi^2}G(t+\omega,\phi)-\frac{\partial^2}{\partial\xi^2}G(t,\phi)\right\|d\xi \\
\le\ & Lr^{\frac{1}{2}}\left(\int_{-r}^0 |b_2(s)|^2\,ds\right)^{\frac{1}{2}}\left(\frac{1}{h}\int_p^h \sup_{t\in[\xi-p,\xi]}|k_2(t+\omega)-k_2(t)|\,d\xi\right),
\end{aligned}
$$

which means that

$$
\lim_{h\to+\infty}\frac{1}{h}\int_p^h \sup_{t\in[\xi-p,\xi]}\sup_{\|\phi\|_{\mathcal{C}_{\frac{1}{2}}}\le L}\left\|\frac{\partial^2}{\partial\xi^2}G(t+\omega,\phi)-\frac{\partial^2}{\partial\xi^2}G(t,\phi)\right\|d\xi=0,
$$

and consequently

$$
G\in PSAP_{\omega,p}(\mathbb{R}^+\times\mathcal{C}_{\frac{1}{2}},X_1). \tag{3.21}
$$

On other side, we have

$$
\begin{aligned}
&\left\|\frac{\partial^2}{\partial\xi^2}G(t,\phi_1)-\frac{\partial^2}{\partial\xi^2}G(t,\phi_2)\right\| \\
\le\ & r^{\frac{1}{2}}\|k_2\|_{C_b([0,+\infty),\mathbb{R}^+)}\left(\int_{-r}^0 |b_2(s)|^2\,ds\right)^{\frac{1}{2}}\left\|\phi_1-\phi_2\right\|_{\mathcal{C}_{\frac{1}{2}}},
\end{aligned}
\tag{3.22}
$$

for any $\phi_1,\ \phi_2\in\mathcal{C}_{\frac{1}{2}}$. Observe that, from (3.19), (3.20), (3.21), and (3.22), we can deduce that the condition (A1), (A2), and (A3) from Section 3 hold. It is immediate that (3.18) implies that the condition (A4) holds with $\|A^{-1/2}\|=1$, $M_{\frac{1}{2}}=\Gamma(1/2)=\sqrt{\pi}$ and $\nu_0=-1$. By Theorem 3.3.1, we conclude that the problem (3.17) has a unique pseudo S-asymptotical ω-periodic $\frac{1}{2}$-mild solution of class p. □

Chapter 4

S-asymptotically Bloch type periodic solutions for abstract fractional equations involving ψ-Hilfer derivatives

The aim of this chapter is to investigate the existence and uniqueness of S-asymptotically Bloch type periodic solutions for a class of the neutral ψ-Hilfer fractional derivative equations with infinite delay. Our approach is based on the semigroup theory, the fractional powers of linear operators, as well as the Banach contraction mapping principle and the Schauder's fixed point theorem. In the end, we present an example to illustrate the applications of the abstract results.

Note: The notations used here are entirely independent of those used in Chapter 3, except for what we specifically mention.

4.1 Introduction

Let $0 < \alpha \leq 1$ and $0 \leq \beta \leq 1$. Consider the following nonlinear fractional neutral functional differential equation with infinite delay

$$\begin{cases} {}^{H}D_{0+}^{\alpha,\beta,\psi}(u(t) - G(t, u_t)) = Au(t) + F(t, u(t), u_t), & t \geq 0, \\ u(t) = \varphi(t), \quad t \leq 0, \end{cases} \tag{4.1}$$

where A is the infinitesimal generator of a uniformly exponentially stable analytic semigroup $(T(t))_{t\geq 0}$ in a Banach space $(X, \|\cdot\|)$. In this study, we define

$$u : \mathbb{R} \longrightarrow X,$$

and u_t denotes the classical history function given by

$$u_t(s) = u(t+s), \ -\infty \leq s \leq 0,$$

while the data φ belongs to a suitable admissible phase space $\mathcal{B}$. In order to furnish a complete study of (4.1), we assume that $G: \mathbb{R}^+ \times \mathcal{B} \to X$ is a continuous function and $F: \mathbb{R}^+ \times X \times \mathcal{B} \to X$ is of a class C^1.

It is necessary to note that the study of existence and uniqueness of Bloch type periodic solutions as parts of the qualitative theory of differential equations have attracted great attention of researchers and have been developed rapidly. Such type of solutions appears in several concrete situations. For instance, it is observed that solutions to equations describing heat or wave propagation in solid-state physics often manifest the Bloch type periodicity, see [54, 77, 86, 112]. Recently, the concept of S-asymptotically Bloch type periodicity was proposed and developed in [31]. This concept can be viewed in some sense as an extension of classical Bloch type periodicity. At this level, we mention that several works have been concerned with the study of the existence and uniqueness of S-asymptotically periodic solutions [56] for ordinary differential equations with finite delay; see [47, 75, 76, 78] and references therein. To explore other perspectives and approaches, we advise the reader to consult [6, 16, 43, 101]. For further information concerning the Bloch-type periodic functions and their applications to evolution equations, we refer the reader to the recent research monographs [30] and [67].

4.2 Preliminaries

According to Remark 1.4.1, we know that if $(T(t))_{t\geq 0}$ is an analytic semigroup generated by A with $0 \in \rho(A)$, then for any $\theta > 0$, the operator $(-A)^{-\theta}$ is well defined and has the following explicit representation

$$(-A)^{-\theta} := \frac{1}{\Gamma(\theta)} \int_0^\infty t^{\theta-1} T(t)\, dt.$$

Moreover, $(-A)^{-\theta}$ is an injective continuous endomorphism of X; see Lemma 1.4.1. Then, we can define $(-A)^{\theta}$ as a closed bijective linear operator in X by

$$(-A)^{\theta} := \left((-A)^{-\theta}\right)^{-1},$$

which is a closed bijective linear operator in X.

Furthermore, the subspace $D\left((-A)^{\theta}\right)$ is dense in X and the expression

$$\|\cdot\|_{\theta} = \left\|(-A)^{\theta} \cdot\right\|$$

defines a norm on $D\left((-A)^{\theta}\right)$ for $x \in D\left((-A)^{\theta}\right)$. For $0 \leq \theta \leq 1$, set

$$X_{\theta} = D((-A)^{\theta}).$$

In the particular situation $\theta = 0$, we consider that $(-A)^0 := I_X$ and $X_0 := X$. As mentioned in Chapter 3, the fractional power space X_{θ} endowed with its natural norm $\|\cdot\|_{\theta}$ is a Banach space. In addition, for $0 \leq \theta_1 \leq \theta_2 \leq 1$, one has

$$X_{\theta_2} \hookrightarrow X_{\theta_1}. \tag{4.2}$$

Remark 4.2.1. *In the rest of this paper, we assume that the function ψ appearing in (1.7) and (1.8) satisfies the following conditions:*

(1) ψ is a non-negative increasing function on $[0,+\infty)$ such that $\psi(0)=0$.

(2) $\psi' \neq 0$ on $[0,+\infty)$.

The technical arguments used in our proofs needs the introducion of the following operators $U_\psi^\alpha(t,s)$ and $V_\psi^\alpha(t,s)$ defined on X as follows

$$U_\psi^\alpha(t,s)x = \int_0^\infty \zeta_\alpha(\tau) T((\psi(t)-\psi(s))^\alpha \tau) x d\tau, \ \ x \in X, \tag{4.3}$$

and

$$V_\psi^\alpha(t,s)x = \alpha \int_0^\infty \tau \zeta_\alpha(\tau) T((\psi(t)-\psi(s))^\alpha \tau) x d\tau, \ \ x \in X, \tag{4.4}$$

for $t \geq s \geq 0$, where $\zeta_\alpha(\cdot)$ is a probability density function defined by (3.3).

Proposition 4.2.1. *For $\theta \geq 0$, we have*

(i) For any fixed $t > s \geq 0$, $U_\psi^\alpha(t,s)$ and $V_\psi^\alpha(t,s)$ are linear bounded operators.

(ii) If $(T(t))_{t\geq 0}$ is a compact, then $U_\psi^\alpha(t,s)$ and $V_\psi^\alpha(t,s)$ are compact operators in X_θ for every $t > s \geq 0$, and hence $U_\psi^\alpha(t,s)$ and $V_\psi^\alpha(t,s)$ are immediately norm-continuous.

(iii) Let $\theta < 1$, we have

$$\int_0^t (\psi(t)-\psi(s))^{\alpha-1} \left\| (-A)^\theta V_\psi^\alpha(t,s) \right\| \psi^{'}(s) ds \leq M_\theta \frac{\Gamma(1-\theta)}{|\nu_0|^{1-\theta}}, \text{ for all } t > 0. \tag{4.5}$$

Proof. (i) See Lemma 3.4 in [85].

(ii) For $R > 0$, we set

$$Y_R = \{x \in X_\theta / \ \|x\|_\theta \leq R\}.$$

We need to show that the sets

$$U_\psi^\alpha(Y_R)(t,s) = \left\{ \int_0^\infty \zeta_\alpha(\tau) T((\psi(t)-\psi(s))^\alpha \tau) x d\tau \, / \, x \in Y_R \right\},$$

and

$$V_\psi^\alpha(Y_R)(t,s) = \left\{ \alpha \int_0^\infty \tau\zeta_\alpha(\tau)T((\psi(t)-\psi(s))^\alpha \tau)xd\tau \,/\, x \in Y_R \right\}$$

are relatively compact for $t > s \geq 0$. Let $t > s \geq 0$ be fixed; for $\delta > 0$, we define the subset $U_{\psi,\delta}^\alpha(Y_R)(t,s)$ in X_θ by

$$U_{\psi,\delta}^\alpha(Y_R)(t,s) := \left\{ \int_\delta^\infty \zeta_\alpha(\tau)T((\psi(t)-\psi(s))^\alpha \tau)xd\tau \,/\, x \in Y_R \right\}.$$

It is immediate that

$$\int_\delta^\infty \zeta_\alpha(\tau)T((\psi(t)-\psi(s))^\alpha \tau)xd\tau = T((\psi(t)-\psi(s))^\alpha \delta) \int_\delta^\infty \zeta_\alpha(\tau)T((\psi(t)-\psi(s))^\alpha (\tau-\delta))xd\tau.$$

According to Lemma 3.3 [81], we can deduce that the set $U_{\psi,\delta}^\alpha(Y_R)(t,s)$ is relatively compact in X_θ for all $\delta > 0$. On the other hand, it follows from Theorem 1.4.2 that

$$\begin{aligned}
\left\| U_\psi^\alpha(t,s)x - U_{\psi,\delta}^\alpha(t,s)x \right\|_\theta &= \left\| \int_0^\delta \zeta_\alpha(\tau)T((\psi(t)-\psi(s))^\alpha \tau)xd\tau \right\|_\theta \\
&\leq \int_0^\delta \zeta_\alpha(\tau) \left\| (-A)^\theta T((\psi(t)-\psi(s))^\alpha \tau)x \right\| d\tau \\
&\leq \int_0^\delta \zeta_\alpha(\tau) \left\| T((\psi(t)-\psi(s))^\alpha \tau)(-A)^\theta x \right\| d\tau \\
&\leq \rho M \int_0^\delta \xi_\alpha(\tau)d\tau \, \|x\|_\theta .
\end{aligned}$$

Then, we conclude that for any $x \in Y_R$

$$\lim_{\delta \to 0} \left\| U_\psi^\alpha(t,s)x - U_{\psi,\delta}^\alpha(t,s)x \right\|_\theta = 0.$$

Consequently, there exist relatively compact sets arbitrarily close to the set $U_\psi^\alpha(Y_R)(t,s)$ for $t > s \geq 0$. As a result, the set $U_\psi^\alpha(Y_R)(t,s)$ for $t > s \geq 0$ is also relatively compact in X_θ. Using the same reasoning, we obtain a similar result for the set $V_\psi^\alpha(Y_R)(t,s)$.

(iii) According to the definition of the operator $V_\psi^\alpha(t,s)$ and Theorem 1.4.2, we have

$$\begin{aligned}
&\int_0^t (\psi(t)-\psi(s))^{\alpha-1} \left\| (-A)^\theta V_\psi^\alpha(t,s) \right\| \psi'(s)ds \\
&= \int_0^t (\psi(t)-\psi(s))^{\alpha-1} \left(\int_0^\infty \tau\zeta_\alpha(\tau) \left\| (-A)^\theta T((\psi(t)-\psi(s))^\alpha \tau) \right\| d\tau \right) \psi'(s)ds
\end{aligned}$$

$$\leq \alpha M_\theta \int_0^t (\psi(t)-\psi(s))^{\alpha-1}\left(\int_0^\infty \frac{\tau^{1-\theta}\zeta_\alpha(\tau)}{(\psi(t)-\psi(s))^{\alpha\theta}} e^{-|\nu_0|(\psi(t)-\psi(s))^\alpha \tau} d\tau\right)\psi^{'}(s)ds,$$

so

$$\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\left\|(-A)^\theta V_\psi^\alpha(t,s)\right\|\psi^{'}(s)ds$$
$$\leq \alpha M_\theta \int_0^\infty \tau^{1-\theta}\zeta_\alpha(\tau)\left(\int_0^t (\psi(t)-\psi(s))^{\alpha(1-\theta)-1} e^{-|\nu_0|(\psi(t)-\psi(s))^\alpha \tau}\psi^{'}(s)ds\right)d\tau.$$

Set

$$\xi := |\nu_0|(\psi(t)-\psi(s))^\alpha \tau;$$

keeping in memory the formula (3.4), we see that the inequality (4.5) is true. □

4.2.1 Notion of phase space

To establish our main results, it is necessary to introduce the notion of a phase space. Let $\mathcal{B}$ be a linear space with a seminorm $\|\cdot\|_{\mathcal{B}}$ consisting of functions from $(-\infty,0]$ into X. As presented in Chapter 1 in [58], the fundamental axioms required on $\mathcal{B}$ are given as follows:

(A): If u is a function mapping $(-\infty,\delta+b]$ into X, $b>0$, such that

$$u\big|_{[\delta,T+\delta]} \in C([\delta,b+\delta];X),$$

and $t\in[\delta,b+\delta]$ and $u_\delta\in\mathcal{B}$, then for every $t\in[\delta,b+\delta]$ the following conditions hold:

(i) $u_t\in\mathcal{B}$ for $t\in[\delta,b+\delta]$,

(ii) There exist a continuous function $\mu_1(t)>0$ and a locally bounded function $\mu_2(t)>0$ from $[0,+\infty)$ into $[0,+\infty)$, for $t\geq 0$, which are independent of v such that

$$\|u_t\|_{\mathcal{B}} \leq \mu_1(t-\delta)\sup_{\delta\leq s\leq t}\|u(s)\| + \mu_2(t-\delta)\|u_\delta\|_{\mathcal{B}},$$

(iii) $\|u(t)\|\leq K\|u_t\|_{\mathcal{B}}$ which is equivalent to $\|\varphi(0)\|\leq K\|\varphi\|_{\mathcal{B}}$ for all $\varphi\in\mathcal{B}$.

(A-1): For the function u in (A), the function $t\mapsto u_t$ is continuous from $[\delta,b+\delta]$ into $\mathcal{B}$.

For the reader convenience, we recall also some basic useful properties of this kind of functional spaces, that is

(B): The space $\mathcal{B}$ is complete.

(C-2): If $(\phi^n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence of continuous functions with compact support and $\phi^n\to\phi$, $n\to\infty$ in the compact open topology, then $\phi\in\mathcal{B}$ and

$$\left\|\phi^n-\phi\right\|\to 0 \text{ as } n\to\infty.$$

Remark 4.2.2. *Since $\mathcal{B}$ satisfies axiom (C-2), the space $C_b((-\infty,0],X)$ is continuously included in $\mathcal{B}$ (cf. [58, Proposition 7.1.1]). Thus, there exists a constant $K' \geq 0$ such that*

$$\|\varphi\|_{\mathcal{B}} \leq K' \sup_{s\leq 0} \|\varphi(s)\|,$$

for every $\varphi \in C_b((-\infty,0],X)$.

Now, for $t \geq 0$, we consider the operator $S(t) : \mathcal{B} \to \mathcal{B}$ given by

$$(S(t)\varphi)(s) = \begin{cases} \varphi(0), & -t \leq s \leq 0, \\ \varphi(t+s), & s < -t, \end{cases}$$

for $\varphi \in \mathcal{B}$. This family of operators is simply a strongly continuous semigroup of bounded linear operators on $\mathcal{B}$ (cf. [58, Proposition 1.2.2]).

Definition 4.2.1. *The phase space $\mathcal{B}$ is called a fading memory space if*

$$\lim_{t\to\infty} \|S_0(t)\varphi\|_{\mathcal{B}} \to 0, \text{ for each } \varphi \in \mathcal{B}^0,$$

where

$$\mathcal{B}^0 = \{\varphi \in \mathcal{B} \,|\, \varphi(0) = 0\},$$

and $S_0(t)$ the restriction of $S(t)$ to $\mathcal{B}^0$.

Example 4.2.1. *Let h be a positive continuous function on $(-\infty,0]$ satisfying the following:*

(g-1) $H(t) = \displaystyle\sup_{s\in(-\infty,-t]} \frac{h(t+s)}{h(s)}$ *is bounded for $t \geq 0$,*

(g-2) $\displaystyle\lim_{s\to-\infty} h(s) = \infty.$

Then, $\mathcal{B} = C_h^0((-\infty,0],X)$ being the space consisting of continuous functions $\varphi : (-\infty,0] \to X$ such that

$$\lim_{s\to-\infty} \frac{\|\varphi(s)\|}{h(s)} = 0$$

is a fading memory space. Moreover, $\|S_0(t)\|_{\mathcal{L}(\mathcal{B})} = H(t)$ for $t \geq 0$.

Remark 4.2.3. *In the case that $\mathcal{B}$ is a fading memory space, one can choose the functions $\mu_1(\cdot)$ and $\mu_2(\cdot)$ in axiom (A-iii) so that $\mu_1(\cdot) = \mu_1$ and $\mu_2(\cdot) = \mu_2$ are constants (cf. [58, Proposition 7.1.5 (i)]).*

4.2.2 S-asymptotically Bloch type periodic mild solution

Let us introduce some functional spaces which play an important role in our study. Further details can be found in [32].

Definition 4.2.2. *A function $f \in C_b(\mathbb{R}^+, X)$ is called S-asymptotically ω-periodic if there exists $\omega > 0$ such that*

$$\lim_{t \longrightarrow +\infty} \|f(t+\omega) - f(t)\| = 0.$$

The set of such functions will be denoted by $SAP_\omega(X)$.

Definition 4.2.3. *A function $f \in C_b(\mathbb{R}^+, X)$ is said to be S-asymptotically Bloch type periodic if, for given $k \in \mathbb{R}$ and $\omega > 0$*

$$\lim_{t \to +\infty} \left\| f(t+\omega) - e^{i\omega k} f(t) \right\| = 0$$

holds for each $t \geq 0$. 4. The collection of such functions will be denoted by $SABP_{\omega,k}(X)$.

Definition 4.2.4. *A function $f \in C_b(\mathbb{R}^+, X)$ is said to be S-asymptotically ω-anti-periodic if there exists $\omega > 0$ such that*

$$\lim_{t \to +\infty} \|f(t+\omega) + f(t)\| = 0.$$

We denote the space of all such functions by $SAAP_\omega(X)$.

Remark 4.2.4. *If $k\omega = 2\pi$, Definition 4.2.3 is equivalent to Definition 4.2.2. Similarly, when $k\omega = \pi$, Definition 4.2.3 can be reduced to Definition 4.2.4.*

Lemma 4.2.1. *Let $f_1, f_2, f \in SABP_{w,k}(X)$. Then the following results hold:*

(i) $f_1 + f_2 \in SABP_{w,k}(X)$, *and* $cf \in SABP_{w,k}(X)$ *for each* $c \in \mathbb{C}$.

(ii) The space $SABP_{w,k}(X)$ is a Banach space with the sup-norm.

Proof. See the proofs of Lemma 3.1 and Theorem 3.2 in [32]. □

Remark 4.2.5. *From (4.2), it is clear that the condition*

$$F : \mathbb{R}^+ \times X_\theta \times \mathcal{B}_\theta \to X$$

is weaker than

$$F : \mathbb{R}^+ \times X \times \mathcal{B} \to X,$$

where $\mathcal{B}_\theta$ stands for the phase space with respect to the space X_θ.

Based on the work of F. Norouzi and G. M. N'guérékata (2021) [85], we define the θ-mild solution for the Cauchy problem (4.1) as follows.

Definition 4.2.5. *A function $u \in C(\mathbb{R}, X_\theta)$ is said to be an θ-mild solution for the Cauchy problem (4.1) if u satisfies*

$$u(t) = \varphi(t), \quad with \quad \varphi \in \mathcal{B}_\theta \quad and \quad t \in (-\infty, 0],$$

and u is given explicitly by

$$\begin{aligned} u(t) &= U_{\psi}^{\alpha}(t,0)\frac{\varphi(0)-G(0,\varphi)}{\Gamma(\gamma)\Gamma(2-\gamma)} + G(t,u_t) \\ &+ \int_0^t \Big((\psi(t)-\psi(s))^{\alpha-1} V_{\psi}^{\alpha}(t,s)AG(s,u_s)\psi'(s)\Big)ds \\ &+ \int_0^t \Big((\psi(t)-\psi(s))^{\alpha-1} V_{\psi}^{\alpha}(t,s)F(s,u(s),u_s)\psi'(s)\Big)ds, \end{aligned} \tag{4.6}$$

for $t \geq 0$, $\gamma = \alpha + \beta(1-\alpha)$, and $\varphi(0) = G(0,\varphi) = 0$. Moreover, if

$$u|_{[0,+\infty)} \in SABP_{\omega,k}(X_\theta),$$

then the vectorial function (4.6) u is called an S-asymptotically Bloch type periodic θ-mild solution for problem (4.1).

Remark 4.2.6. *As a particular case, if $k\omega = \pi$, u is called S-asymptotically ω-anti-periodic θ-mild solution.*

4.3 Existence and uniqueness of solution

In this section we focus ourselves on giving some answers to the questions related to the existence and uniqueness of S-asymptotically Bloch type periodic θ-mild solutions for the problem (4.1).

Lemma 4.3.1. *Assume that $\mathcal{B}_\theta$ is a fading memory space. Let $u : \mathbb{R} \to X_\theta$ be a continuous function with $u_0 \in \mathcal{B}_\theta$ and*

$$u\,|_{\mathbb{R}^+} \in SABP_{w,k}(X_\theta).$$

Then the function $t \mapsto u_t \in SABP_{w,k}(\mathcal{B}_\theta)$.

Proof. Keeping in mind axioms (A-iii)–(A-1) and Remark 4.2.3, we have

$$\|u_t\|_{\mathcal{B}_\theta} \leq \mu_1 \sup_{0\leq s\leq t} \|u(s)\|_\theta + \mu_2 \|u_0\|_{\mathcal{B}_\theta}\,;$$

thus $t \mapsto u_t$ is a bounded continuous function on $[0,+\infty)$. Now, we define the function

$$v(t) = u(t+\omega) - e^{i\omega k}u(t), \quad \text{for all } t \in \mathbb{R}.$$

Observe that $v : \mathbb{R} \to X_\theta$ is a continuos function on $[0,+\infty)$ and satisfies the condition

$$v_0 = u_\omega - e^{i\omega k}u_0 \in \mathcal{B}_\theta.$$

Moreover, we have

$$\lim_{t\to+\infty}\|v(t)\|_\theta = \lim_{t\to+\infty}\left\|u(t+\omega)-e^{i\omega k}u(t)\right\|_\theta = 0;$$

using the results obtained in [58, Proposition 7.1.3], we easily deduce that

$$\lim_{t\to+\infty}\|v_t\|_{\mathcal{B}_\theta} = \lim_{t\to+\infty}\left\|u_{t+\omega}-e^{i\omega k}u_t\right\|_{\mathcal{B}_\theta} = 0.$$

□

Remark 4.3.1. *If X is a real Banach space, then Lemma 4.3.1 hods true only for $k\omega=\pi$ (or 2π).*

Proposition 4.3.1. *For $\theta\in[0,1)$, we assume that $F:\ \mathbb{R}^+\times X_\theta\times\mathcal{B}_\theta\to X$ and $G:\ \mathbb{R}^+\times\mathcal{B}_\theta\to X_1$ are two continuous functions satisfying the following conditions:*

(H1) *For all $(t,x)\in[0,+\infty)\times X_\theta$,*

$$\sup_{t\geq 0}\|F(t,x,0)\|<+\infty \text{ and } \sup_{t\geq 0}\|AG(t,0)\|<+\infty\,.$$

(H2) *There exist L, L_1, $L_2>0$ such that for all $t\geq 0$, $x_1,x_2\in X_\theta$ and $\phi_1,\phi_2\in\mathcal{B}_\theta$,*

$$\left\|F(t,x,\phi_1)-F(t,x_2,\phi_2)\right\|\leq L_1\|x_1-x_2\|_\theta+L_2\left\|\phi_1-\phi_2\right\|_{\mathcal{B}_\theta},$$

and

$$\left\|AG(t,\phi_1)-AG(t,\phi_2)\right\|\leq L\left\|\phi_1-\phi_2\right\|_{\mathcal{B}_\theta}.$$

(H3) *For all $(t,x,\phi)\in[0,+\infty)\times X_\theta\times\mathcal{B}_\theta$, and for a given $k\in\mathbb{R}$ and $\omega\geq 0$,*

$$\lim_{t\to+\infty}\left\|F(t+\omega,x,\phi)-e^{ik\omega}F(t,e^{-ik\omega}x,e^{-ik\omega}\phi)\right\|=0,$$

and

$$\lim_{t\to+\infty}\left\|AG(t+\omega,\phi)-e^{i\omega k}AG(t,e^{-i\omega k}\phi)\right\|=0.$$

Then for each $u\in SABP_{w,k}(X_\theta)$, the function

$$t\mapsto F(t,u(t),u_t)\in SABP_{w,k}(X),$$

and the function

$$t\mapsto G(t,u_t)\in SABP_{w,k}(X_1).$$

Proof. From conditions (H1)–(H2), we see that

$$\sup_{t\geq 0}\|F(t,u(t),u_t)\|\leq L_2\sup_{t\geq 0}\|u_t\|_{\mathcal{B}_\theta}+\sup_{t\geq 0}\|F(t,u(t),0)\|<+\infty, \tag{4.7}$$

and

$$\sup_{t\geq 0}\|AG(t,u_t)\|\leq L\sup_{t\geq 0}\|u_t\|_{\mathcal{B}_\theta}+\sup_{t\geq 0}\|AG(t,0)\|<+\infty. \tag{4.8}$$

Taking into account that $u \in SABP_{w,k}(X_\theta)$ and using Lemma 4.3.1, we conclude that there exists a positive constant $t_{\varepsilon,1}$ sufficiently large such that for $t \geq t_{\varepsilon,1}$,

$$\left\|u(t+\omega) - e^{i\omega k}u(t)\right\|_\theta \leq \varepsilon \ \text{ and } \ \left\|u_{t+\omega} - e^{i\omega k}u_t\right\|_{\mathcal{B}_\theta} \leq \varepsilon. \tag{4.9}$$

At this level, observe that for every $(x,\phi) \in X_\theta \times \mathcal{B}_\theta$, the condition (H3) allows us to deduce that there exists a positive constant $t_{\varepsilon,2}$ such that

$$\left\|F(t+\omega, x, \phi) - e^{ik\omega}F(t, e^{-ik\omega}x, e^{-ik\omega}\phi\right\| \leq \varepsilon,$$

and

$$\left\|AG(t+\omega,\phi) - e^{i\omega k}AG(t, e^{-i\omega k}\phi)\right\| \leq \varepsilon,$$

for any $t \geq t_{\varepsilon,2}$. Since the function $t \mapsto u(t+\omega) \in X_\theta$ and axiom (A-i) implies that

$$t \mapsto u_{t+\omega} \in \mathcal{B}_\theta,$$

for all $t \geq 0$, we conclude that

$$\left\|F(t+\omega, u(t+\omega), u_{t+\omega}) - e^{ik\omega}F(t, e^{-ik\omega}u(t+\omega), e^{-ik\omega}u_{t+\omega})\right\| \leq \varepsilon, \tag{4.10}$$

and

$$\left\|AG(t+\omega, u_{t+\omega}) - e^{i\omega k}AG(t, e^{-i\omega k}u_{t+\omega})\right\| \leq \varepsilon. \tag{4.11}$$

Furthermore, for

$$t \geq t_\varepsilon := \max\left(t_{\varepsilon,1}, t_{\varepsilon,2}\right),$$

it follows from condition (H2) that

$$\begin{aligned}
& \left\|F(t+\omega, u(t+\omega), u_{t+\omega}) - e^{i\omega k}F(t, u(t), u_t)\right\| \\
\leq\ & \left\|F(t+\omega, u(t+\omega), u_{t+\omega}) - e^{ik\omega}F(t, e^{-ik\omega}u(t+\omega), e^{-ik\omega}u_{t+\omega})\right\| \\
& + L_1\left\|u(t+\omega) - e^{i\omega k}u(t)\right\|_\theta + L_2\left\|u_{t+\omega} - e^{i\omega k}u_t\right\|_{\mathcal{B}_\theta} \\
\leq\ & \varepsilon\left(1 + L_1 + L_2\right);
\end{aligned}$$

this gives

$$\lim_{t\to+\infty}\left\|F(t+\omega, u(t+\omega), u_{t+\omega}) - e^{i\omega k}F(t, u(t), u_t)\right\| = 0.$$

Similarly,

$$\begin{aligned}
& \left\|AG(t+\omega, u_{t+\omega}) - e^{i\omega k}AG(t, u_t)\right\| \\
\leq\ & \left\|AG(t+\omega, u_{t+\omega}) - e^{i\omega k}AG(t, e^{-i\omega k}u_{t+\omega})\right\| + L\left\|u_{t+\omega} - e^{i\omega k}u_t\right\|_{\mathcal{B}_\theta} \\
\leq\ & \varepsilon\left(1 + L\right),
\end{aligned}$$

which implies that

$$\lim_{t\to+\infty}\left\|AG(t+\omega, u_{t+\omega}) - e^{i\omega k}AG(t, u_t)\right\| = 0.$$

□

The following existence result for problem (4.1) is based on the use of the Banach contraction mapping principle.

Theorem 4.3.1. *Let A generate a uniformly exponentially stable analytic semigroup $(T(t))_{t\geq 0}$ in a Banach space X, with the growth exponent $\nu_0 < 0$. For $\theta \in [0,1)$, we assume that $\mathcal{B}_\theta$ is a fading memory space, $\varphi \in \mathcal{B}_\theta$, $F:\ \mathbb{R}^+ \times X_\theta \times \mathcal{B}_\theta \to X$ and $G:\ \mathbb{R}^+ \times \mathcal{B}_\theta \to X_1$ are two continuous functions that satisfy the conditions (H1)–(H3) with*

$$\varphi(0) = G(0,\varphi) = 0.$$

If the following condition holds

$$\left(\mu_1 L C_{\theta-1} + M_\theta \frac{\Gamma(1-\theta)(\mu_1 L + \max(L_1, \mu_1 L_2))}{|\nu_0|^{1-\theta}}\right) < 1, \tag{4.12}$$

where

$$C_{\theta-1} := \left\|(-A)^{\theta-1}\right\|,$$

then the problem (4.1) has a unique S-asymptotically Bloch type periodic θ-mild solution.

Proof. Consider the Banach space

$$C_{b,0}(X_\theta) = \left\{x : \mathbb{R} \to X_\theta \big/ \, x\big|_{(-\infty,0]} = 0,\ x\big|_{[0,+\infty)} \in C_b(\mathbb{R}^+, X_\theta)\right\},$$

equipped with the norm

$$\|x\|_{C_{b,0}} = \|x_0\|_{\mathcal{B}_\theta} + \sup_{t\geq 0}\|x(t)\| = \sup_{t\geq 0}\|x(t)\|.$$

According to Lemma 4.2.1, we define the closed subspace of $C_{b,0}(X_\theta)$ denoted by $SABP^0_{\omega,k}(X_\theta)$ as follows

$$SABP^0_{\omega,k}(X_\theta) = \left\{x : \mathbb{R} \to X_\theta \big/ \, x\big|_{(-\infty,0]} = 0,\ x\big|_{[0,+\infty)} \in SABP_{\omega,k}(X_\theta)\right\}.$$

Throughout the proof, $y(\cdot)$ denotes the function defined by

$$y(t) = \begin{cases} 0, & t \geq 0, \\ \varphi(t), & t \leq 0. \end{cases}$$

We introduce the operator the operator

$$N : SABP^0_{\omega,k}(X_\theta) \to SABP^0_{\omega,k}(X_\theta)$$

defined by its action as follows

$$\begin{aligned} Nx(t) \;&=\; U^\alpha_\psi(t,0)\frac{\varphi(0) - G(0,\varphi)}{\Gamma(\gamma)\Gamma(2-\gamma)} + G(t, x_t + y_t) \\ &\quad + \int_0^t \left((\psi(t)-\psi(s))^{\alpha-1} V^\alpha_\psi(t,s) A G(s, x_s + y_s)\psi'(s)\right) ds \qquad (4.13) \\ &\quad + \int_0^t \left((\psi(t)-\psi(s))^{\alpha-1} V^\alpha_\psi(t,s) F(s, x(s)+y(s), x_s + y_s)\psi'(s)\right) ds, \end{aligned}$$

with $t \geq 0$.

We shall show that the operator N has a unique fixed point in $SABP^{0}_{\omega,k}(X_\theta)$.

First of all, we check that N is well defined. Note that, for any $x \in SABP^{0}_{\omega,k}(X_\theta)$, by (4.7) and (4.8), we deduce that there exist two positive constants M_F, M_G such that

$$\left\|F(t,x(t)+y(t),x_t+y_t)\right\| \leq M_F \text{ and } \left\|AG(t,x_t+y_t)\right\| \leq M_G \text{ for all } t \geq 0. \tag{4.14}$$

Therefore, it comes from (4.5) that

$$s \mapsto (\psi(t)-\psi(s))^{\alpha-1} V_{\psi}^{\alpha}(t,s) F(s,x(s)+y(s),x_s+y_s)\psi'(s),$$

$$s \mapsto (\psi(t)-\psi(s))^{\alpha-1} A V_{\psi}^{\alpha}(t,s) G(s,x_s+y_s)\psi'(s),$$

are integrable on $[0,t)$, for every $t \geq 0$, which implies that $t \mapsto Nx(t)$, $t \geq 0$ is a bounded function. Then, it remains to show that

$$\lim_{t\to+\infty} \left\|Nx(t+\omega)-e^{i\omega k}Nx(t)\right\|_\theta = 0.$$

for any $x \in SABP^{0}_{\omega,k}(X_\theta)$. Based on the assumption that ψ is a linear function without loss of generality, a direct computation allows us to write

$$\begin{aligned}
& Nx(t+\omega)-e^{i\omega k}Nx(t) \\
= \; & \left[U_{\psi}^{\alpha}(t+\omega,0)-e^{i\omega k}U_{\psi}^{\alpha}(t,0)\right]\frac{\varphi(0)-G(0,\varphi)}{\Gamma(\gamma)\Gamma(2-\gamma)}+\left[G\left(t+\omega,x_{t+\omega}+y_{t+\omega}\right)-e^{i\omega k}G\left(t,x_t+y_t\right)\right] \\
& +\int_0^\omega \Big((\psi(t+\omega)-\psi(s))^{\alpha-1}V_{\psi}^{\alpha}(t+\omega,s)AG(s,x_s+y_s)\psi'(s)\Big)ds \\
& +\int_0^t \Big((\psi(t)-\psi(s))^{\alpha-1}V_{\psi}^{\alpha}(t,s)\Big(AG(s+\omega,x_{s+\omega}+y_{s+\omega})-e^{i\omega k}AG(s,x_s+y_s)\Big)\psi'(s)\Big)ds \\
& +\int_0^\omega \Big((\psi(t+\omega)-\psi(s))^{\alpha-1}V_{\psi}^{\alpha}(t+\omega,s)F(s,x(s)+y(s),x_s+y_s)\psi'(s)\Big)ds \\
& +\int_0^t \Big((\psi(t)-\psi(s))^{\alpha-1}V_{\psi}^{\alpha}(t,s)\left(F(s+\omega,x(s+\omega)+y(s+\omega),x_{s+\omega}+y_{s+\omega}\right) \\
& -e^{i\omega k}F(s,x(s)+y(s),x_s+y_s)\Big)\psi'(s)\Big)ds \\
= \; & \sum_{i=1}^{6} J_i(t).
\end{aligned}$$

for every $t \geq 0$. So, it is sufficient to prove that

$$\lim_{t\to+\infty}\|J_i(t)\|_\theta = 0, \quad \text{for each } i \in \{1,2,\dots,6\}.$$

Taking into account the uniformly exponentially stability of semigroup $(T(t))_{t\geq 0}$, it can be inferred that $\|T(t)\| \leq Me^{\nu_0 t}$, where $\nu_0 < 0$. Consequently, by combining the definition of the operator U_ψ^α given by (4.3) and (3.4), for every $\varepsilon > 0$, there exists a positive constant t_ε such that

$$\left\|U_\psi^\alpha(t,0)\right\| \leq \frac{\varepsilon}{2} \text{ for all } t \geq t_\varepsilon;$$

it follows that

$$\begin{aligned}
\|J_1(t)\|_\theta &= \left\|\left(U_\psi^\alpha(t+\omega,0) - e^{i\omega k}U_\psi^\alpha(t,0)\right)\frac{\varphi(0)-G(0,\varphi)}{\Gamma(\gamma)\Gamma(2-\gamma)}\right\|_\theta \\
&\leq \left(\left\|U_\psi^\alpha(t+\omega,0)\right\| + \left\|U_\psi^\alpha(t,0)\right\|\right)\left\|\frac{(-A)^\theta\varphi(0)-(-A)^\theta G(0,\varphi)}{\Gamma(\gamma)\Gamma(2-\gamma)}\right\| \\
&\leq \frac{\varepsilon}{\Gamma(\gamma)\Gamma(2-\gamma)}\left\|(-A)^\theta\varphi(0)-(-A)^\theta G(0,\varphi)\right\| \\
&\to 0, \text{ as } \varepsilon \to 0,
\end{aligned}$$

witch means that

$$\lim_{t\to+\infty}\|J_1(t)\|_\theta = 0.$$

According to Lemma 1.4.2 (i) the operator $(-A)^{\theta-1}$ is a bounded in X, and

$$\begin{aligned}
\|J_2(t)\|_\theta &= \left\|G(t+\omega,u_{t+\omega}) - e^{i\omega k}G(t,x_t+y_t)\right\|_\theta \\
&\leq C_{\theta-1}\left\|AG(t+\omega,x_{t+\omega}+y_{t+\omega}) - e^{i\omega k}AG(t,x_t+y_t)\right\|.
\end{aligned}$$

Hence, by Proposition 4.3.1, we obtain

$$\lim_{t\to+\infty}\|J_2(t)\|_\theta = 0.$$

For the terms $J_3(t)$ and $J_5(t)$, from Theorem 1.4.2 (vi), one can see that

$$\left\|(-A)^\theta T\left((\psi(t)-\psi(s))^\alpha\tau\right)\right\| \leq \frac{M_\theta}{\left((\psi(t)-\psi(s))^\alpha\tau\right)^\theta}.$$

Now, by definition of the operator V_ψ^α given by (4.4) and the use of Lemma 2, we confirm that for $z \in X$, one has

$$\begin{aligned}
\left\|(-A)^\theta V_\psi^\alpha(t,s)z\right\| &\leq \alpha\int_0^\infty \tau\zeta_\alpha(\tau)\left\|(-A)^\theta T\left((\psi(t)-\psi(s))^\alpha\tau\right)z\right\|d\tau \\
&\leq \alpha\int_0^\infty \tau\zeta_\alpha(\tau)\left\|(-A)^\theta T\left((\psi(t)-\psi(s))^\alpha\tau\right)\right\|\|z\|d\tau \\
&\leq \frac{\alpha M_\theta}{(\psi(t)-\psi(s))^{\alpha\theta}}\int_0^\infty \tau^{1-\theta}\zeta_\alpha(\tau)d\tau\|z\|.
\end{aligned}$$

Then, it follows from (2.6) that

$$\left\|(-A)^{\theta} V_{\psi}^{\alpha}(t,s)\right\| \leq \frac{\alpha M_{\theta}}{(\psi(t)-\psi(s))^{\alpha\theta}} \frac{\Gamma(1+(1-\theta))}{\Gamma(1+\alpha(1-\theta))},$$

thus

$$\left\|(-A)^{\theta} V_{\psi}^{\alpha}(t,s)\right\| \leq \frac{\eta}{(\psi(t)-\psi(s))^{\alpha\theta}}, \tag{4.15}$$

where

$$\eta = \frac{M_{\theta}\Gamma(1-\theta)}{\Gamma(\alpha(1-\theta))}.$$

Since $s \mapsto G(s, x_s + y_s)$ and $s \mapsto F(s, x(s)+y(s), x_s + y_s)$ are bounded functions on $[0,t)$, then by the fact that

$$\frac{\psi(t+\omega)}{\psi(\omega)}(\psi(\omega)-\psi(s)) \leq \psi(t+\omega)-\psi(s),$$

and (4.15), one obtains

$$\begin{aligned}
\|J_3(t)\|_{\theta} &\leq \int_0^{\omega} \left\|(\psi(t+\omega)-\psi(s))^{\alpha-1} V_{\psi}^{\alpha}(t+\omega,s) A G(s, x_s+y_s)\psi'(s)\right\|_{\theta} ds \\
&\leq \eta M_G \int_0^{\omega} (\psi(t+\omega)-\psi(s))^{\alpha(1-\theta)-1} ds \\
&\leq \eta M_G \frac{\psi(\omega)}{\alpha(1-\theta)} (\psi(t+\omega))^{\alpha(1-\theta)-1},
\end{aligned}$$

which implies that

$$\lim_{t\to+\infty} \|J_3(t)\|_{\theta} = 0.$$

Similarly,

$$\begin{aligned}
\|J_5(t)\|_{\theta} &\leq \int_0^{\omega} \left\|(\psi(t+\omega)-\psi(s))^{\alpha-1} V_{\psi}^{\alpha}(t+\omega,s) F(s, x(s)+y(s), x_s+y_s)\psi'(s)\right\|_{\theta} ds \\
&\leq \eta M_F \int_0^{\omega} (\psi(t+\omega)-\psi(s))^{\alpha(1-\theta)-1} ds \\
&\leq \eta M_F \frac{\psi(\omega)}{\alpha(1-\theta)} (\psi(t+\omega))^{\alpha(1-\theta)-1},
\end{aligned}$$

so that

$$\lim_{t\to+\infty} \|J_5(t)\|_{\theta} = 0.$$

Now, we proceed to show that

$$\lim_{t\to+\infty} \|J_i(t)\|_{\theta} = 0, \; i = 4, 6.$$

Due to Proposition 4.3.1, there exists $t_\varepsilon \geq 0$ such that for any $t > t_\varepsilon$, the following inequality

$$\frac{\psi(t)}{\psi(t_\varepsilon)}\left(\psi(t_\varepsilon)-\psi(s)\right) \leq \psi(t)-\psi(s)$$

holds true for $t_\varepsilon > s$, and

$$\begin{aligned}
&\|J_4(t)\|_\theta \\
\leq\ & \int_0^{t_\varepsilon}\Big((\psi(t)-\psi(s))^{\alpha-1}\left\|(-A)^\theta V_\psi^\alpha(t,s)\right\|\left\|AG(s+\omega,x_{s+\omega}+y_{s+\omega})-e^{i\omega k}AG(s,x_s+y_s)\right\|\psi^{'}(s)\Big)ds \\
&+\int_{t_\varepsilon}^{t}\Big((\psi(t)-\psi(s))^{\alpha-1}\left\|(-A)^\theta V_\psi^\alpha(t,s)\right\|\left\|AG(s+\omega,x_{s+\omega}+y_{s+\omega})-e^{i\omega k}AG(s,x_s+y_s)\right\|\psi^{'}(s)\Big)ds \\
\leq\ & 2\eta M_G\frac{\psi(t_\varepsilon)}{\alpha(1-\theta)}(\psi(t))^{\alpha(1-\theta)-1}+\varepsilon M_\theta\frac{\Gamma(1-\theta)}{|\nu_0|^{1-\theta}}.
\end{aligned}$$

Therefore

$$\lim_{t\to+\infty}\|J_4(t)\|_\theta = 0.$$

Similarly,

$$\begin{aligned}
&\|J_6(t)\|_\theta \\
\leq\ & \int_0^{t_\varepsilon}(\psi(t)-\psi(s))^{\alpha-1}\left\|(-A)^\theta V_\psi^\alpha(t,s)\right\|\Big\|F(s+\omega,x(s+\omega)+y(s+\omega),x_{s+\omega}+y_{s+\omega}) \\
&-e^{i\omega k}F(s,x(s)+y(s),x_s+y_s)\Big\|\psi'(s)\Big)ds \\
&+\int_{t_\varepsilon}^{t}(\psi(t)-\psi(s))^{\alpha-1}\left\|(-A)^\theta V_\psi^\alpha(t,s)\right\|\Big\|F(s+\omega,x(s+\omega)+y(s+\omega),x_{s+\omega}+y_{s+\omega}) \\
&-e^{i\omega k}F(s,x(s)+y(s),x_s+y_s)\Big\|\psi'(s)\Big)ds \\
\leq\ & 2\eta M_F\frac{\psi(t_\varepsilon)}{\alpha(1-\theta)}(\psi(t))^{\alpha(1-\theta)-1}+\varepsilon M_\theta\frac{\Gamma(1-\theta)}{|\nu_0|^{1-\theta}};
\end{aligned}$$

this gives

$$\lim_{t\to+\infty}\|J_6(t)\|_\theta = 0.$$

Combining the above arguments, we can deduce that

$$N : SABP^0_{w,k}(X_\theta) \to SABP^0_{w,k}(X_\theta)$$

is well defined.

Now, we will prove that N is a contraction mapping. Let $x,z \in SABP^0_{w,k}(X_\theta)$; from condition (H2) and axioms (A-iii) with Remark 4.2.3, we get the following estimates:

$$\left\|AG(t,x_t+y_t)-AG(t,z_t+y_t)\right\| \leq \mu_1 L\|x-z\|_{C_{b,0}},$$

$$\left\|F(s,x(s)+y(s),x_s+y_s)-F(s,z(s)+y(s),z_s+y_s)\right\| \leq \max(L_1,\mu_1 L_2)\|x-z\|_{C_{b,0}};$$

thus,

$$\begin{aligned}
&\|Nx(t)-Nz(t)\|_\theta \\
\leq\ & C_{\theta-1}\left\|AG(t,x_t+y_t)-AG(t,z_t+y_t)\right\| \\
&+\int_0^t \left((\psi(t)-\psi(s))^{\alpha-1}\left\|(-A)^\theta V_\psi^\alpha(t,s)\right\|\left\|AG(s,x_s+y_s)-AG(s,z_s+y_s)\right\|\psi^{'}(s)\right)ds \\
&+\int_0^t \left((\psi(t)-\psi(s))^{\alpha-1}\left\|(-A)^\theta V_\psi^\alpha(t,s)\right\|\left\|F(s,x(s)+y(s),x_s+y_s)-F(s,z(s)+y(s),z_s+y_s)\right\|\right. \\
&\left.\psi^{'}(s)\right)ds \\
\leq\ & \mu_1 L C_{\theta-1}\|x-z\|_{C_{b,0}} \\
&\quad +\mu_1 L\int_0^t \left((\psi(t)-\psi(s))^{\alpha-1}\left\|(-A)^\theta V_\psi^\alpha(t,s)\right\|\psi^{'}(s)ds\,\|x-z\|_{C_{b,0}}\right. \\
&\quad +\max(L_1,\mu_1L_2)\int_0^t \left((\psi(t)-\psi(s))^{\alpha-1}\left\|(-A)^\theta V_\psi^\alpha(t,s)\right\|\psi^{'}(s)\right)ds\,\|x-z\|_{C_{b,0}} \\
\leq\ & \left(\mu_1 L C_{\theta-1}+\frac{M_\theta\Gamma(1-\theta)(\mu_1 L+\max(L_1,\mu_1L_2))}{|\nu_0|^{1-\theta}}\right)\|x-z\|_{C_{b,0}},
\end{aligned}$$

with $t \geq 0$. From (4.12) it follows that

$$\begin{aligned}
&\|Nx-Nz\|_{C_{b,0}} \\
\leq\ & \left(\mu_1 L C_{\theta-1}+\frac{M_\theta\Gamma(1-\theta)(\mu_1 L+\max(L_1,\mu_1L_2))}{|\nu_0|^{1-\theta}}\right)\|x-z\|_{C_{b,0}} \\
<\ & \|x-z\|_{C_{b,0}}.
\end{aligned}$$

Then, by the Banach's contraction mapping principle, we deduce that the operator N has a unique fixed point $x \in SABP^0_{w,k}(X_\theta)$. Hence, we can affirm that $u = x+y$ is the S-asymptotically Bloch type periodic θ-mild solution to problem (4.1). □

In what follows, we will show that Proposition 4.3.1 holds true if we replace the conditions (H1) and (H2) with a new condition.

Proposition 4.3.2. *For* $\theta \in [0,1)$, *we assume that* $G : \mathbb{R}^+ \times \mathcal{B}_\theta \to X_1$ *and* $F : \mathbb{R}^+ \times X_\theta \times \mathcal{B}_\theta \to X$ *are two continuous functions that satisfy the conditions (H3) and*

(H4) *There exist* $L', L_1', L_2' > 0$ *such that for all* $x \in X_\theta$, $\phi \in \mathcal{B}_\theta$, *and* $t \geq 0$,

$$\left\|F(t,x,\phi)\right\| \leq L_1'\|x\|_\theta + L_2'\left\|\phi\right\|_{\mathcal{B}_\theta},$$

$$\left\|AG(t,\phi)\right\| \leq L'\left\|\phi\right\|_{\mathcal{B}_\theta}.$$

Then for each $u \in SABP_{w,k}(X_\theta)$, the function $t \mapsto F(t,u(t),u_t) \in SABP_{w,k}(X)$ and the function $t \mapsto G(t,u_t) \in SABP_{w,k}(X_1)$.

Proof. From condition (H4), it is a simple matter to see that the functions $t \mapsto F(t,u(t),u_t)$ and $t \mapsto G(t,u_t)$ are bounded. Indeed,

$$\sup_{t\geq 0}\|F(t,u(t),u_t)\| \leq L_1' \sup_{t\geq 0}\|u(t)\|_\theta + L_2' \sup_{t\geq 0}\|u_t\|_{\mathcal{B}_\theta} < +\infty,$$

$$\sup_{t\geq 0}\|AG(t,u_t)\| \leq L' \sup_{t\geq 0}\|u_t\|_{\mathcal{B}_\theta} < +\infty.$$

According to (4.9) and the continuity of the functions F and H, we have

$$\left\|F(t,e^{-ik\omega}u(t+\omega),e^{-ik\omega}u_{t+\omega}) - F(t,u(t),u_t)\right\| \leq \varepsilon,$$

$$\left\|AG(t,e^{-i\omega k}u_{t+\omega}) - AG(t,u_t)\right\| \leq \varepsilon,$$

for any $t \geq t_{\varepsilon,1}$ and $\varepsilon > 0$. For $t \geq t_\varepsilon$, it follows from (4.10) and (4.11) that

$$\begin{aligned}
& \left\|F(t+\omega,u(t+\omega),u_{t+\omega}) - e^{i\omega k}F(t,u(t),u_t)\right\| \\
\leq\ & \left\|F(t+\omega,u(t+\omega),u_{t+\omega}) - e^{ik\omega}F(t,e^{-ik\omega}u(t+\omega),e^{-ik\omega}u_{t+\omega})\right\| \\
& + \left\|F(t,e^{-ik\omega}u(t+\omega),e^{-ik\omega}u_{t+\omega}) - F(t,u(t),u_t)\right\| \\
\leq\ & 2\varepsilon.
\end{aligned}$$

Hence,

$$\lim_{t\to+\infty}\left\|F(t+\omega,u(t+\omega),u_{t+\omega}) - e^{i\omega k}F(t,u(t),u_t)\right\| = 0.$$

Similarly,

$$\begin{aligned}
& \left\|AG(t+\omega,u_{t+\omega}) - e^{i\omega k}AG(t,u_t)\right\| \\
\leq\ & \left\|AG(t+\omega,u_{t+\omega}) - e^{i\omega k}AG(t,e^{-i\omega k}u_{t+\omega})\right\| \\
& + \left\|AG(t,e^{-i\omega k}u_{t+\omega}) - AG(t,u_t)\right\| \\
\leq\ & 2\varepsilon.
\end{aligned}$$

Then,

$$\lim_{t\to+\infty}\left\|AG(t+\omega,u_{t+\omega}) - e^{i\omega k}AG(t,u_t)\right\| = 0.$$

The proof is complete. □

The following existence result for problem (4.1) is based on the Schauder's fixed point theorem.

Theorem 4.3.2. *Let A generate a compact and uniformly exponentially stable analytic semigroup $(T(t))_{t\geq 0}$ in a Banach space X, with the growth exponent $\nu_0 < 0$. For $\theta \in [0,1)$, we assume that $\mathcal{B}_\theta$ is a fading memory space, $\varphi \in \mathcal{B}_\theta$, $F: \mathbb{R}^+ \times X_\theta \times \mathcal{B}_\theta \to X$ and $G: \mathbb{R}^+ \times \mathcal{B}_\theta \to X_1$*

are continuos functions that satisfy the conditions (H3)–(H4), and $\varphi(0) = G(0,\varphi) = 0$. If the following condition holds

$$\left(L'C_{\theta-1}\mu_1 + (L'\mu_1 + (L_2' + L_1'\mu_1))M_\theta \frac{\Gamma(1-\theta)}{|\nu_0|^{1-\theta}}\right) < 1, \tag{4.16}$$

then the problem (4.1) has an S-asymptotically Bloch type periodic θ-mild solution.

Proof. Throughout the proof, we will use the same notation for the operator N and the Banach space $SABP^0_{\omega,k}(X_\theta)$ as previously defined in the proof of Theorem 4.3.1.

For $\rho > 0$, we define the closed ball of $SABP^0_{\omega,k}(X_\theta)$ whose centre is 0 and radius is ρ as

$$\Omega_\rho = \left\{u \in SABP^0_{\omega,k}(X_\theta) \middle/ \ \|u\|_{C_{b,0}} \le \rho\right\}.$$

We shall show that there exist a positive constant ρ_0 such that $N\left(\Omega_{\rho_0}\right) \subset \Omega_{\rho_0}$. In fact, according to Proposition 4.3.2 and the arguments in the proof of Theorem 4.3.1, it is easy to check that

$$N\left(SABP^0_{w,k}(X_\theta)\right) \subset SABP^0_{w,k}(X_\theta).$$

On the other hand, there exist two positive constant M_F' and M_G' such that

$$\left\|F(t,x(t)+y(s),x_t+y_t)\right\| \le M_F' \ \text{ and } \ \left\|AG(t,x_t+y_t)\right\| \le M_G'\text{ , for all } t \ge 0. \tag{4.17}$$

Note that, from condition (H3) and axioms (A-ii)–(A-iii) with Remark 4.2.3, we obtain

$$\|\varphi(0)\|_\theta \le K\|\varphi\|_{\mathcal{B}_\theta},$$

$$\|G(0,\varphi)\|_\theta \le C_{\theta-1}L'\|\varphi\|_{\mathcal{B}_\theta},$$

$$\left\|AG(t,x_t+y_t)\right\| \le L'\mu_1\|x\|_{C_{b,0}} + L'\mu_2\|\varphi\|_{\mathcal{B}_\theta},$$

$$\left\|F(t,x(t)+y(s),x_t+y_t)\right\| \le (L_2' + L_1'\mu_1)\|x\|_{C_{b,0}} + L_2'\mu_2\|\varphi\|_{\mathcal{B}_\theta}.$$

Now, we assume that for any $\rho > 0$, there exist $x \in \Omega_\rho$ and $t \ge 0$ such that

$$\begin{aligned}
\rho &\le \|Nx(t)\|_\theta \\
&\le \frac{\left\|U_\psi^\alpha(t,0)\right\|}{\Gamma(\gamma)\Gamma(2-\gamma)}\left(\|\varphi(0)\|_\theta + \|G(0,\varphi)\|_\theta\right) + \left\|G(t,x_t+y_t)\right\|_\theta \\
&\quad + \int_0^t \left((\psi(t)-\psi(s))^{\alpha-1}\left\|(-A)^\theta V_\psi^\alpha(t,s)\right\|\left\|AG(t,x_t+y_t)\right\|\psi'(s)\right)ds \\
&\quad + \int_0^t \left((\psi(t)-\psi(s))^{\alpha-1}\left\|(-A)^\theta V_\psi^\alpha(t,s)\right\|\left\|F(t,x(t)+y(s),x_t+y_t)\right\|\psi'(s)\right)ds
\end{aligned}$$

$$\begin{aligned}
&\leq M\left(\frac{K+C_{\theta-1}L'}{\Gamma(\gamma)\Gamma(2-\gamma)}\right)\|\varphi\|_{\mathcal{B}_\theta}+L'C_{\theta-1}\left(\mu_1\|x\|_{C_{b,0}}+\mu_2\|\varphi\|_{\mathcal{B}_\theta}\right)\\
&\quad+L'\left(\mu_1\|x\|_{C_{b,0}}+\mu_2\|\varphi\|_{\mathcal{B}_\theta}\right)\int_0^t\left((\psi(t)-\psi(s))^{\alpha-1}\left\|(-A)^\theta V_\psi^\alpha(t,s)\right\|\psi'(s)\right)ds\\
&\quad+\left((L_2'+L_1'\mu_1)\|x\|_{C_{b,0}}+L_2'\mu_2\|\varphi\|_{\mathcal{B}_\theta}\right)\int_0^t\left((\psi(t)-\psi(s))^{\alpha-1}\left\|(-A)^\theta V_\psi^\alpha(t,s)\right\|\psi'(s)\right)ds\\
&\leq M\left(\frac{K+C_{\theta-1}L'}{\Gamma(\gamma)\Gamma(2-\gamma)}\right)\|\varphi\|_{\mathcal{B}_\theta}+L'C_{\theta-1}\left(\mu_1\rho+\mu_2\|\varphi\|_{\mathcal{B}_\theta}\right)\\
&\quad+\left(L'\left(\mu_1\rho+\mu_2\|\varphi\|_{\mathcal{B}_\theta}\right)+\left((L_2'+L_1'\mu_1)\rho+L_2'\mu_2\|\varphi\|_{\mathcal{B}_\theta}\right)\right)M_\theta\frac{\Gamma(1-\theta)}{|\nu_0|^{1-\theta}}.
\end{aligned}$$

Dividing the both sides by ρ and taking the limits as $\rho\to+\infty$, it results that

$$1\leq L'C_{\theta-1}\mu_1+(L'\mu_1+(L_2'+L_1'\mu_1))M_\theta\frac{\Gamma(1-\theta)}{|\nu_0|^{1-\theta}};$$

this contradicts our assumption (4.16). Therefore, there exists a positive constant ρ_0 such that

$$Nx\subset\Omega_{\rho_0},\ \text{for any } x\in\Omega_{\rho_0}. \tag{4.18}$$

Now, we will prove the compactness of the operator N. To achieve this purpose, we should proceed in three steps as follows.

Step 1. *We show that N is continuous on* Ω_{ρ_0}. *Let* (x^n) *be a sequence such that*

$$\lim_{n\to+\infty}x^n=x,$$

on Ω_{ρ_0}. *Clearly,*

$$\lim_{n\to+\infty}\|x_t^n-x_t\|_{\mathcal{B}_\theta}\to 0.$$

for every $t\geq 0$. *Due to the continuity of the functions F and G, we have*

$$\left\|F(t,x^n(t)+y(t),x_t^n+y_t)-F(t,x(t)+y(t),x_t+y_t)\right\|\to 0,$$

and

$$\lim_{n\to+\infty}\left\|AG(t,x_t^n+y_t)-AG(t,x_t+y_t)\right\|=0.$$

Therefore, from (4.5) and the dominated convergence theorem, we get

$$\begin{aligned}
&\|Nx^n(t)-Nx(t)\|_\theta\\
\leq\ & C_{\theta-1}\left\|AG(t,x_t^n+y_t)-AG(t,x_t+y_t)\right\|\\
&+\int_0^t\left((\psi(t)-\psi(s))^{\alpha-1}\left\|(-A)^\theta V_\psi^\alpha(t,s)\right\|\left\|AG(s,x_s^n+y_s)-AG(s,x_s+y_s)\right\|\psi'(s)\right)ds\\
&+\int_0^t\left((\psi(t)-\psi(s))^{\alpha-1}\left\|(-A)^\theta V_\psi^\alpha(t,s)\right\|\left\|F(s,x^n(s)+y(s),x_s^n+y_s)-F(s,x(s)+y(s),x_s+y_s)\right\|\psi'(s)\right)ds;
\end{aligned}$$

this implies that

$$\lim_{n\to+\infty}\|Nx^n(t)-Nx(t)\|_\theta=0,$$

which means that N is continuous on Ω_{ρ_0}.

Step 2. *We verify the equicontinuity of* $N\left(\Omega_{\rho_0}\right)$ *on* $[0,+\infty)$. *We fix* $t_1\geq 0$ *and suppose that* $t_2>t_1$. *For* $x\in\Omega_{\rho_0}$, *we have*

$$Nx(t_2)-Nx(t_1)=\sum_{i=1}^{5}I_i(t_1,t_2),$$

where

$$I_1(t_1,t_2)=\left(U_\psi^\alpha(t_2,0)-U_\psi^\alpha(t_1,0)\right)\frac{\varphi(0)-G(0,\varphi)}{\Gamma(\gamma)\Gamma(2-\gamma)},$$

$$I_2(t_1,t_2)=G(t_2,x_{t_2}+y_{t_2})-G(t_1,z_{t_1}+y_{t_1}),$$

$$I_3(t_1,t_2)=\int_{t_1}^{t_2}\left((\psi(t_2)-\psi(s))^{\alpha-1}V_\psi^\alpha(t_2,s)\left(AG\left(s,x_s+y_s\right)+F\left(s,x(s)+y(s),x_s+y_s\right)\right)\psi^{'}(s)\right)ds,$$

$$\begin{aligned}I_4(t_1,t_2)\;&=\;-\int_0^{t_1}\Big(\left((\psi(t_1)-\psi(s))^{\alpha-1}-(\psi(t_2)-\psi(s))^{\alpha-1}\right)V_\psi^\alpha(t_1,s)\left(AG\left(s,x_s+y_s\right)\right.\\&\quad\left.+F\left(s,x(s)+y(s),x_s+y_s\right)\right)\psi^{'}(s)\Big)ds,\end{aligned}$$

$$\begin{aligned}I_5(t_1,t_2)\;&=\;\int_0^{t_1}\Big((\psi(t_2)-\psi(s))^{\alpha-1}\left(V_\psi^\alpha(t_2,s)-V_\psi^\alpha(t_1,s)\right)\left(AG\left(s,x_s+y_s\right)+\right.\\&\quad\left.F\left(s,x(s)+y(s),x_s+y_s\right)\right)\psi^{'}(s)\Big)ds.\end{aligned}$$

According to Proposition 4.2.1 (ii) and the continuity of G with axiom (A-1), it is a simple matter to see that

$$\lim_{t_2\to t_1}\|I_i(t_1,t_2)\|_\theta=0,\;\; i=1,2.$$

Since ψ *is an increasing linear function and* $t_2>t_1$, *it follows from* (4.15) *that*

$$\|I_3(t_1,t_2)\|_\theta\leq\eta\left(M'_G+M'_F\right)\int_{t_1}^{t_2}\left((\psi(t_2)-\psi(s))^{\alpha(1-\theta)-1}\psi^{'}(s)\right)ds.$$

Hence,

$$\lim_{t_2\to t_1}\|I_3(t_1,t_2)\|_\theta=0.$$

For the term I_4, *one can see that*

$$\begin{aligned}&\|I_4(t_1,t_2)\|_\theta\\ \leq\;&\eta\left(M'_G+M'_F\right)\int_0^{t_1}\left(\frac{(\psi(t_1)-\psi(s))^{\alpha-1}-(\psi(t_2)-\psi(s))^{\alpha-1}}{(\psi(t_1)-\psi(s))^{\alpha\theta}}\psi^{'}(s)\right)ds\\ \leq\;&\eta\left(M'_G+M'_F\right)\left(\int_0^{t_1}\left(\psi(t_1)-\psi(s))^{\alpha(1-\theta)-1}\psi^{'}(s)\right)ds-\int_0^{t_1}\left((\psi(t_2)-\psi(s))^{\alpha(1-\theta)-1}\psi^{'}(s)\right)ds\right).\end{aligned}$$

Therefore,

$$\lim_{t_2\to t_1} \|I_4(t_1,t_2)\|_\theta = 0.$$

Using the continuity of $t \mapsto \|T(t)\|$, *it comes from Theorem 1.4.2 (vi) and (3.4) that, for every* $s\in[0,t_1)$, *we have*

$$\begin{aligned}
&\left\|(-A)^{\theta} V_{\psi}^{\alpha}(t_2,s)-(-A)^{\theta} V_{\psi}^{\alpha}(t_1,s)\right\| \\
\leq\ & \int_0^\infty \tau\zeta_\alpha(\tau)\left\|\left[T\left(\left(\frac{\psi(t_2-s)^\alpha}{2}+\frac{\psi(t_2-s)^\alpha-\psi(t_1-s)^\alpha}{2}\right)\tau\right)-T\left(\frac{\psi(t_1-s)^\alpha}{2}\tau\right)\right]\right. \\
&\left.(-A)^\theta T\left(\frac{\psi(t_1-s)^\alpha}{2}\tau\right)\right\| d\tau \\
\leq\ & \frac{2\alpha M_\theta}{\psi(t_1-s)^{\alpha\theta}}\int_0^\infty \tau^{1-\theta}\zeta_\alpha(\tau)\left\|T\left(\left(\frac{\psi(t_2-s)^\alpha}{2}+\frac{\psi(t_2-s)^\alpha-\psi(t_1-s)^\alpha}{2}\right)\tau\right)\right. \\
&\left.-T\left(\frac{\psi(t_1-s)^\alpha}{2}\tau\right)\right\| d\tau \\
\to\ & 0, \text{ as } t_2\to t_1.
\end{aligned}$$

By (4.5), it is easily seen that

$$s\mapsto (\psi(t_2)-\psi(s))^{\alpha-1}\left\|(-A)^{\theta} V_{\psi}^{\alpha}(t_2,s)-(-A)^{\theta} V_{\psi}^{\alpha}(t_1,s)\right\|\psi^{'}(s)$$

is integrable on $[0,t_1)$. *Indeed, we have*

$$\begin{aligned}
&\int_0^{t_1}\left((\psi(t_2)-\psi(s))^{\alpha-1}\left\|(-A)^{\theta} V_{\psi}^{\alpha}(t_2,s)-(-A)^{\theta} V_{\psi}^{\alpha}(t_1,s)\right\|\psi^{'}(s)\right)ds \\
\leq\ & \int_0^{t_1}\left((\psi(t_2)-\psi(s))^{\alpha-1}\left\|(-A)^{\theta} V_{\psi}^{\alpha}(t_2,s)\right\|\psi^{'}(s)\right)ds+\int_0^{t_1}\left((\psi(t_1)-\psi(s))^{\alpha-1}\left\|(-A)^{\theta} V_{\psi}^{\alpha}(t_1,s)\right\|\psi^{'}(s)\right)ds \\
\leq\ & 2M_\theta\frac{\Gamma(1-\theta)}{|\nu_0|^{1-\theta}}.
\end{aligned}$$

Hence, from the dominated convergence theorem, it follows that

$$\begin{aligned}
&\|I_5(t_1,t_2)\|_\theta \\
\leq\ & \left(M_G'+M_F'\right)\int_0^{t_1}\left((\psi(t_2)-\psi(s))^{\alpha-1}\left\|(-A)^{\theta} V_{\psi}^{\alpha}(t_2,s)-(-A)^{\theta} V_{\psi}^{\alpha}(t_1,s)\right\|\psi^{'}(s)\right)ds.
\end{aligned}$$

Then,

$$\lim_{t_2\to t_1} \|I_5(t_1,t_2)\|_\theta = 0.$$

Finally, we can conclude that

$$\lim_{t_2\to t_1} \|Nx(t_1)-Nx(t_2)\|_\theta = 0,$$

this shows that $N\left(\Omega_{\rho_0}\right)$ *is equicontinuous on* $[0,+\infty)$.

Step 3. *We check that $N\left(\Omega_{\rho_0}\right)(t)$ is relatively compact in X_θ for all $t \geq 0$. First, it is important to observe that due to the compactness of the embedding $X_1 \hookrightarrow X_\theta$, for $\theta \in (0,1)$, it follows from (4.17) that*

$$\left\{t \mapsto G(t, x_t + y_t) \;\Big/\; x \in \Omega_\rho\right\}$$

is relatively compact set in X_θ.

It is clear that $N\left(\Omega_{\rho_0}\right)(0)$ is relatively compact in X_θ. Let $t > 0$ be a fixed number. For $\varepsilon \in (0,t)$ and $\delta > 0$, we define

$$N^{\varepsilon,\delta}\left(\Omega_{\rho_0}\right)(t) = \left\{N^{\varepsilon,\delta}x(t) \;\Big/\; x \in \Omega_{\rho_0}\right\},$$

where

$$\begin{aligned}
& N^{\varepsilon,\delta}x(t) \\
= \;& U_\psi^\alpha(t,0)\frac{\varphi(0) - G(0,\varphi)}{\Gamma(\gamma)\Gamma(2-\gamma)} + G(t, x_t + y_t) \\
& + \alpha \int_0^{t-\varepsilon}\int_\delta^{\infty} \Big((\psi(t) - \psi(s))^{\alpha-1}\tau\zeta_\alpha(\tau)T\left((\psi(t-s))^\alpha \tau\right)(AG(s, x_s + y_s) + F(s, x(s) + y(s), x_s + y_s)) \\
& \psi'(s)\Big)d\tau ds \\
= \;& U_\psi^\alpha(t)\frac{\varphi(0) - G(0,\varphi)}{\Gamma(\gamma)\Gamma(2-\gamma)} + G(t, x_t + y_t) \\
& + \alpha T\left((\psi(\varepsilon))^\alpha \delta\right)\int_0^{t-\varepsilon}\int_\delta^{\infty} \Big((\psi(t) - \psi(s))^{\alpha-1}\tau\zeta_\alpha(\tau)T\left((\psi(t-s))^\alpha \tau - (\psi(\varepsilon))^\alpha\delta\right)(AG(s, x_s + y_s) \\
& + F(s, x(s) + y(s), x_s + y_s))\,\psi'(s)\Big)d\tau ds,
\end{aligned}$$

for any $x \in \Omega_{\rho_0}$. Then, the set $N^{\varepsilon,\delta}\left(\Omega_{\rho_0}\right)(t)$ is relatively compact in X_θ since the operator

$$T\left((\psi(\varepsilon))^\alpha \delta\right), \;\; (\psi(\varepsilon))^\alpha \delta > 0$$

is compact in X_θ. Furthermore, we have

$$\begin{aligned}
& \left\|Nx(t) - N^{\varepsilon,\delta}x(t)\right\|_\theta \\
\leq \;& \alpha \int_0^{t}\int_0^{\delta} \Big((\psi(t) - \psi(s))^{\alpha-1}\tau\zeta_\alpha(\tau)\left\|(-A)^\theta T\left((\psi(t) - \psi(s))^\alpha \tau\right)\right\|\Big(\left\|AG(s, x_s + y_s)\right\| + \\
& \left\|F(s, x(s) + y(s), x_s + y_s)\right\|\psi'(s)\Big)d\tau ds \\
& + \alpha \int_{t-\varepsilon}^{t}\int_\delta^{+\infty} \Big((\psi(t) - \psi(s))^{\alpha-1}\tau\zeta_\alpha(\tau)\left\|(-A)^\theta T\left((\psi(t) - \psi(s))^\alpha \tau\right)\right\|\Big(\left\|AG(s, x_s + y_s)\right\| + \\
& \left\|F(s, x(s) + y(s), x_s + y_s)\right\|\psi'(s)\Big)d\tau ds
\end{aligned}$$

$$\leq \alpha M_\theta \left(M_G' + M_F'\right) \int_0^\delta \tau^{1-\theta}\zeta_\alpha(\tau) \int_0^t \left((\psi(t)-\psi(s))^{\alpha(1-\theta)-1} e^{-|\nu_0|(\psi(t)-\psi(s)))^\alpha \tau} \psi'(s)\right) ds d\tau$$

$$+\alpha M_\theta \left(M_G' + M_F'\right) \int_{t-\varepsilon}^t (\psi(t)-\psi(s))^{\alpha(1-\theta)-1} \psi'(s) ds \int_\delta^{+\infty} \tau^{1-\theta}\zeta_\alpha(\tau) d\tau$$

$$\leq M_\theta \left(M_G' + M_F'\right) \Gamma(1-\theta) \left(\frac{1}{|\nu_0|^{1-\theta}} \int_0^\delta \zeta_\alpha(\tau) d\tau + \frac{(\psi(\varepsilon))^{\alpha(1-\theta)}}{\Gamma(1+\alpha(1-\theta))} \right).$$

Then, we conclude that

$$\lim_{\varepsilon,\delta\to 0} \left\| Nx(t) - N^{\varepsilon,\delta}x(t) \right\|_\theta = 0.$$

Consequently, $N\left(\Omega_{\rho_0}\right)(t)$ *is relatively compact in* X_θ*.*

Summing up all the steps above, the Arzela-Ascoli theorem guarantees that N is a compact operator on Ω_{ρ_0}. Then, it follows from Theorem 1.7.2 that N has a fixed point $x \in SABP^0_{w,k}(X_\theta)$. Obviously, $u = x + y$ is the S-asymptotically Bloch type periodic θ-mild solution to problem (4.1). □

4.4 Example

In this section, we give an example to illustrate our abstract results obtained in the previous sections. Specifically, we discuss the existence and uniqueness of an S-asymptotically ω-anti-periodic $\frac{1}{2}$-mild solution for the following problem

$$\begin{cases} {}^H D_{0+}^{\frac{1}{2},\beta,\psi} \left(u(t,\xi) - g(t) \int_{-\infty}^{t} \left(\int_0^\xi b(s-t) u(s,\eta) d\eta \right) ds \right) - \dfrac{\partial^2}{\partial \xi^2} u(t,\xi) \\ \qquad = g_2(t) \int_{-\infty}^t a(s-t) u(s,\xi) ds + g_1(t) f(\xi, u(t,\xi)), \ \xi \in [0,\pi], \ t \geq 0, \\ u(t,0) = u(t,\pi) = 0, \ \ t \in [0,+\infty), \\ u(\tau,\xi) = \varphi(\tau)(\xi), \ \ \tau \leq 0; \end{cases} \tag{4.19}$$

here, $0 \leq \beta \leq 1$ and ${}^H D_{0+}^{\frac{1}{2},\beta,\psi}$ is the ψ-Hilfer fractional derivative of order $\frac{1}{2}$ and type β, with respect to the function ψ.

Let $X = L^2([0,\pi])$ and $A : D(A) \subset X \to X$ be the operator defined by

$$\begin{cases} Au = u'', \\ D(A) = \{u \in X /\ u'', u' \in X, \ u(0) = u(\pi) = 0\}. \end{cases}$$

We know that A generates a uniformly exponentially stable analytic semigroup $(T(t))_{t\geq 0}$ on X. Moreover, $-A$ has discrete spectrum $\sigma(A)$ with eigenvalues n^2, $n \in \mathbb{N}$, associated to a normalized eigenvectors $e_n(\xi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(n\xi)$. We note also that $\{e_n \mid n \in \mathbb{N}\}$ is an orthonormal basis of X. Hence, the associated semigroup $(T(t))_{(t\geq 0)}$ is explicitly given by

$$T(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}\langle u, e_n\rangle e_n.$$

Furthermore,

$$\|T(t)\| \leq e^{-t},$$

for all $u \in X$. On other side, the closed linear operator $(-A)^{-\frac{1}{2}}$ is well defined and one has

$$\begin{cases} (-A)^{-\frac{1}{2}} u = \sum_{n=1}^{\infty} n\langle u, e_n\rangle e_n, \\ \\ D((-A)^{\frac{1}{2}}) = \{u \in X / \sum_{n=1}^{\infty} n\langle u, e_n\rangle e_n \in X\}. \end{cases}$$

Note here that $D((-A)^{\frac{1}{2}})$ is the Banach space with the norm $\|u\|_{\frac{1}{2}} = \|u'\|$, for all $u \in X_{\frac{1}{2}}$.

According to [58, Example 7.1.7], we know that if $h(s) = 1 + |s|^n$ for some $n > 0$, then the space $C_h^0\left((-\infty, 0], X_{\frac{1}{2}}\right)$ is a fading memory space. Moreover, it is follows from [58, Theorem 1.3.6] that

$$\mu_1(t) = \sup_{t\leq 0} \frac{1}{1+|s|^n} = 1 \quad \text{and} \quad \mu_2(t) = \sup_{t\leq 0} \frac{1+|s+t|^n}{1+|s|^n} \leq 1.$$

Consider the Banach space

$$\mathcal{B}_{\frac{1}{2}} = C_h^0\left((-\infty, 0], X_{\frac{1}{2}}\right),$$

equipped with its norm

$$\left\|\phi\right\|_{\mathcal{B}_{\frac{1}{2}}} = \sup_{s\leq 0} \frac{\left\|\phi(s)\right\|_{\frac{1}{2}}}{1+|s|^n},$$

which is also equivalent to

$$\left\|\phi\right\|_{\mathcal{B}_{\frac{1}{2}}} = \sup_{s\leq 0} \frac{\left\|\phi'(s)\right\|}{1+|s|^n}.$$

To study the problem (4.19), we need to consider some particular assumptions, that is

- $f : [0, \pi] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function satisfying the following conditions:
  - For $x, y \in X_{\frac{1}{2}}$, there exists $l_1 > 0$ such that
  $$\left\|f(\cdot, x(\cdot)) - f(\cdot, y(\cdot))\right\| \leq l_1 \left\|x - y\right\|_{\frac{1}{2}}.$$
  - Let λ be a complex number with $|\lambda| = 1$,
  $$f(\xi, \lambda x(\xi)) = \lambda f(\xi, x(\xi)), \quad \text{for } \xi \in [0, \pi] \text{ and } x \in X_{\frac{1}{2}}.$$
- The data function $\varphi \in \mathcal{B}_{\frac{1}{2}}$.
- The functions g, g_1 and g_2 belong to the space $C^1([0, +\infty))$ with $g(0) = 0$.

- The functions $s \mapsto (1+|s|^n)a(s)$ and $s \mapsto (1+|s|^n)b(s)$ are integrable functions on $(-\infty, 0]$, and
$$l_2 = \int_{-\infty}^{0} (1+|s|^n)^2 |a(s)|\, ds, \quad l = \int_{-\infty}^{0} (1+|s|^n)^2 |b(s)|\, ds.$$
Now we are in a position to define the functions $F : \mathbb{R}^+ \times X_{\frac{1}{2}} \times \mathcal{B}_{\frac{1}{2}} \to X$ and $G : \mathbb{R}^+ \times \mathcal{B}_{\frac{1}{2}} \to X_1$ as follows

$$F(t,x,\phi)(\xi) = g_2(t) \int_{-\infty}^{0} a(s)\phi(s,\xi)ds + g_1(t) f(\xi, x(\xi)),$$

and
$$G(t,\phi)(\xi) = g(t)\left(\int_{-\infty}^{0} \int_{0}^{\xi} b(s)\phi(s,\eta)d\eta ds \right).$$

According to Theorem 4.3.1, we have the following result.

Proposition 4.4.1. *Suppose that the functions g, g_1 and g_2 belong to $SAP_w(\mathbb{R}^+)$. We assume also that*
$$(1+\pi)\, l \sup_{t\geq 0}|g(t)| + \pi \max\left(l_1 \sup_{t\geq 0}|g_1(t)|, l_2 \sup_{t\geq 0}|g_2(t)| \right) < 1. \tag{4.20}$$
Then, the problem (4.19) has a unique S-asymptotically ω-anti-periodic $\frac{1}{2}$-mild solution.

Proof. It is sufficient to show that the functions F and H satisfies the conditions (H1)–(H3) in Proposition 4.3.1. For $x \in X_{\frac{1}{2}}$, it is clear that

$$\sup_{t\geq 0} \int_{0}^{\pi} |F(t,x,0)(\xi)|^2 d\xi = \sup_{t\geq 0} |g_1(t)| \left(\int_{0}^{\pi} |f(\xi, x(\xi))|^2 d\xi \right) < +\infty,$$

and
$$\sup_{t\geq 0} \int_{0}^{\pi} \left| \frac{\partial^2}{\partial \xi^2} G(t,0)(\xi) \right|^2 d\xi < +\infty;$$

hence, the condition (H1) holds.

In the sequel, we suppose that $k\omega = \pi$. For $t \geq 0$, $x \in X_{\frac{1}{2}}$ and $\phi \in \mathcal{B}_{\frac{1}{2}}$, Since $X_{\frac{1}{2}} \hookrightarrow X$, and thanks to Hölder's inequality and Fubini's theorem, we get

$$\begin{aligned} \int_{0}^{\pi} \left| \int_{-\infty}^{0} a(s)\phi(s,\xi)ds \right|^2 d\xi &\leq \left(\int_{-\infty}^{0} |a(s)|\, ds \right) \left(\int_{-\infty}^{0} |a(s)| \left\| \phi(s,\cdot) \right\|^2 ds \right) \\ &\leq \left(\int_{-\infty}^{0} (1+|s|^n)^2 |a(s)|\, ds \right) \left(\int_{-\infty}^{0} (1+|s|^n)^2 |a(s)| \frac{\left\| \phi(s,\cdot) \right\|_{\frac{1}{2}}^2}{(1+|s|^n)^2} ds \right) \end{aligned}$$

$$
\begin{aligned}
&\leq \left(\int_{-\infty}^{0} (1+|s|^n)^2 |a(s)| \, ds \right)^2 \sup_{s\leq 0} \frac{\left\| \phi(s,\cdot) \right\|_{\frac{1}{2}}^2}{(1+|s|^n)^2} \\
&\leq \left(l_2 \left\| \phi \right\|_{\mathcal{B}_{\frac{1}{2}}} \right)^2 . \qquad (4.21)
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\left\| F(t+\omega, x, \phi) - e^{ik\omega} F(t, e^{-ik\omega} x, e^{-ik\omega} \phi) \right\| \\
\leq \; & l_2 \left| g_2(t+\omega) - g_2(t) \right| \left\| \phi \right\|_{\mathcal{B}_{\frac{1}{2}}} + \left| g_1(t+\omega) - g_1(t) \right| \left\| f(\cdot, x(\cdot)) \right\|,
\end{aligned}
$$

so that

$$
\lim_{t\to+\infty} \left\| F(t+\omega, x, \phi) - e^{ik\omega} F(t, e^{-ik\omega} x, e^{-ik\omega} \phi) \right\| = 0.
$$

Similarly,

$$
\begin{aligned}
\int_0^{\pi} \left| \int_{-\infty}^{0} b(s) \frac{\partial \phi(s,\xi)}{\partial \xi} ds \right|^2 d\xi \; &\leq \left(\int_{-\infty}^{0} |b(s)| \, ds \right) \left(\int_{-\infty}^{0} |b(s)| \left\| \phi'(s,\cdot) \right\|^2 ds \right) \\
&\leq \left(\int_{-\infty}^{0} (1+|s|^n)^2 |b(s)| \, ds \right)^2 \sup_{s\leq 0} \frac{\left\| \phi(s,\cdot) \right\|_{\frac{1}{2}}^2}{(1+|s|^n)^2} \\
&\leq \left(l \left\| \phi \right\|_{\mathcal{B}_{\frac{1}{2}}} \right)^2 ; \qquad (4.22)
\end{aligned}
$$

it follows that

$$
\left\| AG(t+\omega, \phi) - e^{ik\omega} AG(t, e^{-ik\omega} \phi) \right\| \leq l \left| g(t+\omega) - g(t) \right| \left\| \phi \right\|_{\mathcal{B}_{\frac{1}{2}}},
$$

so

$$
\lim_{t\to+\infty} \left\| AG(t+\omega, \phi) - e^{ik\omega} AG(t, e^{-ik\omega} \phi) \right\| = 0,
$$

thus the condition (H3) holds. Observe that, from the inequalities (4.21) and (4.22), one has

$$
\begin{aligned}
&\left\| F(t, x_1, \phi_1) - F(t, x_2, \phi_2) \right\| \\
\leq \; & l_2 \sup_{t\geq 0} |g_2(t)| \left\| \phi_1 - \phi_2 \right\|_{B_{\frac{1}{2}}} + l_1 \sup_{t\geq 0} |g_1(t)| \left\| x_1 - x_2 \right\|_{\frac{1}{2}},
\end{aligned}
$$

and

$$
\left\| AG(t, \phi_1) - AG(t, \phi_2) \right\| \leq l \sup_{t\geq 0} |g(t)| \left\| \phi_1 - \phi_2 \right\|_{B_{\frac{1}{2}}},
$$

which holds true for any $t \in [0, +\infty)$, $x_1, x_2 \in X_{\frac{1}{2}}$ and $\phi_1, \phi_2 \in \mathcal{B}_{\frac{1}{2}}$. This means that the condition (H2) holds. It is immediate that (4.20) implies that (4.12) holds, with

$$
\mu_1 = 1, \; \left\| (-A)^{-\frac{1}{2}} \right\| = 1, \; M_{\frac{1}{2}} = \Gamma(\frac{1}{2}) = \sqrt{\pi},
$$

and $\nu_0 = -1$. Finally, according to Theorem 4.3.1, we conclude that the problem (4.19) has a unique S-asymptotically ω-anti-periodic $\frac{1}{2}$-mild solution. □

Conclusion

In this thesis, we investigated the existence and uniqueness of solutions for fractional-order boundary value problems in non-regular domains, focusing on different classes of abstract differential equations involving fractional operators. Using a combination of semigroup theory, fractional powers of closed operators, interpolation theory, and classical fixed point theorems, we established sufficient conditions for well-posedness in various settings.

In particular, we analyzed fourth-order equations with fractional powers of the negative Laplacian operator under Cauchy-Dirichlet conditions in 3D cusp domains, showing that a transformation of these domains to cylindrical domains facilitates their resolution. Furthermore, we investigated pseudo S-asymptotically periodic mild solutions for neutral evolution equations involving the Caputo fractional operator with finite delay, applying the Banach contraction principle and Krasnoselskii's fixed point theorem. Additionally, we extended this study to S-asymptotically Bloch periodic solutions for neutral evolution equations governed by the ψ-Hilfer fractional operator with infinite delay, utilizing both the Banach contraction principle and Schauder's fixed point theorem.

These interesting results not only deepen the theoretical understanding of fractional differential equations but also provide a foundation for further research into their applications in irregular geometries and complex physical systems. A natural continuation of this study is to consider a nonlinear differential equation involving a fractional operator (such as Riemann–Liouville, Caputo, Hilfer, etc.) with different boundary conditions in non-regular domains. This direction aims to extend the existence and uniqueness results from Chapters 3 and 4 to more general nonlinear settings using similar methods. It may also open the way to investigating qualitative properties, such as periodic-type solutions, for such problems and exploring their implications in scientific and engineering contexts.

Appendices

Definition 4.4.1. ***(Closed operator [61]).*** *A linear operator A on a Banach space X is said to be closed if its graph*

$$G(A) = \{(x, Ax) : x \in D(A)\}$$

is a closed subset of $X \times X$. In other words, if a sequence $\{x_n\} \subset D(A)$ satisfies

$$\begin{cases} x_n \longrightarrow x, \\ Ax_n \longrightarrow y, \end{cases} \Longrightarrow \begin{cases} x \in D(A), \\ Ax = y. \end{cases}$$

Definition 4.4.2. ***(Self-adjoint, positive definite operator [14]).*** *Let H be a Hilbert space. A linear operator $A : D(A) \subset H \to H$ is called* self-adjoint and positive definite *if*

- *$A = A^*$ (self-adjoint),*
- *there exists $a > 0$ such that $\langle Ax, x\rangle \geq a\|x\|^2$ for all $x \in D(A)$. We write $A \geq aI_H$.*

Definition 4.4.3. ***(Positive operator [105]).*** *Let X be a Banach space and let A be a linear closed operator with dense domain of definition $D(A) \subset X$ such that its range is contained in X, too. The operator A is said to be positive, if $(-\infty, 0]$ belongs to the resolvent set of A and there exists a number $C \geq 0$ such that*

$$\left\|(A - \lambda I)^{-1}\right\| \leq \frac{C}{1 + |\lambda|}, \quad \textit{for } \lambda \leq 0.$$

Remark 4.4.1. *[105] Any self-adjoint, positive definite operator A on a Hilbert space is a positive operator in the sense of the above definition.*

Lemma 4.4.1. ***(Bochner's theorem [30]).*** *A measurable function $f : I \longrightarrow X$ is Bochner integrable if and only if $\|f\|$ is Lebesgue integrable. Furthermore, if f is Bochner integrable, then*

$$\left\|\int_I f(t)dt\right\| \leq \int_I \|f(t)\|\, dt.$$

Lemma 4.4.2. *[30] Let A be a closed linear operator on X. Let $f : I \longrightarrow X$ be Bochner integrable. Suppose that $f(t) \in D(A)$ for all $t \in I$ and $A \cdot f : I \longrightarrow X$ is Bochner integrable. Then*

$$\int_I f(t)dt \in D(A) \quad \textit{and} \quad A\int_I f(t)dt = \int_I Af(t)dt.$$

Lemma 4.4.3. ***(Dominated convergence theorem [30]).*** *Let $f_n : I \longrightarrow X$ be Bochner integrable functions. Assume that:*

(i) There exists an integrable function $g : I \longrightarrow \mathbb{R}$ such that

$$\|f_n(t)\| \leq g(t),$$

a.e. on I for all $n \in \mathbb{N}$.

(ii) $f(t) := \lim_{n\longrightarrow+\infty} f_n(t)$ exists a.e. on I.

Then f is Bochner integrable and

$$\int_I f(t)dt = \int_I \lim_{n\longrightarrow+\infty} f_n(t)dt.$$

Furthermore,

$$\int_I \|f_n(t) - f(t)\| dt \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

Lemma 4.4.4. ***(Fubini's theorem [30]).*** *Let $I = I_1 \times I_2$ be a rectangle in $\mathbb{R}^2$. Let $f : I \longrightarrow X$ be measurable, and suppose that*

$$\int_{I_1}\int_{I_2} \|f(s,t)\| dtds < +\infty.$$

Then f is Bochner integrable and the repeated integrals

$$\int_{I_1}\int_{I_2} \|f(s,t)\| dtds < +\infty \quad \text{and} \quad \int_{I_2}\int_{I_1} \|f(s,t)\| dsdt < +\infty$$

exist and are equal, and they coincide with the double integral

$$\int_I \|f(s,t)\| d(t,s).$$

Theorem 4.4.1. ***(Cauchy-Schwarz's inequality [7]).*** *Let H be a Hilbert space. Then, for x, $y \in H$*

$$\left| < x, y >_H \right| \leq \|x\|_H \left\|y\right\|_H.$$

Theorem 4.4.2. ***(Holder's inequality [7]).*** *Let Ω be open set of $\mathbb{R}^n$, and let $1 < p < \infty$, with p' denoting the conjugate exponent, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. If $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$, then $u \cdot v \in L^1(\Omega)$, and*

$$\int_\Omega |u(x)v(x)| dx \leq \|u\|_{L^p(\Omega)} \|u\|_{L^{p'}(\Omega)}.$$

Bibliography

[1] S. Abbas, M. Benchohra, J. R. Graef, and J. Henderson, *Implicit fractional differential and integral equations: existence and stability*, vol. 26, Walter de Gruyter GmbH & Co KG, 2018.

[2] S. Abbas, M. Benchohra, and G. M. N'Guérékata, *Topics in fractional differential equations*, vol. 27, Springer Science & Business Media, 2012.

[3] S. Abbas, M. Benchohra, G. M. N'Guérékata, and B. A. Slimani, *Darboux problem for fractional-order discontinuous hyperbolic partial differential equations in Banach algebras*, Complex Variables and Elliptic Equations. **57** (2012), no. 2-4, 337–350.

[4] T. Abdeljawad, *On conformable fractional calculus*, Journal of computational and Applied Mathematics. **279** (2015), 57–66.

[5] M. S. Abdo, S. K. Panchal, and A. M. Saeed, *Fractional boundary value problem with ψ-Caputo fractional derivative*, Proceedings-Mathematical Sciences. **129** (2019), 65, DOI 10.1007/s12044-019-0514-8.

[6] O. Abu Arqub, H. Alsulami, and M. Alhodaly, *Numerical Hilbert space solution of fractional Sobolev equation in* $(1+1)$*-dimensional space*, Mathematical Sciences. **18** (2022), no. 2, 217–228, DOI 10.1007/s40096-022-00495-9.

[7] R. A. Adams and J. J. Fournier, *Sobolev spaces*, vol. 140, Pure and Applied Mathematics, Elsevier/Academic Press Amsterdam, 2003.

[8] B. Ahmad and S. Sivasundaram, *Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations*, Nonlinear Analysis: Hybrid Systems. **3** (2009), no. 3, 251–258, DOI 10.1016/j.nahs.2009.01.008.

[9] E. S. Al-Aidarous, A. R. Aliev, E. S. Rzayev, and H. A. Zedan, *Fourth order elliptic operator-differential equations with unbounded operator boundary conditions in the Sobolev-type spaces*, Boundary Value Problems. **2015** (2015), 65, DOI 10.1186/s13661-015-0453-y.

[10] A. Ali, K. Shah, F. Jarad, V. Gupta, and T. Abdeljawad, *Existence and stability analysis to a coupled system of implicit type impulsive boundary value problems of fractional-order differential equations*, Advances in difference equations. **2019** (2019), 101, DOI 10.1186/s13662-019-2047-y.

[11] A. R. Aliev and E. S. Rzayev, *Solvability of boundary value problem for elliptic operator-differential equations of fourth order with operator boundary conditions*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **40** (2014), 13–22.

[12] R. Almeida, *A Caputo fractional derivative of a function with respect to another function*, Communications in Nonlinear Science and Numerical Simulation. **44** (2017), 460–481, DOI 10.1016/j.cnsns.2016.09.006.

[13] B. Alqahtani, H. Aydi, E. Karapınar, and V. Rakočević, *A solution for Volterra fractional integral equations by hybrid contractions*, Mathematics. **7** (2019), no. 8, 694, DOI 10.3390/math7080694.

[14] H. Amann, *Linear and quasilinear parabolic problems*, vol. 89, Birkhäuser Verlag, Basel· Boston· Berlin, 1995.

[15] H. Aydi, M. Jleli, and B. Samet, *On positive solutions for a fractional thermostat model with a convex–concave source term via ψ-Caputo fractional derivative*, Mediterranean Journal of Mathematics. **17** (2020), 16, DOI 10.1007/s00009-019-1450-7.

[16] H. Badawi, N. Shawagfeh, and O. Abu Arqub, *Fractional conformable stochastic integrodifferential equations: Existence, uniqueness, and numerical simulations utilizing the shifted Legendre spectral collocation algorithm*, Mathematical Problems in Engineering. **2022** (2022), DOI 10.1155/2022/5104350.

[17] E. A. Baderko, *The solvability of boundary value problems for higher order parabolic equations in domains with curvilinear lateral boundaries*, Differentsial'nye Uravneniya. **12** (1976), no. 10, 1781–1792.

[18] E. Balcı, İ. Öztürk, and S. Kartal, *Dynamical behaviour of fractional order tumor model with Caputo and conformable fractional derivative*, Chaos, Solitons & Fractals. **123** (2019), 43–51, DOI 10.1016/j.chaos.2019.03.032.

[19] D. Baleanu, S. Etemad, and S. Rezapour, *A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions*, Boundary Value Problems. **2020** (2020), 64, DOI 10.1186/s13661-020-01361-0.

[20] D. Baleanu, A. Jajarmi, H. Mohammadi, and S. Rezapour, *A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative*, Chaos, Solitons & Fractals. **134** (2020), 109705, DOI 10.1016/j.chaos.2020.109705.

[21] K. Belahdji, *Problèmes elliptiques dans des domaines à points cuspides*, Ph.D. thesis, Ecully, Ecole centrale de Lyon, 1996.

[22] S. Ben Chikh, A. Amara, S. Etemad, and S. Rezapour, *On Ulam–Hyers–Rassias stability of a generalized Caputo type multi-order boundary value problem with four-point mixed integro-derivative conditions*, Advances in difference equations. **2020** (2020), 680, DOI 10.1186/s13662-020-03139-1.

[23] M. Benchohra, S. Hamani, and S. Ntouyas, *Boundary value problems for differential equations with fractional order and nonlocal conditions*, Nonlinear Analysis: Theory, Methods & Applications. **71** (2009), no. 7-8, 2391–2396, DOI 10.1016/j.na.2009.01.073.

[24] F. D. Bezerra and L. A. Santos, *Fractional powers approach of operators for abstract evolution equations of third order in time*, Journal of Differential Equations. **269** (2020), no. 7, 5661–5679, DOI 10.1016/j.jde.2020.04.020.

[25] F. Boutaous, *Study of a class of fractional order non linear neutral abstract Volterra integro-differential equations with deviated arguments*, Chaos, Solitons & Fractals. **187** (2024), 115341, DOI 10.1016/j.chaos.2024.115341.

[26] R. F. Brown, *A topological introduction to nonlinear analysis*, Birhaüser, 1993.

[27] T. A. Burton, *A fixed-point theorem of Krasnoselskii*, Applied Mathematics Letters. **11** (1998), no. 1, 85–88, DOI 10.1016/S0893-9659(97)00138-9.

[28] T. Carleman, *Über das neumann-poincarésche problem für ein gebiet mit ecken*, Almqvist & Wiksells, 1916.

[29] F. Chabane, S. Abbas, M. Benbachir, M. Benchohra, and G. N'Guérékata, *Existence of concave positive solutions for nonlinear fractional differential equation with p-Laplacian operator*, Vietnam Journal of Mathematics. **51** (2023), 505–543, DOI 10.1007/s10013-022-00552-9.

[30] Y. K. Chang, G. M. N'Guérékata, and R. Ponce, *Bloch-type periodic functions: Theory and applications to evolution equations*, vol. 22, World Scientific, Singapore, 2022.

[31] Y. K. Chang and Y. Wei, *S-asymptotically Bloch type periodic solutions to some semi-linear evolution equations in Banach spaces*, Acta Mathematica Scientia. **41** (2021), 413–425, DOI 10.1007/s10473-021-0206-1.

[32] Y. K. Chang and J. Zhao, *Pseudo S-asymptotically (ω, c)-periodic solutions to some evolution equations in Banach spaces*, Banach Journal of Mathematical Analysis. **17** (2023), 34, DOI 10.1007/s43037-023-00260-7.

[33] B. Chaouchi, L. Benaissa, and M. Kostić, *On abstract third-order differential equation and its applications*, Acta Mathematica Sinica -English Series. **39** (2023), no. 3, 399–411, DOI 10.1007/s10114-023-1530-2.

[34] B. Chaouchi, M. Benbachir, and M. Kostic, *On the solvability of time conformable fractional equation set on singular domain of* $\mathbb{R}^{N+1}$, Advances in Computational Mathematics and Modelling. **1** (2023), no. 1, 102–115.

[35] B. Chaouchi and F. Boutaous, *An abstract approach for the study of an elliptic problem in a nonsmooth cylinder*, Arabian Journal of Mathematics. **3** (2014), 325–340, DOI 10.1007/s40065-014-0103-8.

[36] B. Chaouchi and M. Kostić, *An efficient abstract method for the study of an initial boundary value problem on singular domain*, Afrika Matematika. **30** (2019), 551–562, DOI 10.1007/s13370-019-00665-4.

[37] B. Chaouchi and M. Kostić, *On the study of an initial value problem for a second order differential equation set in a singular cylindrical domain*, Sarajevo J. Math. **15** (2019), no. 28, 1, DOI 10.5644/SJM.15.01.07.

[38] B. Chaouchi and M. Kostić, *An abstract approach for the study of the Dirichlet problem for an elliptic system on a conical domain*, Mat. Vesn. **73** (2021), no. 2, 131–140.

[39] B. Chaouchi, R. Labbas, and B. K. Sadallah, *Laplace equation on a domain with a cuspidal point in little Hölder spaces*, Mediterranean journal of mathematics. **10** (2013), no. 1, 157–175, DOI 10.1007/s00009-012-0181-9.

[40] M. Cheggag, *Problèmes de Sturm-Liouville abstrait pour une équation différentielle abstraite compléte elliptique du second ordre dans divers espaces*, Ph.D. thesis, Université d'Oran1-Ahmed Ben Bella, 2008.

[41] N. Chegloufa, B. Chaouchi, F. Boutaous, and M. Kostić, *S-asymptotically Bloch type periodic solutions for abstract fractional equations involving* ψ*-Hilfer derivatives*, Journal of Applied Nonlinear Dynamics. **14** (2025), no. 2, 343–354.

[42] N. Chegloufa, B. Chaouchi, M. Kostić, and F. Boutaous, *On a class of abstract fourth-order differential equations set on cusp domains*, Journal of Prime Research in Mathematics. **21** (2025), no. 1, 22–39.

[43] N. Chegloufa, B. Chaouchi, M. Kostić, and W. S. Du, *On the study of pseudo* S-*asymptotically periodic mild solutions for a class of neutral fractional delayed evolution equations*, Axioms. **12** (2023), no. 8, 800, DOI 10.3390/axioms12080800.

[44] S. Cherfaoui, A. Kessab, and A. Kheloufi, *Well-posedness and regularity results for a 2m-th order parabolic equation in symmetric conical domains of* $\mathbb{R}^{N+1}$, Mathematical methods in the applied sciences. **40** (2017), no. 16, 6035–6047, DOI 10.1002/mma.4451.

[45] J. W. Cholewa and T. Dlotko, *Global attractors in abstract parabolic problems*, vol. 278, Cambridge University Press, 2000.

[46] J. Dasht, J. Engström, A. Kufner, and L. E. Persson, *Rothe's method for parabolic equations on non-cylindrical domains*, Advances in Algebra and Analysis. **1** (2006), no. 1, 59–80.

[47] W. Dimbour, G. Mophou, and G. M. N'Guérékata, *S-asymptotically periodic solutions for partial differential equations with finite delay*, Electronic Journal of Differential Equations. **2011** (2011), no. 117, 1–12.

[48] V. S. Erturk, A. Ahmadkhanlu, P. Kumar, and V. Govindaraj, *Some novel mathematical analysis on a corneal shape model by using Caputo fractional derivative*, Optik. **261** (2022), 169086, DOI 10.1016/j.ijleo.2022.169086.

[49] A. Favini, R. Labbas, K. Lemrabet, S. Maingot, and H. D. Sidibé, *Transmission problem for an abstract fourth-order differential equation of elliptic type in UMD spaces*, Adv. Differential Equations. **15** (2010), no. 1-2, 43–72.

[50] A. Favini, R. Labbas, A. Medeghri, and A. Menad, *Analytic semigroups generated by the dispersal process in two habitats incorporating individual behavior at the interface*, Journal of Mathematical Analysis and Applications. **471** (2019), no. 1-2, 448–480, DOI 10.1016/j.jmaa.2018.10.085.

[51] C. L. Fefferman, K. W. Hajduk, and J. C. Robinson, *Simultaneous approximation in Lebesgue and Sobolev norms via eigenspaces*, Proceedings of the London Mathematical Society. **125** (2022), no. 4, 759–777, DOI 10.1112/plms.12469.

[52] A. Granas and J. Dugundji, *Fixed point theory*, vol. 14, Springer, 2003.

[53] M. Hachama and F. Boutaous, *A fractional osmosis model for image fusion*, Advances in Computational Mathematics. **50** (2024), 7, DOI 10.1007/s10444-023-10103-6.

[54] M. F. Hasler and G. M. N'Guérékata, *Bloch-periodic functions and some applications*, Nonlinear Studies. **21** (2014), no. 1, 21–30.

[55] H. Henríquez, M. Pierri, and V. Rolnik, *Pseudo S-asymptotically periodic solutions of second-order abstract Cauchy problems*, Applied Mathematics and Computation. **274** (2016), 590–603, DOI 10.1016/j.amc.2015.11.034.

[56] H. R. Henríquez, M. Pierri, and P. Táboas, *On S-asymptotically ω-periodic functions on Banach spaces and applications*, Journal of Mathematical Analysis and Applications. **343** (2008), no. 2, 1119–1130, DOI 10.1016/j.jmaa.2008.02.023.

[57] R. Hilfer, *Applications of fractional calculus in physics*, World scientific, 2000.

[58] Y. Hino, S. Murakami, and T. Naito, *Functional differential equations with infinite delay*, Springer-Verlag, Berlin, 1991.

[59] S. Hofmann and J. L. Lewis, *The L^p Neumann problem for the heat equation in non-cylindrical domains*, Journal of Functional Analysis. **220** (2005), no. 1, 1–54, DOI 10.1016/j.jfa.2004.10.016.

[60] U. O. Kalemkush, *On a boundary value problem for fourth-order operator-differential equations with a variable coefficient*, Azerbaijan Journal of Mathematics. **10** (2020), no. 1, 181–192.

[61] T. Kato, *Perturbation theory for linear operators*, vol. 132, Springer-Verlag, Berlin, 1980.

[62] G. A. Khalili, K. K. Ali, R. Yilmazer, and M. K. Kaabar, *Economic models involving time fractal*, Journal of Mathematics and Modeling in Finance. **1** (2021), no. 1, 159–178.

[63] S. Ali Khan, K. Shah, G. Zaman, and F. Jarad, *Existence theory and numerical solutions to smoking model under Caputo-Fabrizio fractional derivative*, Chaos: An Interdisciplinary Journal of Nonlinear Science. **29** (2019), no. 1, DOI 10.1063/1.5079644.

[64] A. Kheloufi and B. K. Sadallah, *Parabolic equations with Robin type boundary conditions in a non-rectangular domain.*, Electronic Journal of Differential Equations. **2010** (2010), no. 25, 1–14.

[65] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, vol. 204, Elsevier Science, Amsterdam, 2006.

[66] V. A. Kondrat'ev and O. A. Oleinik, *Boundary-value problems for partial differential equations in non-smooth domains*, Russian Mathematical Surveys. **38** (1983), no. 2, 1–86.

[67] M. Kostić, *Selected topics in almost periodicity*, vol. 84, Walter de Gruyter, Berlin, 2022.

[68] K. Kuliev and L. E. Persson, *An extension of Rothe's method to non-cylindrical domains*, Applications of Mathematics. **52** (2007), 365–389, DOI 10.1007/s10492-007-0021-6.

[69] P. Kumar, V. S. Erturk, and C. Harley, *A novel study on a fractional-order heat conduction model for the human head by using the least-squares method*, International Journal of Dynamics and Control. **11** (2023), 1040–1049, DOI 10.1007/s40435-022-01051-y.

[70] P. Kumar, V. Govindaraj, V. S. Erturk, K. S. Nisar, and M. Inc, *Fractional mathematical modeling of the stuxnet virus along with an optimal control problem*, Ain Shams Engineering Journal. **14** (2023), no. 7, 102004, DOI 10.1016/j.asej.2022.102004.

[71] P. Kumar, V. Govindaraj, and M. Murillo-Arcila, *The existence, uniqueness, and stability results for a nonlinear coupled system using ψ-Caputo fractional derivatives*, Boundary Value Problems. **2023** (2023), 75, DOI 10.1186/s13661-023-01769-4.

[72] R. Labbas, A. Medeghri, and B. K. Sadallah, *On a parabolic equation in a triangular domain*, Applied Mathematics and Computation. **130** (2002), no. 2-3, 511–523, DOI 10.1016/S0096-3003(01)00113-8.

[73] R. Labbas, A. Medeghri, and B. K. Sadallah, *Sur une équation parabolique dans un domaine non cylindrique*, Comptes Rendus Mathematique. **335** (2002), no. 12, 1017–1022, DOI 10.1016/S1631-073X(02)02592-X.

[74] R. Labbas, A. Medeghri, and B. K. Sadallah, *An-approach for the study of degenerate parabolic equations*, Electronic Journal of Differential Equations. **2005** (2005), no. 36, 1–20.

[75] F. Li, J. Liang, and H. Wang, *S-asymptotically ω-periodic solution for fractional differential equations of order $q \in (0,1)$ with finite delay*, Advances in Difference Equations. **2017** (2017), 83, DOI 10.1186/s13662-017-1137-y.

[76] F. Li and H. Wang, *S-asymptotically ω-periodic mild solutions of neutral fractional differential equations with finite delay in Banach space*, Mediterranean Journal of Mathematics. **14** (2017), 57, DOI 10.1007/s00009-017-0855-4.

[77] Q. Li, W. Wang, K. Teng, and X. Wu, *Ground states for fractional Schrödinger equations with electromagnetic fields and critical growth*, Acta Mathematica Scientia. **40** (2020), 59–74, DOI 10.1007/s10473-020-0105-0.

[78] Q. Li and M. Wei, *Existence and asymptotic stability of periodic solutions for neutral evolution equations with delay*, Evolution Equations & Control Theory. **9** (2020), no. 3, 753–772, DOI 10.3934/eect.2020032.

[79] J. L. Lions, *Théorèmes de trace et d'interpolation*, Annali della Scuola Normale Superiore di Pisa-Scienze Fisiche e Matematiche. **13** (1959), no. 4, 389–403.

[80] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, vol. 181, Springer-Verlag, Berlin-New York, 1972.

[81] H. Liu and J. C. Chang, *Existence for a class of partial differential equations with nonlocal conditions*, Nonlinear Analysis: Theory, Methods & Applications. **70** (2009), no. 9, 3076–3083, DOI 10.1016/j.na.2008.04.009.

[82] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, vol. 16, Birkhäuser Verlag, Basel, 1995.

[83] H. Mohammadi, S. Kumar, S. Rezapour, and S. Etemad, *A theoretical study of the Caputo–Fabrizio fractional modeling for hearing loss due to mumps virus with optimal control*, Chaos, Solitons & Fractals. **144** (2021), 110668, DOI 10.1016/j.chaos.2021.110668.

[84] J. van Neerven, *The asymptotic behaviour of semigroups of linear operators*, vol. 88, Birkhäuser Verlag, Basel, 1996.

[85] F. Norouzi and G. M. N'guérékata, *Existence results to a ψ-Hilfer neutral fractional evolution equation with infinite delay*, Nonautonomous Dynamical Systems. **8** (2021), 101–124.

[86] E. R. Oueama-Guengai and G. M. N'Guérékata, *On S-asymptotically ω-periodic and Bloch periodic mild solutions to some fractional differential equations in abstract spaces*, Mathematical Methods in the Applied Sciences. **41** (2018), no. 18, 9116–9122, DOI 10.1002/mma.5062.

[87] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, vol. 44, Springer-Verlag, New York, 1983.

[88] M. Pierri and V. Rolnik, *On pseudo-asymptotically periodic functions*, Bulletin of the Australian Mathematical Society. **87** (2013), no. 2, 238–254, DOI 10.1017/S0004972712000950.

[89] R. Poovarasan, P. Kumar, K. S. Nisar, and V. Govindaraj, *The existence, uniqueness, and stability analyses of the generalized Caputo-type fractional boundary value problems*, AIMS Mathematics. **8** (2023), no. 7, 16757–16772, DOI 10.3934/math.2023857.

[90] K. R. Prasad, M. Khuddush, and D. Leela, *Existence, uniqueness and Hyers–Ulam stability of a fractional order iterative two-point boundary value problems*, Afrika Matematika. **32** (2021), 1227–1237, DOI 10.1007/s13370-021-00895-5.

[91] Da. Prato and P. Grisvard, *Sommes d'opérateurs linéaires et équations différentielles opérationelles*, J. Math. pures appl. **54** (1975), 305.

[92] S. Rezapour, S. Etemad, and H. Mohammadi, *A mathematical analysis of a system of Caputo–Fabrizio fractional differential equations for the anthrax disease model in animals*, Advances in Difference Equations. **2020** (2020), 481, DOI 10.1186/s13662-020-02937-x.

[93] S. Rezapour, P. Kumar, V. S. Erturk, and S. Etemad, *A study on the 3D Hopfield neural network model via nonlocal Atangana–Baleanu operators*, Complexity. **2022** (2022), 1–13, DOI 10.1155/2022/6784886.

[94] E. Rothe, *Zweidimensionale parabolische randwertaufgaben als grenzfall eindimensionaler randwertaufgaben*, Mathematische Annalen. **102** (1930), 650–670.

[95] B. K. Sadallah, *Une équation 2m-parabolique dans un domaine plan singulier*, CR Acad. Sci. Paris Ser. A-B. **287** (1978), 1025–1027.

[96] B. K. Sadallah, *Etude d'un probleme 2m-parabolique dans des domaines plan non rectangulaires*, Boll. Un. Mat. Ital. **5** (1983), no. 2-B, 51–112.

[97] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach Science Publishers, 1993.

[98] L. A. Santos, *Fractional powers approach of operators for abstract evolution equations of third order in time*, Ph.D. thesis, Federal University of Paraíba, Brazil, 2020.

[99] J. V. C. Sousa and E. C. De Oliveira, *On the ψ-Hilfer fractional derivative*, Communications in Nonlinear Science and Numerical Simulation. **60** (2018), 72–91, DOI 10.1016/j.cnsns.2018.01.005.

[100] X. Su, *Boundary value problem for a coupled system of nonlinear fractional differential equations*, Applied Mathematics Letters. **22** (2009), no. 1, 64–69, DOI 10.1016/j.aml.2008.03.001.

[101] H. Sweis, N. Shawagfeh, and O. Abu Arqub, *Fractional crossover delay differential equations of Mittag-Leffler kernel: Existence, uniqueness, and numerical solutions using the Galerkin algorithm based on shifted Legendre polynomials*, Results in Physics. **41** (2022), 105891, DOI 10.1016/j.rinp.2022.105891.

[102] S. Thabet, S. Etemad, and S. Rezapour, *On a coupled Caputo conformable system of pantograph problems*, Turkish Journal of Mathematics. **45** (2021), no. 1, 496–519, DOI 10.3906/mat-2010-70.

[103] S. T. Thabet, S. Etemad, and S. Rezapour, *On a new structure of the pantograph inclusion problem in the Caputo conformable setting*, Boundary value problems. **2020** (2020), 171, DOI 10.1186/s13661-020-01468-4.

[104] C. C. Travis and G. F. Webb, *Existence, stability, and compactness in the α-norm for partial functional differential equations*, Transactions of the American Mathematical Society. **240** (1978), 129–143.

[105] H. Triebel, *Interpolation theory, function spaces, differential operators*, vol. 18, North-Holland Publishing Company, Amsterdam, 1978.

[106] J. Wang, A. Zada, and H. Waheed, *Stability analysis of a coupled system of nonlinear implicit fractional anti-periodic boundary value problem*, Mathematical Methods in the Applied Sciences. **42** (2019), no. 18, 6706–6732, DOI 10.1002/mma.5773.

[107] J. Wang and Y. Zhou, *A class of fractional evolution equations and optimal controls*, Nonlinear Analysis: Real World Applications. **12** (2011), no. 1, 262–272, DOI 10.1016/j.nonrwa.2010.06.013.

[108] M. Wei and Q. Li, *Existence and uniqueness of S-asymptotically periodic α-mild solutions for neutral fractional delayed evolution equation*, Applied Mathematics-A Journal of Chinese Universities. **37** (2022), no. 2, 228–245, DOI 10.1007/s11766-022-4393-x.

[109] Z. Xia, *Pseudo asymptotically periodic solutions of two-term time fractional differential equations with delay*, Kodai Mathematical Journal. **38** (2015), no. 2, 310–332, DOI 10.2996/kmj/1436403893.

[110] M. Yang and Q. Wang, *Pseudo asymptotically periodic solutions for fractional integro-differential neutral equations*, Science China Mathematics. **62** (2019), 1705–1718, DOI 10.1007/s11425-017-9222-2.

[111] W. Zhang and W. Liu, *Existence and Ulam's type stability results for a class of fractional boundary value problems on a star graph*, Mathematical Methods in the Applied Sciences. **43** (2020), no. 15, 8568–8594, DOI 10.1002/mma.6516.

[112] X. Zheng, Y. Shang, and P. Xiaoming, *Orbital stability of periodic traveling wave solutions to the generalized Zakharov equations*, Acta Mathematica Scientia. **37** (2017), no. 4, 998–1018, DOI 10.1016/S0252-9602(17)30054-1.