People's Democratic Republic of Algeria
Ministry of Higher Education
and Scientific Research

Blida 1 University

Faculty of Technology

ANALYSIS 1

for first-year sciences and technology

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Preface

Foreword, this document is a teaching aid intended for students enrolled at university in the first year of engineering in the Science and Technology field. It is a course illustrating the basic notions of mathematical analysis in order to acquire and understand the fundamentals of mathematical reasoning, which is essential for understanding the rest of the teaching in science and technology.

In this course, the definitions of mathematical tools and their properties are presented, along with important remarks that help to assimilate these notions, as well as basic theorems and propositions, illustrating everything with detailed examples. At the end of each chapter exercises of varying degrees of difficulty are presented.

This document describes the program for the subject Analysis1, taught in the first semester to first year engineering students, it consists of six chapters, in the first we introduce the field of real numbers then we pass to the chapter of real sequences and their properties, then, the third chapter deals with real functions with one real variable, in particular the notions of continuity and derivability, as well as the study of inverse trigonometric and hyperbolic functions and their inverse, followed by the chapter on the development of Taylor series and ending with the chapter on integral and the calculation of primitives.

References

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Contents

Chapitre 1

Real numbers

1.1 Introduction

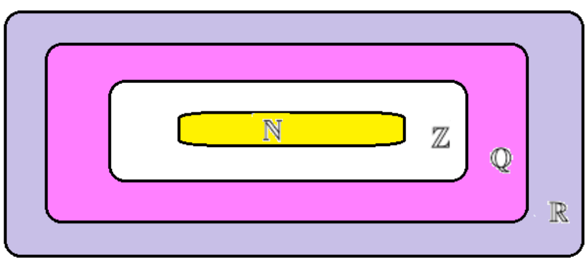
Calculus is based on the real number system. Real numbers are numbers that can be expressed as decimals. We distinguish three special subsets of real numbers:

(a) The set of natural numbers, denoted by $\mathbb{N} = \{1, 2, 3, ...\}$.

(b) The set of integer numbers, denoted by $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.

(c) The set of rational numbers, denoted by $\mathbb{Q} = \{\frac{m}{n},\ m \in \mathbb{Z},\ n \in \mathbb{Z}^* \text{ with } GCD(m, n) = 1\}$.
Every rational number can be written both as a ratio of integers and as a decimal that terminates or begins to repeat.

The set of real numbers, denoted by $\mathbb{R}$. The set of real numbers includes both rational and irrational numbers $2, \frac{-3}{5}, \sqrt{3} \in \mathbb{R}$. As consequence $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.



1.2 Fundamental properties of real.

1.2.1 Intervals and operations on real numbers

Definition 1.2.1

Let $a, b \in \mathbb{R}$.
• A closed interval is a subset of the form:

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\} \subset \mathbb{R}.$$

• An open interval is a subset of the form

$$]a, b[= \{x \in \mathbb{R} \mid a < x < b\} \subset \mathbb{R}.$$

Remark 1.2.1

Let $a \in \mathbb{R}$, then

- $\varnothing =]a, a[$ is an open interval.
- $\mathbb{R} =]-\infty; +\infty[$ is an open interval.

Proposition 1.2.1

Let $a, b, c \in \mathbb{R}$, then

- $a + b = b + a$ and $ab = ba$,
- $(a + b)c = ac + bc$.

Remark 1.2.2

Let a, b be real numbers, then $a \leq b$ or $b \leq a$ i.e the real set $\mathbb{R}$ is totally ordered.

Proposition 1.2.2

$\forall\, a, b, c \in \mathbb{R}$, we have

- $a \leq b \implies a + c \leq b + c,$
- $a \leq b \implies \begin{cases} a^2 \leq b^2 & \text{if} \quad 0 < a \leq b, \\ b^2 \leq a^2 & \text{if} \quad a \leq b < 0, \\ ac \leq bc & \text{if} \quad c > 0, \\ ac \geq bc & \text{if} \quad c < 0, \end{cases}$
- $0 < a \leq b \implies 0 < \dfrac{1}{b} \leq \dfrac{1}{a},$
- $a \leq b < 0 \implies \dfrac{1}{b} \leq \dfrac{1}{a} < 0.$

1.2.2 Upper and Lower bound

Definition 1.2.2

Let A be a non-empty subset of $\mathbb{R}$ and a, b two real numbers.

(a) We say that a is a lower-bound of A if $\forall x \in A,\ x \geq a$.

(b) We say b is an upper-bound of A if $\forall x \in A,\ x \leq b$.

Example 1.2.1.

(a) $-66, -10, -8, -7$ *are lower-bounds of* $]-7, 11[$.

(b) $11, 12, 20, 100$ *are upper-bounds of* $]-7, 11[$.

(c) $-6, -9, -3$ *are lower-bounds of* $\{0, -2, 3, -1, 6, 4\}$.

(d) $7, 11$ *are upper-bounds of* $\{0, -2, 3, -1, 6, 4\}$.

(e) The set $C =]-\infty, 0[$ *does not have a lower-bounds.*

(f) The set $D =]2, +\infty[$ *does not have an upper-bounds.*

Definition 1.2.3

Let A be a non-empty subset of $\mathbb{R}$.

(a) If there exists a lower-bound of A, we say that A is bounded from below.

(b) If there exists an upper-bound of A, we say that A is bounded from above.

(c) We say that A is bounded if both bounds (upper and lower) exist.

Example 1.2.2.

(a) Let $A =]-4, +\infty[$. *A is bounded from below since* $-5 \in \mathbb{R}$ *such that* $\forall x \in A,\ x \geq -5$. *Note that A is not bounded from above.*

(b) Let $B =]-\infty, 8[$. *B is bounded from above since* $9 \in \mathbb{R}$ *such that* $\forall x \in B,\ x \leq 9$. *Note that B is not bounded from below.*

(c) Let $C =]-4, 8[$. *C is bounded since* $-5, 9 \in \mathbb{R}$ *such that* $\forall x \in C,\ -5 \leq x \leq 9$.

(d) $\mathbb{N}$ *is bounded from below (each number* $k < 0$ *is a lower bound), but not from above.*

1.2.3 Infimum, Supremum, Minimum and Maximum

Definition 1.2.4

Let A be a subset in $\mathbb{R}$ and a_0, b_0 be two real.

(a) We say that a_0 is the greatest lower-bound or the infimum of A if a_0 is a lower-bounded of A and satisfies $a \leq a_0$ for every lower-bounded $a \in \mathbb{R}$.
This infimum is not necessarily belonging to A. We write $a_0 := inf(A)$.

(b) We say that b_0 is the least upper-bound or the supremum of A if b_0 is a lower-bounded of A and satisfies $b \geq b_0$ for every lower-bounded $a \in \mathbb{R}$. This supremum is not necessarily belonging to A. We write $b_0 := sup(A)$.

Definition 1.2.5

Let A be a subset of $\mathbb{R}$ and a_0, b_0 two real.

(a) We say that a_0 is the minimum of A if a_0 is a lower-bounded of A and it an element of A. We write

$$a_0 = min(A).$$

(b) We say that b_0 is the maximum of A if b_0 is a upper-bounded of A and it an element of A. We write

$$b_0 = max(A).$$

Example 1.2.3.

(a) If $A = [2, 5]$ *then,* $inf(A) = min(A) = 2$ *and* $sup(A) = max(A) = 5$.

(b) If $A = [1, 7[$ *then* $inf(A) = min(A) = 1$ *and* $sup(A) = 7$.

(c) If $A =]-3, 2]$ *then* $inf(A) = -3$ *and* $sup(A) = max(A) = 2$.

(d) If $A =]0, 8[$ *then* $inf(A) = 0$ *and* $sup(A) = 8$.

Example 1.2.4.

(a) If $A = \{-3, 2, , 0, -4, 8, -1\}$ *then,* $min(A) = -4$ *and* $max(A) = 8$.

(b) If $A = [2, 5]$ *then,* $min(A) = 2$ *and* $max(A) = 5$.

(c) If $A = [1, 7[$ *then* $min(A) = 1$ *and* $max(A)$ *doesn't exist.*

(d) If $A =]-3, 2]$ *then* $min(A)$ *doesn't exist and* $max(A) = 2$.

(e) If $A =]0, 8[$ *then* $min(A)$ *doesn't exist and* $max(A)$ *doesn't exist.*

Theorem 1.2.1

Let A be a non-empty subset of $\mathbb{R}$, we have

$$(a)\ \ M = \sup(A) \iff \begin{cases} \forall x \in A,\ x \leq M, \\ \forall \varepsilon > 0,\ \exists x \in A,\ M - \varepsilon < \varepsilon. \end{cases}$$

$$(b)\ \ m = \inf(A) \iff \begin{cases} \forall x \in A,\ x \geq m, \\ \forall \varepsilon > 0,\ \exists x \in A,\ x < m + \varepsilon. \end{cases}$$

Example 1.2.5. *Let* $A = \left\{1 + \frac{1}{n},\ n \in \mathbb{N}^*\right\}$ *and* $B = \left\{2 - \frac{1}{n},\ n \in \mathbb{N}^*\right\}$ *are tow sets. We have*

$$A = \left\{1 + \frac{1}{n},\ n \in \mathbb{N}^*\right\} = \left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \cdots\right\},$$

and

$$B = \left\{2 - \frac{1}{n},\ n \in \mathbb{N}^*\right\} = \left\{1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \cdots\right\}.$$

Yields,

$$\inf(A) = 1,\ \max(A) = \sup(A) = 2 \textit{ and } \min(A) \textit{ doesn't exists}.$$

and

$$\inf(B) = \min(B) = 1,\ \sup(B) = 2 \textit{ and } \max(B) \textit{ doesn't exists}.$$

1.3 Absolute value, radicals and integer part

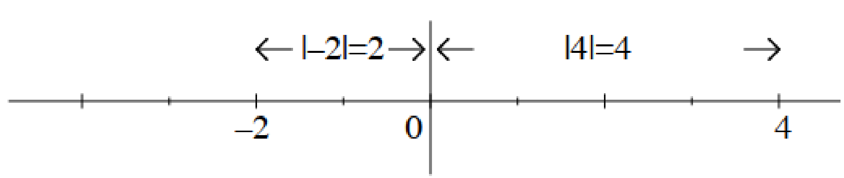
1.3.1 The absolute value of real

Definition 1.3.1

Let x be a real number. The absolute value of x denoted $|x|$ is defined as follows

$$|x| = \begin{cases} x & \text{indeterminate form } x \geq 0, \\ -x & \text{indeterminate form } x < 0. \end{cases}$$

Example 1.3.1. $|4| = 4$ *since* 4 *is positive, but* $|-2| = 2$ *since* -2 *is negative.*



Proposition 1.3.1

Let x, y be real numbers, we have the following properties:

(a) $|x| \geq 0$ and $|x| = |-x|$,

(b) $\sqrt{x^2} = |x| = \begin{cases} x & \text{si } x \geq 0, \\ -x & \text{si } x < 0. \end{cases}$

(c) $|x \cdot y| = |x| \cdot |y|$ and if $y \in \mathbb{R}^*$, $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$,

(d) $|x + y| \leq |x| + |y|$, (Triangular inequality),

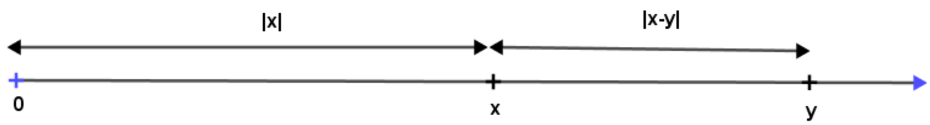
(e) $||x| - |y|| \leq |x - y|$.

(f) $\forall r \geq 0,\ |x| = r \iff (x = r \vee x = -r)$.

(g) $\forall r > 0,\ |x| \leq r \iff -r \leq x \leq r$.

Remark 1.3.1

Geometrically, $|x|$ is the distance from x to 0 and $|x - y|$ is the distance from x to y on the numberline.



Note that $|x - y| = |y - x|$ and $|x - y| = \begin{cases} x - y & \text{indeterminate form } x \geq y, \\ y - x & \text{indeterminate form } x < y. \end{cases}$

Example 1.3.2. *We have*

$$|x - 3| = \begin{cases} x - 3 & \textit{indeterminate form } x \geq 3, \\ 3 - x & \textit{indeterminate form } x < 3. \end{cases}, \quad |x - e| = \begin{cases} x - e & \textit{indeterminate form } x \geq e, \\ e - x & \textit{indeterminate form } x < e. \end{cases}$$

and

$$|x + 2| = |x - (-2)| = \begin{cases} x + 2 & \textit{indeterminate form } x \geq -2, \\ -2 - x & \textit{indeterminate form } x < -2. \end{cases}$$

Example 1.3.3. *solve in* $\mathbb{R}$ *the following equations:*

$$|\ln(x) - 3| = 2,\ |x^2 - 3x| = |x|.$$

Solution

$$\begin{aligned} |\ln(x) - 3| = 2 &\iff \ln(x) - 3 = 2 \ \vee\ \ln(x) - 3 = -2, \\ &\iff \ln(x) = 5 \ \vee\ \ln(x) = 1, \\ &\iff x = e^5 \ \vee\ x = e. \end{aligned}$$

Thus, the set set of solutions is the following $\mathcal{S} = \{e^5, e\}$.

$$\begin{aligned} |x^2 - 3x| = |x| &\iff x^2 - 3x = x \ \vee\ x^2 - 3x = -x, \\ &\iff x^2 - 4x = 0 \ \vee\ x^2 - 2x = 0, \\ &\iff x = 0 \ \vee\ x = 4 \ \vee\ x = 2. \end{aligned}$$

Then, the set set of solutions is the following $\mathcal{S} = \{0, 2, 4\}$.

1.3.2 Radicals

Definition 1.3.2

Let x be a positive real number and $n \geq 2$ be a natural number, then the principal square root of a number x is defined as

$$\sqrt{x} = y \text{ if and only if } x = y^2.$$

where $y \geq 0$ and $\sqrt{\ }$ is the radical symbol and x is called the radicand. In general,

$$\sqrt[n]{x} = y \text{ if and only if } x = y^n.$$

If n is even, then x and y must be greater than or equal to zero. If n is odd, then x and y can be any real number.

Remark 1.3.2

If $x < 0$ the following roots $\sqrt[2]{x}$, $\sqrt[4]{x}$, ... $\sqrt[2k]{x}$ $k \in \mathbb{N}^*$ do note exist.

Example 1.3.4. $\sqrt[3]{27} = 3$, $\sqrt[3]{-8} = -2$, $\sqrt[4]{16} = 2$ *and* $\sqrt[6]{-101}$ *do not exists.*

1.3.3 Integer part of a real number

Definition 1.3.3

Let x be a real number, the largest integer less than or equals to x is called the integer part of x and denoted by $E(x)$ or $\lfloor x \rfloor$.

Example 1.3.5. $\lfloor 1,74 \rfloor = 1$, $\lfloor e \rfloor = 2$, $\lfloor -2,44 \rfloor = -3$.

Proposition 1.3.2

Let x be a real number, then $\lfloor x \rfloor$ satisfies:

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1 \text{ or } x - 1 < \lfloor x \rfloor \leq x.$$

Moreover, for $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, we have

- $\lfloor x \rfloor = n \iff n \leq x < n + 1$,
- $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.
- $\lfloor x \rfloor + \lfloor x + \frac{1}{n} \rfloor + \lfloor x + \frac{2}{n} \rfloor + \cdots + \lfloor x + \frac{n-1}{n} \rfloor = \lfloor nx \rfloor$.

Example 1.3.6. *solve in* $\mathbb{R}$ *the following equations* $\lfloor x^2 \rfloor = 1$ *and* $\lfloor e^x - 3 \rfloor = -2$.

Solution

$$\begin{aligned}\lfloor x^2 \rfloor = 1 &\iff 1 \leq x^2 < 2\\ &\iff 1 \leq |x| < \sqrt{2}\\ &\iff x \in]-\sqrt{2}, -1] \cup [1, \sqrt{2}[.\end{aligned}$$

$$\begin{aligned}\lfloor e^x - 3 \rfloor = -2 &\iff -2 \leq e^x - 3 < -1\\ &\iff 1 \leq e^x < 2\\ &\iff 0 \leq x < \ln(2)\\ &\iff x \in [0, \ln(2)[.\end{aligned}$$

1.4 Some basic formulas in $\mathbb{R}$

1.4.1 Sum

Definition 1.4.1

The sigma notation $\sum$ is the most easiest way of writing a very large sum of elements of a sequence in a simple manner. A sequence is a collection of terms that follow a pattern and sigma notation $\sum$ is used to represent the sum of such elements. For example, the sum of elements a_i such that $i \in \mathbb{N}$ is written as follows

$$\sum_{i\geq 0} a_i = a_0 + a_1 + a_2 + \cdots$$

Moreover, if $k, n \in \mathbb{N}$, we read the sum $\sum_{k=0}^{n} e^k$ as follows: the sum of e^k from $k = 0$ to $k = n$, and we write it as

$$\sum_{k=0}^{n} e^k = e^0 + e^1 + e^2 + \cdots + e^n.$$

Example 1.4.1.

(a) We read $\sum_{k=0}^{2} 3^k$, the sum for k, star from 0 to 2 of 3 power k. Its abbreviated writing is

$$\sum_{k=0}^{2} 3^k = 3^0 + 3^1 + 3^2 = 13.$$

(b) We read $\sum_{k=1}^{3} 5k$, the sum for k, star from 1 to 3 of 5 times k. Its abbreviated writing is

$$\sum_{k=1}^{3} 5k = 5 \times 1 + 5 \times 2 + 5 \times 3 = 30.$$

Remark 1.4.1

A result of sum does not depending of the summation index, we can write

$$\sum_{k=0}^{2} 3^k = \sum_{p=0}^{2} 3^p = \sum_{r=0}^{2} 3^r \ldots$$

Proposition 1.4.1

(a) $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$

(b) $\sum_{k=1}^{n}(c \cdot a_k) = c \cdot \sum_{k=1}^{n} a_k.$

(c) $\sum_{k=1}^{n} c = n \cdot c.$

(d) $(a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}.$

Example 1.4.2.

$$\sum_{k=0}^{3}(2k - k^2) = 2\sum_{k=0}^{k=3} k - \sum_{k=0}^{k=3} k^2 = 2(0+1+2+3) - (0^2+1^2+2^2+3^2) = -2$$

$$\sum_{i=1}^{4} 3\ln(i) = 3\sum_{i=1}^{4}\ln(i) = 3(\ln(1)+\ln(2)+\ln(3)+\ln(4)) = 4(\ln(1\times 2\times 3\times 4)) = 3\ln(4!).$$

$$\sum_{t=1}^{4}\frac{(-1)^t}{t} = \frac{(-1)^1}{1} + \frac{(-1)^2}{2} + \frac{(-1)^3}{3} + \frac{(-1)^4}{4} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = \frac{-7}{12}.$$

Summation formulas

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Example 1.4.3.

$$\begin{aligned}\sum_{k=0}^{n}(2+3k) &= \sum_{k=0}^{n} 2 + 3\sum_{k=0}^{n} k\\ &= (\underbrace{2+2+\cdots+2}_{(n+1)\ times}) + 3(\underbrace{1+2+3+\cdots+n}_{arithmetic\ sequence})\\ &= 2(n+1) + \frac{3n(n+1)}{2}.\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{n} 3\cdot 2^k &= 3\sum_{k=1}^{n} 2^k\\ &= 3(\underbrace{2^1+2^2+2^3+\cdots+2^n}_{geometric\ sequence})\\ &= 3\frac{2^n-1}{2-1} = 3(2^n-1).\end{aligned}$$

1.4.2 Product.

Definition 1.4.2

The pi notation $\prod$ is used to represent the product of a bunch of terms. It is used in the same way as the Sigma symbol described above, except that succeeding terms are multiplied instead of added:

$$\prod_{i\geq 0} a_i = a_0 \times a_1 \times a_2 \times \cdots \tag{1.1}$$

Example 1.4.4.

$$\prod_{k=1}^{6} k = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720,$$

$$\prod_{k=1}^{4}(k-1) = (1-1)\times(2-1)\times(3-1)\times(4-1) = 0,$$

$$\prod_{i=-3}^{-1} i^2 = (-3)^2 \times (-2)^2 \times (-1)^2 = 36,$$

$$\prod_{k=2}^{5}\frac{1}{k} = \frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5} = \frac{1}{120}$$

Proposition 1.4.2

(a) $\prod_{k=1}^{n}(a_k \cdot b_k) = \left(\prod_{k=1}^{n} a_k\right)\left(\prod_{k=1}^{n} b_k\right).$

(b) $\prod_{k=1}^{n} a_k^m = \left(\prod_{k=1}^{n} a_k\right)^m.$

(c) $\prod_{k=1}^{n} c = c^n.$

Example 1.4.5.

$$\begin{aligned}\prod_{k=1}^{5} ke^{(-1)^k} &= \left(\prod_{k=1}^{5} k\right)\left(\prod_{k=1}^{5} e^{(-1)^k}\right)\\ &= (1\times 2\times 3\times 4\times 5)(e^{(-1)^1}e^{(-1)^2}e^{(-1)^3}e^{(-1)^4}e^{(-1)^5})\\ &= 5!e^{-1}ee^{-1}ee^{-1}\\ &= 120e^{-1}.\end{aligned}$$

Example 1.4.6. *Let* $n, k \in \mathbb{N}$.

$$\begin{aligned}\prod_{i=1}^{n}\left(2+\frac{2}{i}\right)^{k} &= \prod_{i=1}^{n}\left[2^{k}\left(i+\frac{1}{i}\right)^{k}\right] \\ &= \left[\prod_{i=1}^{n}2^{k}\right]\left[\prod_{i=1}^{n}\left(i+\frac{1}{i}\right)^{k}\right] \\ &= (2^{k})^{n}\prod_{i=1}^{n}\left(\frac{1+i}{i}\right)^{k} \\ &= 2^{nk}\left[\left(\frac{1+1}{1}\right)^{k}\left(\frac{1+2}{2}\right)^{k}\cdots\left(\frac{1+n}{n}\right)^{k}\right] \\ &= 2^{nk}(n+1)^{k}\end{aligned}$$

1.5 Practice exercises

Exercise 1.1

Find the set of lower-bounds, the set of upper-bounds, the minimum, the maximum, the infimum and the supremum if there exist of each set of the following:

$$A=[-3,2],\ B=]0,4[,\ C=[-2,+\infty[, D=]-\infty,1[,\ E=\{1+\frac{1}{n},\ n\in\mathbb{N}^{*}\},$$

$$F=\{2+\frac{(-1)^{n}}{n},\ n\in\mathbb{N}^{*}\}.$$

Exercise 1.2

Find the set of lower-bounds, the set of upper-bounds, the minimum, the maximum, the infimum and the supremum if there exist of each set of the following:

$$A=\{2,4,-7,0,-2\},\ B=]2,e[,\ C=[-\sqrt{2},+\infty[,\ E=\{\sin(x),\ x\in\mathbb{R}\},$$

$$F=\left\{\frac{n}{n-1},\ n\in\mathbb{N}\text{ and } n\geq 2\right\},\ G=\left\{\frac{m+n}{m},\ (m,n)\in\mathbb{N}^{*}\times\mathbb{N}\right\}.$$

Exercise 1.3

Consider the set $A=\left\{\frac{1}{m}+\frac{1}{n},\ m,n\in\mathbb{N}^{*}\right\}$.

(a) Show that A in bounded.

(b) Find the minimum, the maximum, the infimum and the supremum of the set A if there exist.

Exercise 1.4

Let $K = \left\{ \dfrac{2x+3}{x+2},\ x \in]-\infty; -3] \right\}$.
Determine the infimum and supremum of the set K.

Exercise 1.5

Solve in $\mathbb{R}$ the following equalities and inequalities:

$$|e^x - 2| = 1,\ |20 - x^2| \leq 16,\ |2x+3| = |x-1|,\ |x-2| \leq |x+4|.$$

Exercise 1.6

Show that $\forall x, y \in \mathbb{R}$ we have:

- $|x+y| \leq |x| + |y|$,
- $||x| - |y|| \leq |x-y|$,
- $2|xy| \leq x^2 + y^2$,
- $\max\{x,y\} = \dfrac{x+y+|x-y|}{2}$, $\min\{x,y\} = \dfrac{x+y-|x-y|}{2}$.

Exercise 1.7

Using the inequality

$$||x| - |y|| \leq |x-y|$$

to show that $\forall x, y \in \mathbb{R}$,

$$\sqrt{x^4 - 2|x^3 \cdot y| + x^2y^2} \leq |x^2 - xy|.$$

Exercise 1.8

(a) Solve in $\mathbb{R}$ the following equalities:

$$\lfloor e^x \rfloor = 1,\ \ \lfloor \ln(x) \rfloor = -1,\ \ \lfloor x^2 - 3x \rfloor = -2.$$

(b) Let $x, y \in \mathbb{R}$. Show that

$$\lfloor x \rfloor + \lfloor y \rfloor \leq x + y \leq \lfloor x \rfloor + \lfloor y \rfloor + 1.$$

(c) Prove that $\forall n \in \mathbb{N}^*$, and $\forall x \in \mathbb{R}$,

$$\left\lfloor \frac{\lfloor nx \rfloor}{n} \right\rfloor = \lfloor x \rfloor.$$

Exercise 1.9

(a) Compute the exact value of

$$\sum_{k=1}^{98}\left(\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n+2}}\right).$$

(b) Let $n\in\mathbb{N}$. Calculate

$$\sum_{k=0}^{n}(2k-1),\quad \sum_{k=1}^{n}3^k,\quad \sum_{k=1}^{n}\frac{(-1)^k}{k}+\sum_{k=0}^{n}\frac{1}{2k+1},\quad \sum_{k=1}^{n}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{2k}.$$

Exercise 1.10

(a) Let $n\in\mathbb{N}$. Calculate

$$\prod_{k=1}^{n}\frac{ke^k}{k+1},\quad \prod_{k=0}^{2n}\sqrt{nk},\quad \prod_{k=2}^{n}\left(\frac{2\ln(k!)}{\sum_{i=1}^{k}\ln(i)}\right).$$

(b) Show that

$$\frac{\sum_{k=1}^{n}n!\ln\left(\frac{1}{k}\right)}{\prod_{i=1}^{n}k}+\ln(n!)=0,\quad \text{where } n!=1\times 2\times\cdots\times n.$$

Chapitre 2

Sequences

2.1 Overview

2.1.1 Definitions

Definition 2.1.1

A sequence is a mapping

$$\begin{array}{rcl} u_n : \mathbb{N} & \longrightarrow & \mathbb{R} \\ n & \longmapsto & u_n. \end{array}$$

It is denoted by $(u_n)_{n\in\mathbb{N}}$ or $(u_n)_{n\geq 0}$. u_n is called the general term of the sequence (u_n) and

$$(u_n)_{n\in\mathbb{N}} = \{u_0,\ u_1,\ u_2,\ u_3,\ \cdots\}.$$

Example 2.1.1.

(a) Let $u_n = \dfrac{2^n - 3n}{n}$. *The sequence* (u_n) *is defined when* $n \geq 1$.

(b) Let $v_n = \sqrt{n-3}$. *The sequence* (v_n) *is defined when* $n \geq 3$.

(c) Let $w_n = \dfrac{1}{1+n^2}$. *The sequence* (w_n) *is defined for any* n *in* $\mathbb{N}$.

Remark 2.1.1

The ways in which a sequence can be defined.

- By an explicit definition of the general term of the sequence (u_n) i.e.: Express u_n in terms of n. For example, $u_n = n^3 + 2n - 1$.
- By a recurrence formula, i.e. a relationship that links any term in the sequence to the one that precedes it. In this case, to calculate u_n, you need to calculate all the terms that precede it. For example: $\begin{cases} u_3 = 1, \\ u_{n+1} = 4 - u_n. \end{cases}$

Example 2.1.2.

(i) *Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence defined as follows:* $u_n = \dfrac{n+1}{n^2+1}$.

Then, $u_0 = 1,\ u_1 = 2,\ u_2 = \dfrac{3}{5}, \cdots$

(ii) *Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence defined as follows:* $\begin{cases} u_0 = 1, \\ u_{n+1} = \sqrt{3+u_n} \end{cases}$

Then, $u_1 = \sqrt{3+u_0} = 2,\ u_2 = \sqrt{3+u_1} = \sqrt{5},\ \cdots$

Example 2.1.3. *Consider the following sequences:*

$$u_n = \frac{2^n - n}{3^n},\ v_n = \begin{cases} v_1 = 1, \\ v_{n+1} = \dfrac{v_n^2}{1+v_n} \end{cases} \quad \text{and } w_n = \begin{cases} w_0 = 1,\ w_1 = 1, \\ w_{n+1} = 5w_n - 6w_{n-1}. \end{cases}$$

We have

$$u_0 = \frac{2^0 - 0}{3^0} = 1,\ \ u_1 = \frac{2^1 - 1}{3^1} = \frac{1}{3},\ \ u_2 = \frac{2^2 - 2}{3^2} = \frac{2}{9}\cdots$$

$$v_2 = \frac{v_1^2}{1+v_1} = \frac{1^2}{1+1} = \frac{1}{2},\ \ v_3 = \frac{v_2^2}{1+v_2} = \frac{\left(\frac{1}{2}\right)^2}{1+\frac{1}{2}} = \frac{1}{6}\cdots$$

$$w_2 = 5w_1 - 6w_0 = 5 - 6 = -1,\ \ w_3 = 5w_2 - 6w_1 = -11\cdots$$

2.1.2 Boundedness and monotony of a sequence

Definition 2.1.2

(a) A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded below if $\exists m \in \mathbb{R},\ \forall n \in \mathbb{N},\ u_n \geq m$.

(b) A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded above if $\exists M \in \mathbb{R},\ \forall n \in \mathbb{N},\ u_n \leq M$.

(c) A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded if $\exists m,\ M \in \mathbb{R},\ \forall n \in \mathbb{N},\ m \leq u_n \leq M$ or,

$$\exists C \in \mathbb{R},\ \forall n \in \mathbb{N},\ |u_n| \leq C.$$

Example 2.1.4. *The sequence* $u_n = \dfrac{n}{n+1}$ *is bounded.*

Indeed, $\forall n \in \mathbb{N}$ *we have* $0 \leq n < n+1 \implies 0 \leq \dfrac{n}{n+1} < 1 \implies 0 \leq u_n < 1$.

Definition 2.1.3

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence.

(a) We say that $(u_n)_{n\in\mathbb{N}}$ is increasing (resp. decreasing) if

$$\forall n \in \mathbb{N}, u_n \leq u_{n+1} \text{ (resp.} u_n \geq u_{n+1}).$$

(b) We say that $(u_n)_{n\in\mathbb{N}}$ is constant if

$$\forall n \in \mathbb{N}, u_n = u_{n+1}.$$

Remark 2.1.2

We say that $(u_n)_{n\in\mathbb{N}}$ is monotone if it's increasing or decreasing.

Example 2.1.5. *Studying the monotony of the following sequence (u_n) such that:*

$$u_n = \frac{n+1}{2n+1}.$$

• *The monotony of the sequence (u_n).*

$$\begin{aligned} u_{n+1} - u_n &= \frac{n+2}{2n+3} - \frac{n+1}{2n+1} = \frac{(n+2)(2n+1)}{(2n+3)(2n+1)} \\ &= \frac{2n^2+5n+2-2n^2-5n-3}{(2n+3)(2n+1)} \\ &= \frac{-1}{(2n+3)(2n+1)} < 0 \end{aligned}$$

Then, the sequence (u_n) in decreasing (monotone).

2.1.3 Arithmetic and geometric sequences.

2.1.3.1 Arithmetic sequences.

Definition 2.1.4

A sequence $(u_n)_{n\in\mathbb{N}}$ is said to be arithmetic if and only if

$$\exists r \in \mathbb{R},\ \forall n \in \mathbb{N},\ u_{n+1} - u_n = r.$$

Here, r noted the common difference of the sequence (u_n).

Proposition 2.1.1

If $(u_n)_{n\in\mathbb{N}}$ is an arithmetic sequence of first term u_0 and common difference r, then we have

(a) $\forall n \in \mathbb{N},\ u_n = u_0 + nr.$

(b) $\forall n \in \mathbb{N} : u_n = \frac{1}{2}(u_{n+1} + u_{n-1}).$

(c) If $r = 0$ then, (u_n) is constant.

(d) If $r = 0$ then, (u_n) is convergent and if $r \neq 0$, (u_n) is divergent.

(e) If $S_n = u_0 + u_1 + \cdots + u_n$ then, $S_n = \dfrac{(u_0 + u_n)(n+1)}{2}$.

Example 2.1.6. *Consider an arithmetic sequence (u_n) such that* $\begin{cases} u_0 = 13, \\ 3u_3 - u_5 = 18. \end{cases}$

(a) To determine the common difference r of the sequence (u_n), we have

$$\begin{aligned} 3u_3 - u_5 = 18 &\iff 3(u_0 + 3r) - (u_0 + 5r) = 18 \\ &\iff 2u_0 + 4r = 18 \\ &\iff 4r = 18 - 26 \\ &\iff r = -2. \end{aligned}$$

(b) The general term of the sequence (u_n) is $u_n = 13 - 2n$.

(c) Writing the sum $S_n = \sum_{k=0}^{n-1} u_k$ in terms of n. We have

$$\begin{aligned} S_n &= \sum_{k=0}^{n-1} u_k = u_0 + u_1 + u_2 + \cdots + u_{n-1} \\ &= \frac{n}{2}(u_0 + u_{n-1}) \\ &= n(12 - n). \end{aligned}$$

2.1.3.2 Geometric sequences.

Definition 2.1.5

A sequence $(u_n)_{n\in\mathbb{N}}$ is said to be geometric if and only if

$$\exists q \in \mathbb{R}^*, \ \forall n \in \mathbb{N}, \ u_{n+1} = q \cdot u_n.$$

Here, q noted the common ratio of the sequence (u_n).

Proposition 2.1.2

If $(u_n)_{n\in\mathbb{N}}$ is a geometric sequence of first term u_0 and common ratio q, then we have

(a) $\forall n \in \mathbb{N}, \ u_n^2 = u_{n+1}u_{n-1}$.

(b) u_n is:
-convergent if $0 < |q| < 1$.

-divergent if $|q| > 1$.
-constant and hence convergent if $q = 1$.

(c) If $S_n = u_0 + u_1 + \cdots + u_n$ then,
- $S_n = (n+1) \cdot u_0$ if $q = 1$.
- $S_n = u_0 \dfrac{1-q^{n+1}}{1-q}$ if $q \neq 1$.

2.2 Convergence.

Definition 2.2.1

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence and ℓ be a real number. We say that:

(a) $(u_n)_{n\in\mathbb{N}}$ is convergent (or converge to ℓ) iff it has a finite limit ℓ.

(b) $(u_n)_{n\in\mathbb{N}}$ is divergent if it has more than one limit or $\lim_{n\to+\infty} u_n = \pm\infty$.

Example 2.2.1. *Consider the following sequences*

$$u_n = \log(\frac{n+1}{n}),\ v_n = (-1)^n,\ w_n = \sin(2n),\ \text{and } t_n = \sqrt{e^n}.$$

• *The sequence (u_n) is convergent since $\lim_{n\to+\infty} u_n = 0$ (exists and unique).*

• *The sequence (v_n) does not converge since $\lim_{n\to+\infty} v_n = \begin{cases} -1 & \text{indeterminate form } n \text{ is odd}, \\ 1 & \text{indeterminate form } n \text{ is even}, \end{cases}$ (does not exist).*

• *The sequence (w_n) does not converge since $\lim_{n\to+\infty} w_n$ is not exists.*

• *The sequence (t_n) is divergent since $\lim_{n\to+\infty} t_n = +\infty$.*

Theorem 2.2.1

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.

(a) If $(u_n)_{n\in\mathbb{N}}$ is increasing and bounded above, then it is convergent.

(b) If $(u_n)_{n\in\mathbb{N}}$ is decreasing and bounded below, then it is convergent.

Example 2.2.2. *Let $(u_n)_{n\geq 1}$ be a sequence defined as follows:*

$$\begin{cases} u_1 = \sqrt{2}, \\ u_{n+1} = \sqrt{2 + u_n}. \end{cases}$$

(a) We have $0 \leq u_n \leq 2$ *for all* $n \geq 1$. *Indeed, let* $\mathcal{P}(n)$ *the proposition defined by*

$$\mathcal{P}(n): \; 0 \leq u_n \leq 2.$$

We prove by induction that $\mathcal{P}(n)$ *is true for all* $n \in \mathbb{N}$.

- $\mathcal{P}(0)$: *is true since* $0 \leq u_0 = \sqrt{2} \leq 2$.
- *Suppose that* $\mathcal{P}(n)$ *is true and we prove that* $\mathcal{P}(n+1)$ *is true. We have*

$$\begin{aligned} 0 \leq u_n \leq 2 &\implies 2 \leq 2 + u_n \leq 4 \\ &\implies \sqrt{2} \leq \sqrt{2+u_n} \leq 2 \\ &\implies 0 \leq u_{n+1} \leq 2. \end{aligned}$$

  Thus, $\mathcal{P}(n+1)$ *is true.*
- *We conclude that* $\forall n \geq 1, \;\; 0 \leq u_n \leq 2$.

(b) The monotony of the sequence (u_n). *We have*

$$\begin{aligned} u_{n+1} - u_n &= \sqrt{2+u_n} - u_n \\ &= \frac{(\sqrt{2+u_n} - u_n)(\sqrt{2+u_n} + u_n)}{\sqrt{2+u_n} + u_n} \\ &= \frac{2 + u_n - u_n^2}{\sqrt{2+u_n} + u_n} \\ &= \frac{-(u_n+1)(u_n-2)}{\sqrt{2+u_n} + u_n} \geq 0. \;\; (\textit{ since } 0 \leq u_n \leq 2). \end{aligned}$$

Therefore, (u_n) *is increasing.*

(c) We have (u_n) *is bounded above by* 2 *and it's increasing. Then,* (u_n) *is convergent. Moreover,*

$$\begin{aligned} \lim_{n \to +\infty} u_{n+1} = \lim_{n \to +\infty} u_n = \ell &\implies \sqrt{2+\ell} = \ell \\ &\implies 2 + \ell - \ell^2 = 0, \\ &\implies \ell = 2 \vee \ell = -1, \\ &\implies \ell = 2 \; (\textit{ as } \ell \geq 0). \end{aligned}$$

Then, (u_n) *converges to* 2.

2.2.1 Properties

Proposition 2.2.1

Let $(u_n)_{n \in \mathbb{N}}, \; (v_n)_{n \in \mathbb{N}}$ two convergent sequences such that $\lim_{n \to +\infty} u_n = \ell$ and $\lim_{n \to +\infty} v_n = \ell'$. The properties of limits are summarized as follows:

- $\lim_{n \to +\infty} \lambda u_n = \lambda \ell$ for any $\lambda \in \mathbb{R}$.
- $\lim_{n \to +\infty} (u_n + v_n) = \ell + \ell'$.
- $\lim_{n \to +\infty} u_n v_n = \ell \cdot \ell'$.
- If $u_n \neq 0$ for $n \geq n_0$ and $\ell \neq 0$ then $\lim_{n \to +\infty} \frac{1}{u_n} = \frac{1}{\ell}$.
- If $v_n \neq 0$ for $n \geq n_0$ and $\ell' \neq 0$ then $\lim_{n \to +\infty} \frac{u_n}{v_n} = \frac{\ell}{\ell'}$.

Theorem 2.2.2

If $(u_n)_n \in \mathbb{N}$ is a convergent sequence, then $(u_n)_{n \in \mathbb{N}}$ is a bounded sequence.

Proof. Suppose that $u_n \to \ell$. Then for every number $\varepsilon > 0$ there is an integer N so that

$$|u_n - \ell| < \varepsilon$$

whenever $n \geq N$. In particular we could take just one value of ε, say ε, and find a number N so that

$$|u_n - \ell| < 1$$

whenever $n \geq N$. From this we see that

$$|u_n| = |u_n - \ell + \ell| \leq |u_n - \ell| + |\ell| \leq 1 + |\ell|,$$

for all $n \geq N$. This number $1 + |\ell|$ would be an upper bound for all the numbers $|u_n|$ except that we have no indication of the values for $|u_0|,\ |u_1|,\ |u_2|, \cdots, |u_{N-1}|$.
Thus if we write

$$M = \max\{|u_0|,\ |u_1|,\ |u_2|, \cdots, |u_{N-1}|, 1 + |\ell|\}$$

we must have

$$|u_n| \leq M,$$

for every value of n. This is an upper bound, proving the theorem. □

Proposition 2.2.2

Let $(u_n)_{n \in \mathbb{N}}$ be a real sequence.

(a) If (u_n) is increasing and not bounded above, then it is divergent and $\lim_{n \to +\infty} u_n = +\infty$.

(b) If (u_n) is decreasing and not bounded below, then it is divergent and $\lim_{n \to +\infty} u_n = -\infty$.

Theorem 2.2.3

Let $(u_n)_{n\in\mathbb{N}},\ (v_n)_{n\in\mathbb{N}}$ be two convergent sequences such that

$$\exists n_0 \in \mathbb{N},\ \ \forall n \geq n_0,\ u_n \leq v_n.$$

Then

$$\lim_{n\to+\infty} u_n \leq \lim_{n\to+\infty} v_n.$$

The next theorem is another useful variant on these themes. Here an unknown sequence is sandwiched between two convergent sequences, allowing us to conclude that that sequence converges. This theorem is often taught as **the squeeze theorem**.

Theorem 2.2.4

Let ℓ be a real number and $(u_n)_{n\in\mathbb{N}},\ (v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ three sequences such that

$$\begin{cases} \exists n_0 \in \mathbb{N},\ \forall n \geq n_0,\ v_n \leq u_n \leq w_n, \\ \lim\limits_{n\to+\infty} v_n = \lim\limits_{n\to+\infty} w_n = \ell, \end{cases}$$

then the sequence $(u_n)_n$ is convergent

$$\lim_{n\to+\infty} u_n = \ell.$$

Proof. Let ℓ be the limit of the two sequences (v_n) and (w_n). Choose n_1 so that

$$|v_n - \ell| < \varepsilon$$

if $n \geq n_1$ and choose n_2 so that

$$|w_n - \ell| < \varepsilon$$

if $n \geq n_2$. Set $N = \max\{n_1, n_2\}$. Note that

$$v_n - \ell \leq u_n - \ell \leq w_n - \ell,$$

for all n and so

$$-\varepsilon < v_n - \ell \leq u_n - \ell \leq w_n - \ell < \varepsilon,$$

if $n \geq N$. From this we see that

$$-\varepsilon < u_n - \ell < \varepsilon,$$

or, to put it in a more familiar form,

$$|u_n - \ell| < \varepsilon,$$

proving the statement of the theorem. □

Example 2.2.3. *Determine if the sequence* $(u_n)_{n>0}$ *where* $u_n = \dfrac{\sin(n)}{n}$, *converges or diverges.*

Recall that for all $n \in \mathbb{N},\ -1 \leq \sin(n) \leq 1 \implies \dfrac{-1}{n} \leq \dfrac{\sin(n)}{n} \leq \dfrac{1}{n}$.

Choosing $v_n = \dfrac{-1}{n}$ *and* $w_n = \dfrac{1}{n}$. *We have* $\lim_{n\to+\infty} v_n = \lim_{n\to+\infty} w_n = 0$. *Therefore,* $\lim_{n\to+\infty} \dfrac{\sin(n)}{n} = 0$ *(exists and unique). So* (u_n) *converges.*

Example 2.2.4. *Let* $(u_n)_{n>0}$ *be a sequence and* $x \in \mathbb{R}$. *Define* (u_n) *as follows*

$$u_n = \frac{1}{n^2}\sum_{k=1}^{n}\lfloor kx \rfloor.$$

(a) $\forall n > 0$ *we have* $\dfrac{x(n+1)}{2n} - \dfrac{1}{n} \leq u_n \leq \dfrac{x(n+1)}{2n}$. *Indeed, according to the properties of the floor, (see section 1.3.3), we have: for all* $k = 1, \cdots, n$ *and for all* $x \in \mathbb{R}$:

$$kx - 1 < \lfloor kx \rfloor \leq kx.$$

By the sum from $k = 1$ *to* $k = n$ *we obtain*

$$\begin{aligned}
\sum_{k=1}^{n}(kx-1) < \sum_{k=1}^{n}\lfloor kx \rfloor \leq \sum_{k=1}^{n} kx &\implies x\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 < \sum_{k=1}^{n}\lfloor kx \rfloor \leq x\sum_{k=1}^{n} k \\
&\implies x\frac{n(n+1)}{2} - n < \sum_{k=1}^{n}\lfloor kx \rfloor \leq x\frac{n(n+1)}{2} \\
&\implies x\frac{n+1}{2n} - \frac{1}{n} < \frac{1}{n^2}\sum_{k=1}^{n}\lfloor kx \rfloor \leq x\frac{n+1}{2n} \\
&\implies x\frac{n+1}{2n} - \frac{1}{n} < u_n \leq x\frac{n+1}{2n}
\end{aligned}$$

(b) Since

$$\lim_{n\to+\infty}\left(x\frac{n+1}{2n} - \frac{1}{n}\right) = \frac{x}{2}$$

and

$$\lim_{n\to+\infty}\left(x\frac{n+1}{2n}\right) = \frac{x}{2}.$$

Then,

$$\lim_{n\to+\infty} u_n = \frac{x}{2}.$$

Proposition 2.2.3

Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two sequences such that $\exists n_0 \in \mathbb{N},\ \forall n \geq n_0,\ u_n \leq v_n$. Then,

(a) $\lim_{n\to+\infty} u_n = +\infty \implies \lim_{n\to+\infty} v_n = +\infty.$

(b) $\lim_{n\to+\infty} v_n = -\infty \implies \lim_{n\to+\infty} u_n = -\infty.$

Theorem 2.2.5

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence. Then,

$$\lim_{n\to+\infty} |u_n| = 0 \implies \lim_{n\to+\infty} u_n = 0.$$

Example 2.2.5. *Determine if the sequence* $(u_n)_{n\geq 0}$ *where* $u_n = \dfrac{(-1)^n}{n^2+2}$, *converges or diverges.*
Using the Absolute Value Theorem we see that

$$\lim_{n\to+\infty} |u_n| = \lim_{n\to+\infty} \left|\frac{(-1)^n}{n^2+2}\right| = \lim_{n\to+\infty} \frac{|(-1)^n|}{|n^2+2|} = \lim_{n\to+\infty} \frac{1}{n^2+2} = 0.$$

Therefore, $\lim_{n\to+\infty} |u_n| = 0$ *which implies* $\lim_{n\to+\infty} u_n = 0$.

2.3 Particular sequences

2.3.1 Adjacent sequences

Definition 2.3.1

Two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are said to be adjacent if and only if:
• one of them is increasing and the other is decreasing.
• $\lim_{n\to+\infty} (u_n - v_n) = 0.$

Theorem 2.3.1

If two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent, then they are convergent and possess the same limit ℓ i.e

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n = \ell$$

Example 2.3.1. *Consider two sequences* $(u_n)_{n\in\mathbb{N}^*}$ *and* $(v_n)_{n\in\mathbb{N}^*}$ *such that*

$$u_n = 1 + \frac{1}{n^2} \text{ and } v_n = 1 - \frac{1}{n^2}.$$

(a) *We have* (u_n) *is decreasing. Indeed,* $\forall n \in \mathbb{N}^*$:

$$u_{n+1} - u_n = \left(1 + \frac{1}{(n+1)^2}\right) - \left(1 + \frac{1}{n^2}\right) = \frac{-2n-1}{n^2(n+1)^2} \leq 0.$$

(b) *We have* (v_n) *is increasing. Indeed,* $\forall n \in \mathbb{N}^*$:

$$v_{n+1} - v_n = \left(1 - \frac{1}{(n+1)^2}\right) - \left(1 - \frac{1}{n^2}\right) = \frac{2n+1}{n^2(n+1)^2} \geq 0.$$

(c) $\lim\limits_{n\to+\infty}(u_n - v_n) = \lim\limits_{n\to+\infty} \frac{2}{n^2} = 0.$
So, from (a), (b) *and* (c), *we conclude that* (u_n) *and* (v_n) *are adjacent sequences and possess the same limit* $\ell = 1$.

Example 2.3.2. *Consider two sequences* $(u_n)_{n\in\mathbb{N}^*}$ *and* $(v_n)_{n\in\mathbb{N}^*}$ *such that*

$$u_n = \ln\left(\frac{n+1}{n}\right) \text{ and } v_n = \ln\left(\frac{n}{n+1}\right)$$

(u_n) *and* (v_n) *are adjacent sequences. Indeed, we have*

$$\begin{aligned} u_{n+1} - u_n &= \ln\left(\frac{n+2}{n+1}\right) - \ln\left(\frac{n+1}{n}\right) \\ &= \ln\left(\frac{(n+2)n}{(n+1)^2}\right) \\ &= \ln\left(\frac{n^2+2n}{n^2+2n+1}\right) < 0 \;\; \textit{since} \;\; \frac{n^2+2n}{n^2+2n+1} < 1. \end{aligned}$$

So, (u_n) *is decreasing. Moreover,*

$$\begin{aligned} v_{n+1} - v_n &= \ln\left(\frac{n+1}{n+2}\right) - \ln\left(\frac{n}{n+1}\right) \\ &= \ln\left(\frac{n^2+2n+1}{n^2+2n}\right) > 0 \;\; \textit{since} \;\; \frac{n^2+2n+1}{n^2+2n} > 1. \end{aligned}$$

Thus, (v_n) *is decreasing.*
Concerning the deference $u_n - v_n$ *we have*

$$\begin{aligned} u_n - v_n &= \ln\left(\frac{n+1}{n}\right) - \ln\left(\frac{n}{n+1}\right) \\ &= \ln\left(\frac{(n+1)^2}{n^2}\right) \end{aligned}$$

Then, $\lim\limits_{n\to+\infty}(u_n - v_n) = \ln\left(\frac{(n+1)^2}{n^2}\right) = 0.$
From, the preceding results, we deduce that (u_n) *and* (v_n) *are adjacent sequences.*

2.3.2 Subsequence

Definition 2.3.2

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence and (k_n) be a strictly increasing sequence of natural numbers. Then, the sequence (u_{k_n}) is called a subsequence of $(u_n)_{n\in\mathbb{N}}$.

Example 2.3.3. *Consider the sequence*

$$(u_n)_{n\geq 0} = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, \cdots\}.$$

• *The sequence (v_n) defined as*

$$v_0 = u_1,\ v_1 = u_3,\ v_2 = u_5 \cdots$$

that means $v_k = u_{2k+1}$ is a subsequence of (u_n).

• *The sequence (w_n) defined as*

$$w_0 = u_0,\ w_1 = u_2,\ w_2 = u_4 \cdots$$

that means $w_k = u_{2k}$ is a subsequence of (u_n).

• *The sequence (t_n) defined as*

$$t_0 = u_0,\ t_1 = u_1,\ t_2 = u_4, t_3 = u_9 \cdots$$

that means $t_k = u_{k^2}$ is a subsequence of (u_n).

Theorem 2.3.2

Suppose that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to the real number ℓ. Then every subsequence of (u_n) also converges to ℓ.

Example 2.3.4. *Define a sequence $(u_n)_{n\in\mathbb{N}^*}$ by $u_n = \ln(ne^{-2n} + 1)$.*

(a) The sequence (u_n) is convergent. Indeed,

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \ln(ne^{-2n} + 1) = 0.$$

(b) Defined two sequences $(v_n) = (u_{3n})$ and $(w_n) = (u_{n^2})$.

(i) (v_n) and (w_n) are subsequences of (u_n) since $n \mapsto 3n$ and $n \mapsto n^2$ are strictly increasing functions on n.

(ii) We deduce that (v_n) and (w_n) are convergent and

$$\lim_{n\to+\infty} v_n = \lim_{n\to+\infty} w_n = \lim_{n\to+\infty} u_n = 0.$$

Theorem 2.3.3

Every sequence contains a monotonic subsequence.

This theorem about the existence of monotonic subsequences, we give now the theorem of **Bolzano-Weierstrass** about the existence of convergent subsequences

Theorem 2.3.4

Every bounded sequence contains a convergent subsequence.

Proof. By Theorem 2.3.3 every sequence contains a monotonic subsequence. Here that subsequence would be both monotonic and bounded, and hence convergent. □

Theorem 2.3.5

From any bounded real sequence, we can extract a convergent sequence.

Example 2.3.5. *Let* $(u_n)_{n\in\mathbb{N}^*}$ *be a sequence such that:* $u_n = \cos\left(\frac{n\pi}{2}\right) + \frac{1}{n}$.

(a) (u_n) *is divergent since*

$$\begin{aligned}\lim_{n\to+\infty} u_n &= \lim_{n\to+\infty}\left(\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n}\right)\\ &= \lim_{n\to+\infty}\cos\left(\frac{n\pi}{2}\right) \quad (\textit{ doesn't exists. })\end{aligned}$$

(b) (u_n) *is a bounded sequence. Indeed, we have for all* $n \in \mathbb{N}^*$:

$$-1 \le \cos\left(\frac{n\pi}{2}\right) \le 1 \ \textit{ and } 0 < \frac{1}{n} \le 1. \tag{2.1}$$

Then, for all $n \in \mathbb{N}^*$:

$$-1 \le \cos\left(\frac{n\pi}{2}\right) + \frac{1}{n} \le 2.$$

(c) According Bolzano-Weierstrass Theorem, we have for any bounded real sequence, we can extract a convergent sequence.
Since (u_n) *is bounded, we have for example* (v_n) *and* (w_n) *are subsequences from* (u_n) *defined by*

$$v_n = u_{4n} = \cos(2n\pi) + \frac{1}{4n} \ \textit{ converges to } 1.$$

$$w_n = u_{2n+1} = \cos\left(\frac{(2n+1)\pi}{2}\right) + \frac{1}{2n+1} \ \textit{ converges to } 0.$$

$$\vdots$$

Proposition 2.3.1

If a subsequence of (u_n) converges to ℓ_1 and another subsequence of (u_n) converges to ℓ_2 where $\ell_1 \neq \ell_2$, then the sequence (u_n) does not converge.

Example 2.3.6. *Consider the sequence*

$$(u_n) = \left\{1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \cdots\right\}.$$

This sequence has the constant subsequence $(v_n) = \{1, 1, 1, 1, \cdots\}$ *that converges to* 1 *and the subsequence* $(w_n) = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\right\}$ *that converges to* 0.
Thus,

$$(u_n) \textit{ does not converge.}$$

Example 2.3.7. *Consider a sequence* $(u_n)_{n\in\mathbb{N}}$ *such that*

$$u_n = (-1)^n + e^{-n}$$

Defined two subsequences (v_n) *and* (w_n) *of* (u_n) *such that*

$$v_n = u_{2n} = (-1)^{2n} + e^{-2n} \textit{ and } w_n = u_{2n+1} = (-1)^{2n+1} + e^{-2n-1}.$$

We have

(a) (v_n) *converges to* 1 *and* (w_n) *converges to* -1.

(b) Conclusion: (v_n) *and* (w_n) *are subsequences of* (u_n) *and* $\lim_{n\to+\infty} v_n \neq \lim_{n\to+\infty} w_n$.
Then,

$$(u_n) \textit{ does not converge.}$$

Proposition 2.3.2

If a sequence (u_n) has divergent subsequence, then one can conclude that (u_n) diverges.
If a sequence (u_n) has convergent subsequence, then one cannot conclude that (u_n) converges.

2.4 Practice Exercises

Exercise 2.1

Consider an arithmetic sequence $(u_n)_{n\geq 1}$ such that

$$\begin{cases} u_6 + u_8 + u_{10} + u_{12} = 132, \\ u_2 + u_4 = 18. \end{cases}$$

(a) Determine the common difference r and the first term u_1 of the sequence (u_n).

(b) Express (u_n) in terms of n.

(c) Calculate the sum $S_n = \sum_{k=1}^{2n} u_k$ in terms of n.

Exercise 2.2

Let the sequence $(u_n)_{n\in\mathbb{N}}$ be recursively defined by:

$$\begin{cases} u_0 = \alpha \in \mathbb{R}, \\ u_{n+1} = \dfrac{u_n}{u_n^2 + 1}, \ \forall n. \end{cases}$$

(a) Determine α, such that the sequence (u_n) be constant.

(b) Suppose that $\alpha < 0$.

  (i) Prove that, $\forall n \in \mathbb{N}, \ u_n < 0$.

  (ii) Show that the sequence (u_n) is strictly increasing.

(c) Suppose that $\alpha > 0$.

  (a) Prove that, $\forall n \in \mathbb{N}, \ u_n > 0$.

  (b) Show that the sequence (u_n) is strictly decreasing.

Exercise 2.3

Consider a sequence (u_n) such that

$$\begin{cases} u_0 = \alpha \in \mathbb{R}, \\ 5u_{n+1} = u_n - 10. \end{cases}$$

(a) Determine α such that (u_n) is constant.

(b) Let $\alpha = \frac{1}{2}$. Consider a sequence (v_n) such that

$$\forall n \in \mathbb{N}, \ v_n = u_n - \lambda \text{ where } \lambda \in \mathbb{R}.$$

(c) Determine λ such that (v_n) be a geometric sequence and then calculate the common ratio q and the first term v_0.

(d) Express (v_n) in terms of n and then deduce the expression of (u_n) in terms of n.

(e) Calculate the sum $S_n = \sum_{k=0}^{2n+1} v_k$ and then deduce the sum $S'_n = \sum_{k=0}^{2n+1} u_k$.

(f) Calculate $T_n = \sum_{k=0}^{n} v_k^2$, $T'_n = \sum_{k=0}^{n} \dfrac{1}{v_k}$ and $P_n = \prod_{k=0}^{n} v_k$.

Exercise 2.4

Calculate the limit, if there exists of each of the following sequences and determine which sequences are convergent and which are not.

$$u_n = \frac{3n-2}{5n+1},\ v_n = \cos(\pi n),\ w_n = \frac{(-1)^n}{2} \text{ and } t_n = ln(n) - e^n.$$

Exercise 2.5

Let $(u_n)_{n\geq 0}$ be a sequence defined as follows: $\begin{cases} u_0 = \alpha \in \mathbb{R}, \\ u_{n+1} = \dfrac{u_n + 4}{3}. \end{cases}$

I. Determine α such that (u_n) is a constant sequence.
II. Suppose that $\alpha = -1$.

(a) Show that $\forall n \geq 0,\ u_n \leq 2$.

(b) Study the monotony of the sequence (u_n).

(c) Show that (u_n) is convergent and then determine its limit.

Exercise 2.6

Let $(u_n)_{n\geq 1}$ be a sequence defined as follows: $u_n = \sum_{k=1}^{n} \dfrac{1}{k^2}$.

(a) Study the monotony of the sequence (u_n).

(b) Show that $\forall n \geq 1,\ u_n < 2$. $\left(\text{indication } \forall m > 0,\ \frac{1}{m} - \frac{1}{m+1} = \frac{1}{m(m+1)}.\right)$.

(c) Is (u_n) convergent? Justify.

Exercise 2.7

Study the convergence of the following sequences:

$$1)\ u_n = 1-(-1)^n+\frac{1}{n},\quad 2)\ u_n=\frac{cos(n)+(-1)^n}{n^3+2},\quad 3)\ u_n=\frac{2^n-1}{2^n+1},$$

$$4)\ \frac{3^n-5^n}{3^n+5^n},\quad 5)\ \sqrt{n^2+n}-n,\quad 6)\ u_n=\sum_{k=1}^{n}\frac{k}{n^3},$$

$$7)\ u_n=\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}},\quad 8)\sum_{k=1}^{n}\ln(n^k),\quad 9)\ u_n=\prod_{k=1}^{n}n^2e^{-2k}.$$

Exercise 2.8

Let $(u_n)_{n\geq 1}$, $(v_n)_{n\geq 1}$ and $(w_n)_{n\geq 0}$ be three sequences defined as follows:

$$u_n=\left(1+\frac{1}{n}\right)^n,\ v_n=\left(1+\frac{1}{n}\right)^{n^2}\quad \text{and}\quad w_n=\frac{n!+5^n}{n-3^n}.$$

1. Let $x\in\mathbb{R}$. Use the induction proof to show that $\forall n\in\mathbb{N}^*,\ (1+x)^n\geq 1+nx$.
2. Show that the sequence (u_n) is monotonically increasing.
3. Prove that $\forall n\in\mathbb{N},\ v_n\geq 2^n$ and deduce $\lim\limits_{n\to+\infty} v_n$.
4. Using the fact that $n<3^n, \forall n\in\mathbb{N}$ to show that $\forall n\in\mathbb{N},\ w_n\leq \dfrac{\frac{n}{3^n}+\left(\frac{5}{3}\right)^n}{\frac{n}{3^n}-1}$.
5. Is the sequence (w_n) convergent ? Justify.

Exercise 2.9

Consider the sequences $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$, $(w_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ defined as follows

$$\begin{cases} u_0=12, \\ u_{n+1}=\dfrac{u_n+2v_n}{3}\end{cases},\ \begin{cases} v_0=1, \\ v_{n+1}=\dfrac{u_n+3v_n}{4}\end{cases},\ w_n=u_n-v_n,\quad \text{and}\quad t_n=3u_n+8v_n.$$

(a) Show that $(w_n)_{n\in\mathbb{N}}$ is a geometric sequence and then express (w_n) in terms of n.

(b) Calculate $S_n=\sum\limits_{k=0}^{n}w_k$ and $P_n=\prod\limits_{k=0}^{n-1}w_k$ in terms of n.

(c) Show that $(t_n)_{n\in\mathbb{N}}$ a constant sequence and for all n we have $t_n=44$.

(d) Show that the sequence (u_n) is decreasing and the sequence (v_n) is increasing.

(e) Conclude that (u_n) and (v_n) are adjacent sequences.

(f) Express (u_n) and (v_n) in terms of n and then deduce $\lim\limits_{n\to+\infty} u_n$ and $\lim\limits_{n\to+\infty} v_n$.

Exercise 2.10

Defined a sequence $(u_n)_{n\geq 1}$ by $u_n = \sum\limits_{k=1}^{n} \dfrac{1}{k^2}$ and we put $v_n = u_n + \dfrac{1}{n}$.
Show that (u_n) and (v_n) are adjacent sequences.

Exercise 2.11

Use the subsequence result and show that the sequence (u_n) such that $u_n = \sin(\frac{n\pi}{2})$ does not converge.

Exercise 2.12

Consider two sequences $(u_n)_{n>0}$ and $(v_n)_{n\geq 0}$ where

$$u_n = \frac{\cos(n\pi) + \sin(n\pi)}{n\pi} \text{ and } v_n = \cos\left(\frac{n\pi}{4}\right).$$

1. Show that the sequence (u_n) is convergent and determine its limit.
2. Let $(v_n) = (u_{3n})$ and $(w_n) = (u_{5n+1})$. Are the sequences (v_n) and (w_n) convergent? Justify.
3. Let $(t_n) = (v_{4n})$. Shaw that the sequence (t_n) is divergent and then deduce the convergence of (v_n).

Exercise 2.13

We define a sequence $(u_n)_{n\geq 1}$ by $u_n = \sum\limits_{k=1}^{n} \dfrac{(-1)^{k+1}}{k}$. and put $v_n = u_{2n}$ and $w_n = u_{2n+1}$.
Show that (v_n) and (w_n) are adjacent sequences.

Chapitre 3
Real valued functions of a real variable

3.1 Overview

3.1.1 Definitions

Definition 3.1.1

Let I and J be two intervals of $\mathbb{R}$. A function f from I to J is a rule that assigns to each element x in I exactly one element y in J.

Example 3.1.1. $\begin{array}{rcl} f:\mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \ln(2x) \end{array}$ *and* $\begin{array}{rcl} g:\mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & 1+3e^x \end{array}$ *are functions.*

Definition 3.1.2

The domain $\mathcal{D}_f$ of a function $f: I \to J$ is the set

$$\boxed{\mathcal{D}_f = \{x \in I,\ f(x) \text{ exists}\}}.$$

Example 3.1.2.
Determine the domain of the following functions:

$$f(x) = \frac{2-e^x}{e^{-x}-1} \quad \textit{and } g(x) = \ln(\sqrt{3-x}-1).$$

Solution

(a) Determine $\mathcal{D}_f$:

$$\mathcal{D}_f = \{x \in \mathbb{R} :\ e^{-x}-1 \neq 0\} = \{x \in \mathbb{R} :\ -x \neq 0\} = \mathbb{R}^*.$$

(b) Determine $\mathcal{D}_g$:

$$\mathcal{D}_g = \{x \in \mathbb{R} :\ 3-x \geq 0 \ \textit{and}\ \sqrt{3-x}-1 > 0\}.$$

$$3-x \geq 0 \implies x \leq 3. \ \textit{Thus, } x \in]-\infty, 3].$$

$$\begin{aligned} \sqrt{3-x}-1 > 0 &\implies 3-x > 1 \\ &\implies x < 2. \ \textit{Thus, } x \in]-\infty, 2[. \end{aligned}$$

Then,

$$\mathcal{D}_g =]-\infty, 3] \cap]-\infty, 2[=]-\infty, 2[.$$

Graph of a function

Definition 3.1.3

Let f be a function defined on $\mathcal{D}_f$. The graph, or representative curve, of a function f is the set

$$\mathcal{C}_f = \{(x, f(x)),\ x \in \mathcal{D}_f\}.$$

formed by the points $(x, f(x)) \in \mathbb{R}^2$ in the plane with an orthonormal reference $(o,\ \vec{i},\ \vec{j})$.

3.1.2 Parity and periodicity of a function

Definition 3.1.4

Let I be an interval or union of intervals in $\mathbb{R}$. I is said to be symmetric at the origin iff

$$\forall x \in \mathbb{R},\ x \in I \implies (-x) \in I.$$

Definition 3.1.5

Let f be a function, then

◇ f is said to be even if:

$$\boxed{\forall x \in \mathcal{D}_f,\ (-x) \in \mathcal{D}_f\ \wedge\ f(-x) = f(x).}$$

◇ f is said to be odd if:

$$\boxed{\forall x \in \mathcal{D}_f,\ (-x) \in \mathcal{D}_f\ \wedge\ f(-x) = -f(x).}$$

Example 3.1.3.

1. Let f be a function defined by $f(x) = \dfrac{1+e^{2x}}{e^x}$.

$$\mathcal{D}_f = \{x \in \mathbb{R} :\ e^x \neq 0\} = \mathbb{R}.$$

- *$\mathbb{R}$ it is symmetric at the origin i.e $\forall x \in \mathcal{D}_f$ we have $(-x) \in \mathcal{D}_f$.*
- *$\forall x \in \mathcal{D}_f$ we have*

$$\begin{aligned} f(-x) &= \frac{1+e^{-2x}}{e^{-x}} \\ &= \frac{\dfrac{e^{2x}+1}{e^{2x}}}{e^{-x}} \\ &= \frac{1+e^{2x}}{e^x} \\ &= f(x). \end{aligned}$$

Then, f is an even function.

2. Let g be a function defined by $g(x) = \dfrac{\sqrt{1-x^2}}{x^3}$.

$$\mathcal{D}_g = \{x \in \mathbb{R},\ 1 - x^2 \geq 0 \ \text{ and } x^3 \neq 0\}.$$

- $1 - x^2 \geq 0 \implies x \in [-1, 1]$.
- $x^3 \neq 0 \implies x \neq 0$ *that means* $x \in \mathbb{R}^*$.

Thus

$$\mathcal{D}_g = [-1, 1] \cap \mathbb{R}^* = [-1, 0[\cup]0, 1].$$

- $[-1, 0[\cup]0, 1]$ *is symmetric at the the origin i.e* $\forall x \in \mathcal{D}_g$ *we have* $(-x) \in \mathcal{D}_g$.
- $\forall x \in \mathcal{D}_g$ *we have*

$$\begin{aligned} g(-x) &= \frac{\sqrt{1-(-x)^2}}{(-x)^3} \\ &= -\frac{\sqrt{1-x^2}}{x^3} \\ &= -g(x). \end{aligned}$$

Then g is an odd function.

Definition 3.1.6

Let f be a function and $T > 0$ is a positive real number. The function f is said to be $T-$periodic if

$$\bullet\ \forall x \in \mathcal{D}_f,\ \ x + T \in \mathcal{D}_f.$$
$$\bullet\ \forall x \in \mathcal{D}_f,\ f(x+T) = f(x).$$

Here, T is called a period.

Example 3.1.4. *We have:*

(a) $\cos(x + 2\pi) = \cos(x),\ \ \forall x \in \mathbb{R}$, *then* $x \mapsto \cos(x)$ *is* $2\pi-$ *periodic function.*

(b) $\sin(2(x + \pi)) = \sin(2x),\ \ \forall x \in \mathbb{R}$, *then* $x \mapsto \sin(2x)$ *is* $\pi-$*periodic function.*

Base Period

Proposition 3.1.1

Let f be a periodic function, then f has a minimal period or a smallest period, which is often called either the **fundamental period** or the **base period**.

Example 3.1.5. *Find the base period of the following function*

$$f(x) = \cos\left(\frac{2x}{5}\right) - \sin\left(\frac{3x}{2}\right).$$

Solution

We have $f(x) = \cos\left(\frac{2x}{5}\right) - \sin\left(\frac{3x}{2}\right)$, *the periods of these two part are* 5π *and* $\frac{4\pi}{3}$. *So, the base period is the least common multiple of these two numbers i.e* 20π.

3.1.3 Bounded and monotonic functions

Definition 3.1.7

Let f be a function defined on $\mathcal{D}_f \subseteq \mathbb{R}$. We say that:

(a) f is upper-bounded or bounded above on $\mathcal{D}_f$ if $\boxed{\exists M \in \mathbb{R},\ \forall x \in \mathcal{D}_f,\ f(x) \leq M,}$

(b) f is lower-bounded or bounded bellow on $\mathcal{D}_f$ if $\boxed{\exists m \in \mathbb{R},\ \forall x \in \mathcal{D}_f,\ f(x) \geq m,}$

(c) f is bounded on $\mathcal{D}_f$ if $\boxed{\exists M, m \in \mathbb{R},\ \forall x \in \mathcal{D}_f,\ m \leq f(x) \leq M}$.

Example 3.1.6. *The function* $x \mapsto 1 + 2\sin(x)$ *is bounded on* $\mathbb{R}$. *Indeed*

$$\begin{aligned} \forall x \in \mathbb{R},\ -1 \leq \sin(x) \leq 1 &\implies -2 \leq 2\sin(x) \leq 2 \\ &\implies -1 \leq 1 + 2\sin(x) \leq 3. \end{aligned}$$

Definition 3.1.8

Let f be a function defined on $\mathcal{D}_f \subseteq \mathbb{R}$. Then

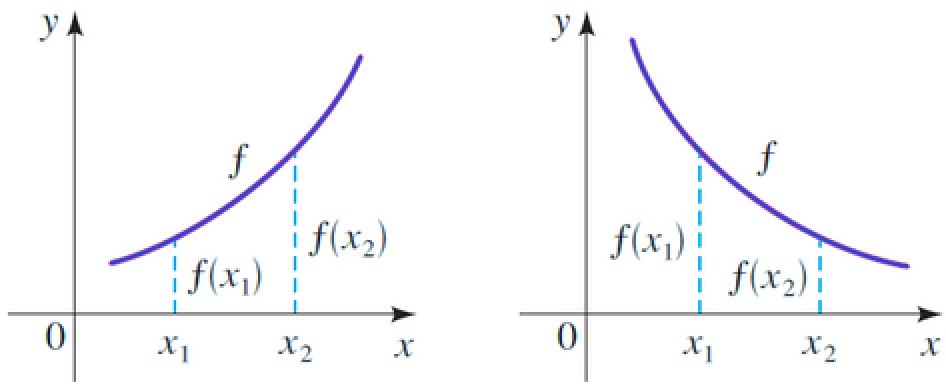
(a) f is strictly increasing on $\mathcal{D}_f$ if $\boxed{\forall x_1,\ x_2 \in \mathcal{D}_f,\ x_1 < x_2 \implies f(x_1) < f(x_2)}$.

(b) f is strictly decreasing on $\mathcal{D}_f$ if $\boxed{\forall x_1,\ x_2 \in \mathcal{D}_f,\ x_1 < x_2 \implies f(x_1) > f(x_2)}$.

(c) f is non-decreasing on $\mathcal{D}_f$ if $\boxed{\forall x_1,\ x_2 \in \mathcal{D}_f,\ x_1 < x_2 \implies f(x_1) \leq f(x_2)}$.

(d) f is non-increasing on $\mathcal{D}_f$ if $\boxed{\forall x_1,\ x_2 \in \mathcal{D}_f,\ x_1 < x_2 \implies f(x_1) \geq f(x_2)}$.

(e) f is constant on $\mathcal{D}_f$ if $\boxed{\forall x_1,\ x_2 \in \mathcal{D}_f,\ x_1 < x_2 \implies f(x_1) = f(x_2)}$.



f is increasing $\qquad$ f is decreasing

3.1.4 Composition of functions

Definition 3.1.9

Let f and g be two functions defined on $\mathcal{D}_f$ and $\mathcal{D}_g$ respectively and such that $\forall x \in \mathcal{D}_f,\ \ f(x) \in \mathcal{D}_g$. **The composition** of functions f and g is an operation consisting of replacing the first function's independent variable by the expression representing the second function's dependent variable. The composite g of f is denoted $g \circ f$ and is called the composite of g by f.

Example 3.1.7. *Let the function f be defined by $f(x) = 2x + 3$ and the function g be defined by $g(x) = x^2$. At first, since f and g are polynomials we have $\mathcal{D}_f = \mathcal{D}_g = \mathbb{R}$. So, we can calculate $f \circ g$ and $g \circ f$.*
• *The composite $(f \circ g)$ is calculated as follows:*

$$\forall x \in \mathbb{R},\ f \circ g(x) = f(g(x)) = 2x^2 + 3.$$

• *The composite $(g \circ f)$ is calculated as follows:*

$$\forall x \in \mathbb{R},\ g \circ f = g(f(x)) = (2x+3)^2 = 4x^2 + 12x + 9. \tag{3.1}$$

Remark 3.1.1

Generally, the composition of functions is not commutative i.e $f \circ g \neq g \circ f$.

Proposition 3.1.2

Let f and g be two functions defined on $\mathcal{D}_f$ and $\mathcal{D}_g$ respectively and such that $\forall x \in \mathcal{D}_f, f(x) \in \mathcal{D}_g$. Then

(a) The function $g \circ f$ is increasing if f and g have the same monotonicity on their domains.

(b) The function $g \circ f$ is decreasing if f and g do not have the same monotonicityon their domains.

Example 3.1.8.

(a) The function $x \mapsto e^{x^3+5}$ is increasing on $\mathbb{R}$ since $x \mapsto e^x$ and $x \mapsto x^3 + 5$ are increasing.

(b) The function $x \mapsto \dfrac{1}{\ln(x)}$ is decreasing on $]0, 1[\cup]1, +\infty]$ since the function $x \mapsto \dfrac{1}{x}$ is decreasing and $x \mapsto \ln(x)$ is increasing.

3.2 Power functions

3.2.1 Domain of a power function

Definition 3.2.1

Let f and g be two functions defined on $\mathcal{D}_f$ and $\mathcal{D}_g$ respectively. A power function h is defined as follows

$$\boxed{h(x) = [f(x)]^{g(x)}}.$$

Moreover, the domain of the power function h is

$$\boxed{\mathcal{D}_h = \mathcal{D}_f \cap \mathcal{D}_g \cap \{x \in \mathbb{R}, f(x) > 0\}}.$$

Example 3.2.1. *Determine the domain of the following functions:*

$$h(x) = \Big(\ln(x)\Big)^{\sqrt{7-x}}, \quad k(x) = \left(\frac{1-x}{1+x}\right)^{\sqrt{x}}.$$

(a) Determine Domain of h. We have

$$\mathcal{D}_h = \{x \in \mathbb{R} : \ x > 0 \text{ and } 7 - x \geq 0 \text{ and } \ln(x) > 0\}.$$

$$\begin{aligned} x > 0 &\implies x \in]0, +\infty[, \\ 7 - x \geq 0 &\implies x \leq 7 \implies x \in]-\infty, 7] \\ \ln(x) > 0 &\implies x > 1 \implies x \in]1, +\infty[. \end{aligned}$$

Then,

$$\mathcal{D}_h =]0, +\infty[\cap] - \infty, 7] \cap]1, +\infty[=]1, 7].$$

(b) Determine Domain of k. We have

$$\mathcal{D}_k = \{x \in \mathbb{R} : \ 1 + x \neq 0 \textit{ and } x \geq 0 \textit{ and } \frac{1-x}{1+x} > 0\}.$$

$$\begin{aligned} 1 + x \neq 0 &\implies x \in] - \infty, -1[\cup] - 1, +\infty[, \\ x \geq 0 &\implies x \in [0, +\infty[\\ \frac{1-x}{1+x} > 0 &\implies x \in] - 1, 1[. \end{aligned}$$

Then,

$$\mathcal{D}_k = (] - \infty, -1[\cup] - 1, +\infty[) \cap [0, +\infty[\cap]1, +\infty[= [0, 1[.$$

3.3 Limit of a function

3.3.1 Neighborhood

Definition 3.3.1

A neighbourhood $\mathcal{V}$ of a real point x_0 except possibly at x_0 itself, is any open set containing the element x_0. In particular $]x_0 - \delta, \ x_0 + \delta[$ is called the $\delta-$ neighbourhood of x_0 and $]x_0 - \delta, x_0[\cup]x_0, \ x_0 + \delta[$ is called the deleted $\delta-$neighbourhood of x_0.

Geometrical sense.
Imagine the real number line. The point x_0 is somewhere on this line. Now, you go δ units to the left and δ units to the right. The interval between these two points, excluding the points themselves, is the $\delta-$ neighborhood of x_0.

3.3.2 Limit at a point

The concept of a neighborhood is closely tied to the idea of a limit. When we say:

$$\lim_{x \to x_0} f(x) = \ell.$$

It means that for every positive ε, no matter how small, there exists a positive number δ such that if x is within the $\delta-$neighborhood of x_0 (excluding x_0 itself), then $f(x)$ is within the $\varepsilon-$neighborhood of ℓ. This is a way to formalize the idea that as x gets close to x_0, $f(x)$ gets close to ℓ.
In the following we give the $\varepsilon - \delta$ definition of the limit of a function.

Definition 3.3.2

Let $f : I \to \mathbb{R}$ be a function defined on an interval I of $\mathbb{R}$. Let $\ell \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ be a point of I or an extremity of I. We say that f has ℓ as its limit at x_0 iff,

$$\forall \varepsilon > 0,\ \exists \alpha > 0,\ \forall x \in \mathcal{D}_f,\ |x - x_0| < \alpha \implies |f(x) - \ell| < \varepsilon,$$

and we write $\lim\limits_{x \to x_0} f(x) = \ell$.

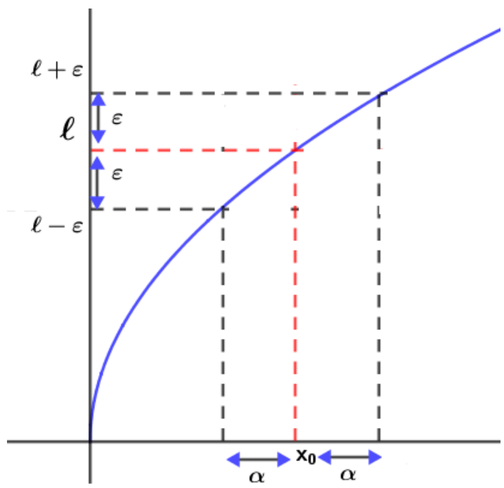


Illustration of a finite limit at a point

Example 3.3.1. *Show that* $\lim\limits_{x \to 1}(2x + 1) = 3$.

Solution: *We have*

$\lim\limits_{x \to 1}(2x + 1) = 3 \iff \forall \varepsilon > 0,\ \exists \alpha > 0,\ \forall x \in \mathbb{R},\ |x - 1| < \alpha \implies |f(x) - 3| < \varepsilon.$

$$\begin{aligned} |f(x) - 3| &= |2x + 1 - 3| \\ &= |2x - 2| \\ &= 2|x - 1|. \end{aligned}$$

So, if $|x - 1| < \dfrac{\varepsilon}{2}$ *then we have* $|f(x) - 3| < \varepsilon$ *i.e*

$$\forall \varepsilon > 0, \exists \alpha = \frac{\varepsilon}{2} : |x - 1| < \alpha \implies |(2x + 1) - 3| < \varepsilon.$$

That shows that $\lim\limits_{x \to 1}(2x + 1) = 3$.

Remark 3.3.1

When x goes to infinity, by definition,

- $\lim_{x\to+\infty} f(x) = \ell$, if $\forall\varepsilon > 0,\ \exists A > 0 :\ x > A \implies |f(x) - \ell| < \varepsilon$.
- $\lim_{x\to-\infty} f(x) = \ell$, if $\forall\varepsilon > 0,\ \exists A > 0 :\ x < -A \implies |f(x) - \ell| < \varepsilon$.

Uniqueness of the limit.

Theorem 3.3.1

The limit l of a real function, if it exists, is unique.

Proof. We prove by contradiction. We assume that there be two limits ℓ_1 and ℓ_2 such that $\ell_1 \neq \ell_2$. Our goal is to get a contradiction. By definition limit, we have

$$\forall\varepsilon > 0,\ \exists\alpha_1 > 0,\ \forall x \in \mathcal{D}_f,\ |x - x_0| < \alpha_1 \implies |f(x) - \ell_1| < \varepsilon/2,$$
$$\forall\varepsilon > 0,\ \exists\alpha_2 > 0,\ \forall x \in \mathcal{D}_f,\ |x - x_0| < \alpha_2 \implies |f(x) - \ell_2| < \varepsilon/2.$$

Let $\alpha = \min(\alpha_1,\ \alpha_2) > 0$. Then, there exists $x \in \mathcal{D}_f$ such that $|x - x_0| < \alpha$. It follows that

$$|\ell_1 - \ell_2| \leq |\ell_1 - f(x)| + |f(x) - \ell_2| < \varepsilon.$$

Since this holds for arbitrary $\varepsilon > 0$ we choose $\varepsilon = |\ell_1 - \ell_2|$, we must have $|\ell_1 - \ell_2| < |\ell_1 - \ell_2|$. Which is a contradiction, hence ℓ_1 and ℓ_2 must be the same. □

3.3.3 Infinite limit.

Definition 3.3.3

Let $x \in I$, f be a function defined on I and x_0 is a real number. Then

$a)$ $\lim_{x\to x_0} f(x) = +\infty$, if $\forall A > 0,\ \exists\alpha > 0,\ |x - x_0| \leq \alpha \implies f(x) > A$.
$b)$ $\lim_{x\to x_0} f(x) = -\infty$, if $\forall A > 0,\ \exists\alpha > 0,\ |x - x_0| \leq \alpha \implies f(x) < -A$.
$c)$ $\lim_{x\to+\infty} f(x) = +\infty$, if $\forall A > 0,\ \exists B > 0,\ x > B \implies f(x) > A$.
$d)$ $\lim_{x\to-\infty} f(x) = +\infty$, if $\forall A > 0,\ \exists B > 0,\ x < -B \implies f(x) > A$.
$e)$ $\lim_{x\to+\infty} f(x) = -\infty$, if $\forall A > 0,\ \exists B > 0,\ x > B \implies f(x) < -A$.
$f)$ $\lim_{x\to-\infty} f(x) = -\infty$, if $\forall A > 0,\ \exists B > 0,\ x < -B \implies f(x) < -A$.

Example 3.3.2. *Compute the following limits* $\lim_{x\to 0} \frac{1}{x}$, $\lim_{x\to-\infty} x^2$ *and* $\lim_{x\to+\infty} x^3 + 2$.

Solution.

• *Looking that if* $x \to 0$, *we have have two cases,* $x \to 0^-$ *or* $x \to 0^+$. *Then*

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \quad \textit{and} \quad \lim_{x\to 0^+} \frac{1}{x} = +\infty.$$

• $\lim\limits_{x\to -\infty} x^2 = +\infty.$
• $\lim\limits_{x\to +\infty} x^3 + 2 = +\infty.$

3.3.4 Left hand and right hand limits

Definition 3.3.4

Let f be a function defined on $I \subset \mathbb{R}$, $x \in I$ and x_0 be a real number. Then,
a. We say that f has the left limit ℓ as x tends to x_0 iff

$$\forall \varepsilon > 0,\ \exists \alpha > 0,\ \forall x \in \mathcal{D}_f,\ (x_0 - \alpha < \ x \ < x_0) \implies |f(x) - \ell| < \varepsilon.$$

The left limit of f as x tends to x_0 is denoted $\lim\limits_{x \overset{<}{\to} x_0} f(x)$ or $\lim\limits_{x\to x_0^-} f(x)$.
b. We say that f has the right limit ℓ as x tends to x_0 iff

$$\forall \varepsilon > 0,\ \exists \alpha > 0,\ \forall x \in \mathcal{D}_f,\ (x_0 < \ x \ < x_0 + \alpha) \implies |f(x) - \ell| < \varepsilon.$$

The right limit of f as x tends to x_0 is denoted $\lim\limits_{x \overset{>}{\to} x_0} f(x)$ or $\lim\limits_{x\to x_0^+} f(x)$.

Proposition 3.3.1

Let f be a function defined on $I \subset \mathbb{R}$, $x \in I$ and x_0 be a real number. Then $\lim\limits_{x\to x_0} f(x)$ exists if and only if:

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = \ell.$$

So, $\lim\limits_{x\to x_0} f(x) = \ell$.

Example 3.3.3. *Are the following functions have a limit at* x_0*?*

$$f(x) = \begin{cases} x - 3 & x > 3; \\ \ln(4 - x) & x < 3; \end{cases} \quad x_0 = 3 \ \textit{ and } g(x) = \frac{|x|}{x} \ x_0 = 0.$$

Solution: *we have*

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \ln(4 - x) = 0.$$

and

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x - 3) = 0.$$

We have $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 0$. *Then* $\lim_{x\to 3} f(x) = 0$.
Concerning the function g *we have*

$$\begin{aligned}\lim_{x\to 0^-} g(x) &= \lim_{x\to 0^-} \frac{|x|}{x} \\ &= \lim_{x\to 0^-} \frac{-x}{x} = -1,\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 0^+} g(x) &= \lim_{x\to 0^+} \frac{|x|}{x} \\ &= \lim_{x\to 0^+} \frac{x}{x} = 1.\end{aligned}$$

We have $\lim_{x\to 0^-} g(x) \neq \lim_{x\to 0^+} g(x)$. *Thus* $\lim_{x\to 0} g(x)$ *does not exists.*

3.3.5 Operations of limits

Consider two functions f and g such that $\lim_{x\to x_0} f(x) = \ell$ and $\lim_{x\to x_0} f(x) = m$ where $x_0, \ell, m \in \mathbb{R}$. Then
• $\lim_{x\to x_0} (f(x) \pm g(x)) = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x)$ and $\lim_{x\to x_0} f(x) \times g(x) = \lim_{x\to x_0} f(x) \lim_{x\to x_0} g(x)$. Moreover,

$\lim_{x\to x_0} f(x)$	$\lim_{x\to x_0} g(x)$	$\lim_{x\to x_0} (f+g)(x)$	$\lim_{x\to x_0} (f\times g)(x)$
ℓ	m	$\ell + m$	$\ell \times m$
$+\infty$	m	$+\infty$	$\begin{cases} +\infty & \text{if } m > 0 \\ -\infty & \text{if } m < 0 \\ \text{Indeterminate} & \text{if } m = 0 \end{cases}$
$-\infty$	m	$-\infty$	$\begin{cases} -\infty & \text{if } m > 0 \\ +\infty & \text{if } m < 0 \\ \text{Indeterminate} & \text{if } m = 0 \end{cases}$
$+\infty$	$+\infty$	$+\infty$	$+\infty$
$-\infty$	$-\infty$	$-\infty$	$+\infty$
$-\infty$	$+\infty$	Indeterminate	$-\infty$

• $\forall \lambda \in \mathbb{R},\ \lim_{x\to x_0} (\lambda f(x)) = \lambda \lim_{x\to x_0} f(x)$.

• The limit of quotient of two functions is equal to quotient of their limits i.e $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$. Moreover

$\lim\limits_{x\to x_0} f(x)$	$\lim\limits_{x\to x_0} g(x)$	$\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)}$
ℓ	m	$\begin{cases} \dfrac{\ell}{m} \text{ if } m \neq 0, \\ \pm\infty \text{ if } \ell \neq 0 \text{ and } m = 0, \\ \text{indeterminate if } \ell = 0 \text{ and } m = 0. \end{cases}$
$\pm\infty$	m	$\pm\infty$
ℓ	$\pm\infty$	0
$\pm\infty$	$\pm\infty$	indeterminate

Limit of composition.
Suppose that $\lim\limits_{x\to x_0} g(x) = \ell$ and $\lim\limits_{x\to \ell} f(x) = m$. Then

$$\lim_{x\to x_0} f(g(x)) = m.$$

3.3.6 Indeterminate forms

An indeterminate form is an expression involving two functions whose limit cannot be determined solely from the limits of the individual functions. Note that

$$+\infty - \infty,\ 0 \times \mp\infty,\ \frac{\mp\infty}{\mp\infty},\ \frac{0}{0},\ (0^+)^0, (+\infty)^0,\ \text{ and } 1^{\pm\infty}.$$

are an indeterminate forms.

Proprieties: Define some usual limits

$$\lim_{x\to 0^+} \ln(x) = -\infty,\ \lim_{x\to +\infty} \ln(x) = +\infty,\ \lim_{x\to +\infty} \frac{\ln(x)}{x} = 0,$$

$$\lim_{x\to 0^+} x\ln(x) = 0,\ \lim_{x\to 0} \frac{\ln(x+1)}{x} = 1,\ \lim_{x\to 1} \frac{\ln(x)}{x-1} = 1,$$

$$\lim_{x\to -\infty} e^x = 0,\ \lim_{x\to +\infty} e^x = +\infty,\ \lim_{x\to +\infty} \frac{e^x}{x} = +\infty,$$

$$\lim_{x\to -\infty} xe^x = 0,\ \lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

Example 3.3.4. *Calculate* $\lim\limits_{x\to +\infty} (\sqrt{x^2+6x+1} - x)$ *and* $\lim\limits_{x\to +\infty} \dfrac{\sqrt{x^2-1}}{x+1}$.
Solution:

- $\lim_{x\to+\infty}(\sqrt{x^2+6x+1}-x)$ *is an indermined form of type* $\infty-\infty$*. So, we use the conjugate we get*

$$\begin{aligned}\lim_{x\to+\infty}\sqrt{x^2+6x+1}-x &= \lim_{x\to+\infty}\frac{(\sqrt{x^2+6x+1}-x)(\sqrt{x^2+6x+1}+x)}{\sqrt{x^2+6x+1}+x}\\ &= \lim_{x\to+\infty}\frac{6x+1}{|x|\sqrt{1+\frac{6}{x}+\frac{1}{x^2}}+x}.\\ &= \lim_{x\to+\infty}\frac{x(6+\frac{1}{x})}{x\left(\sqrt{1+\frac{6}{x}+\frac{1}{x^2}}+1\right)}=3\end{aligned}$$

- $\lim_{x\to+\infty}\dfrac{\sqrt{x^2-1}}{x+1}$ *is an indeterminate form of type* $\frac{\pm\infty}{+\infty}$*. Then,*

$$\begin{aligned}\lim_{x\to+\infty}\frac{\sqrt{x^2-1}}{x+1} &= \lim_{x\to+\infty}\frac{\sqrt{x^2(1-\frac{1}{x^2})}}{x(1+\frac{1}{x})}\\ &= \lim_{x\to+\infty}\frac{|x|\sqrt{(1-\frac{1}{x^2})}}{x(1+\frac{1}{x})}\\ &= \lim_{x\to+\infty}\frac{x\sqrt{(1-\frac{1}{x^2})}}{x(1+\frac{1}{x})}=1\end{aligned}$$

3.3.6.1 Indeterminate forms of type $(0^+)^0$ and $(+\infty)^0$

Proposition 3.3.2

Let x_0 be a real number or equals to $\pm\infty$. If $\lim_{x\to x_0}[f(x)]^{g(x)}$ is an indeterminate form of type $(0^+)^0$ or $(+\infty)^0$ we use the equality $f^g=e^{g\ln(f)}$ to eliminate these indeterminate forms.

Example 3.3.5. *Calculate* $\lim_{x\to 0^+}x^{\sin(x)}$ *and* $\lim_{x\to-\infty}(x^2)^{e^x}$.
Solution:

• $\lim_{x \to 0^+} x^{\sin(x)}$ *is an indeterminate form of type* 0^0. *Then*

$$\begin{aligned}\lim_{x \to 0^+} x^{\sin(x)} &= \lim_{x \to 0^+} e^{\sin(x)\ln(x)} \\ &= e^{\lim_{x \to 0^+}\left(\frac{\sin(x)}{x}\right)\left(\lim_{x \to 0^+} x\ln(x)\right)} \\ &= e^0 \\ &= 1.\end{aligned}$$

3.3.6.2 Indeterminate form of type $1^{\pm\infty}$:

Proposition 3.3.3

Let x_0 be a real number or equals to $\pm\infty$. If $\lim_{x \to x_0} [f(x)]^{g(x)}$ is an indeterminate form of type $1^{\pm\infty}$, we have

$$\lim_{x \to x_0} [f(x)]^{g(x)} = 1^{\pm\infty} \iff \lim_{x \to x_0} [f(x)]^{g(x)} = e^{\lambda},$$

where $\lambda = \lim_{x \to x_0} [(f(x) - 1)g(x)]$.

Example 3.3.6. *Calculate* $\lim_{x \to 0} \left(x + \sqrt{x+1}\right)^{\frac{1}{x}}$, $\lim_{x \to 0} \left(\frac{1+x}{1-x}\right)^{\frac{1}{\sin(x)}}$ *and* $\lim_{x \to 1^+} (2 - \sqrt{x})^{\frac{1}{x-1}}$.

Solution:

• $\lim_{x \to 0} \left(x + \sqrt{x+1}\right)^{\frac{1}{x}}$ *is an indeterminate form of type* 1^{∞}. *Then,* $\lim_{x \to 0} \left(x + \sqrt{x+1}\right)^{\frac{1}{x}} = e^{\lambda}$ *such that*

$$\begin{aligned}\lambda &= \lim_{x \to 0} \frac{x - 1 + \sqrt{x+1}}{x} \quad \textit{is an indeterminate form of type } \frac{0}{0}, \\ &= \lim_{x \to 0} \frac{(x - 1 + \sqrt{x+1})(x - 1 - \sqrt{x+1})}{x(x - 1 - \sqrt{x+1})} \\ &= \lim_{x \to 0} \frac{(x-1)^2 - x - 1}{x(x - 1 - \sqrt{x+1})} \\ &= \lim_{x \to 0} \frac{x^2 - 3x}{x(x - 1 - \sqrt{x+1})} \\ &= \lim_{x \to 0} \frac{x - 3}{x - 1 - \sqrt{x+1}} \\ &= \frac{3}{2}\end{aligned}$$

Thus,

$$\lim_{x \to 0} \left(x + \sqrt{x+1}\right)^{\frac{1}{x}} = e^{\frac{3}{2}}.$$

- $\lim\limits_{x\to 0}\left(\frac{1+x}{1-x}\right)^{\frac{1}{\sin(x)}}$ *is an indeterminate form of type* 1^{∞}*. Then,* $\lim\limits_{x\to 0}\left(\frac{1+x}{1-x}\right)^{\frac{1}{\sin(x)}} = e^{\lambda}$ *such that*

$$\begin{aligned}
\lambda &= \lim_{x\to 0}\left(\frac{1+x}{1-x}-1\right)\frac{1}{\sin(x)}\\
&= \lim_{x\to 0}\frac{2x}{1-x}\frac{1}{\sin(x)}\\
&= 2\lim_{x\to 0}\frac{1}{1-x}\frac{x}{\sin(x)}\\
&= 2\lim_{x\to 0}\frac{1}{\frac{\sin(x)}{x}}\\
&= 2
\end{aligned}$$

Thus,

$$\lim_{x\to 0}\left(\frac{1+x}{1-x}\right)^{\frac{1}{\sin(x)}} = e^{2}.$$

- $\lim\limits_{x\to 1^+}(2-\sqrt{x})^{\frac{1}{x-1}}$ *is an indeterminate form of type* $1^{+\infty}$*. Then,* $\lim\limits_{x\to 1^+}(2-\sqrt{x})^{\frac{1}{x-1}} = e^{\lambda}$ *such that*

$$\begin{aligned}
\lambda &= \lim_{x\to 1^+}\left((2-\sqrt{x}-1)\frac{1}{x-1}\right) \text{ is an indeterminate form of type } 0\cdot\infty\\
&= \lim_{x\to 1^+}\frac{1-\sqrt{x}}{x-1} \text{ (FI de type } \frac{0}{0}\text{)}\\
&= \lim_{x\to 1^+}\frac{(1-\sqrt{x})(1+\sqrt{x})}{(x-1)(1+\sqrt{x})}\\
&= \lim_{x\to 1^+}\frac{(1-x)}{(x-1)(1+\sqrt{x})}\\
&= \frac{-1}{2}
\end{aligned}$$

Thus,

$$\lim_{x\to 1^+}\left(2-\sqrt{x}\right)^{\frac{1}{x-1}} = e^{\frac{-1}{2}} = \frac{1}{\sqrt{e}}.$$

Calculate a limit by using the substitution.

Example 3.3.7. $\lim\limits_{x\to 1}(x-1)\frac{1}{\sin(\pi x)}$ *is an indeterminate form of type* $0\times+\infty$ *.*

Put $y = x - 1$, *thus if* $x \to 1$ *we get* $y \to 0$. *So*

$$\begin{aligned}
\lim_{x\to 1}(x-1)\frac{1}{\sin(\pi x)} &= \lim_{y\to 0} y\frac{1}{\sin(\pi(y+1))} \\
&= \lim_{y\to 0} \frac{y}{\sin(\pi y + \pi)} \\
&= \lim_{y\to 0} -\frac{1}{\pi}\frac{\pi y}{\sin(\pi y)} \\
&= \lim_{y\to 0} -\frac{1}{\pi}\frac{\pi y}{\sin(\pi y)} \\
&= \lim_{z\to 0} -\frac{1}{\pi}\frac{z}{\sin(z)} = \frac{-1}{\pi}. \ (\textit{Here } z = \pi y).
\end{aligned}$$

Example 3.3.8. $\lim_{x\to+\infty}\left(\sin\left(\frac{\pi x+2}{2x}\right)\right)^{x^2}$ *is an indeterminate form of type* $1^{+\infty}$.
Put $y = \frac{1}{x}$, *if* $x \to +\infty$ *we get* $y \to 0$. *So*

$$\lim_{x\to+\infty}\left(\sin\left(\frac{1}{x}+\frac{\pi}{2}\right)\right)^{x^2} = \lim_{y\to 0}\left(\sin\left(y+\frac{\pi}{2}\right)\right)^{1/y^2} = e^{\lambda} \ (\textit{ since we have } 1^{\infty}).$$

where

$$\begin{aligned}
\lambda &= \lim_{y\to 0}\left(\sin\left(y+\frac{\pi}{2}\right)-1\right)\frac{1}{y^2} \ (\textit{ indeterminate form of type } 0) \\
&= \lim_{y\to 0}\frac{\cos(y)-1}{y^2} \ (\textit{ since } \sin\left(y+\frac{\pi}{2}\right) = \cos(y)) \\
&= \lim_{y\to 0}\frac{(\cos(y)-1)(\cos(y)+1)}{y^2(\cos(y)+1)} \\
&= \lim_{y\to 0}\frac{\cos^2(y)-1}{y^2\underbrace{(\cos(y)+1)}_{=2}} \\
&= \frac{1}{2}\lim_{y\to 0}\frac{-\sin^2(y)}{y^2} = \frac{-1}{2}, \ (\textit{since} \lim_{y\to 0}\frac{\sin(y)}{y} = 1).
\end{aligned}$$

Thus, $\lim_{y\to 0}\left(\sin\left(y+\frac{\pi}{2}\right)\right)^{1/y^2} = e^{-1/2}$. *Therefore,* $\lim_{x\to+\infty}\left(\sin\left(\frac{\pi x+2}{2x}\right)\right)^{x^2} = \frac{1}{\sqrt{e}}$.

Corollary 3.3.1

If $\lim_{x\to x_0} f(x) = 0$ and g is a bounded function. Then $\lim_{x\to x_0}(f\cdot g)(x) = 0$.

Example 3.3.9. *Compute* $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Since we have: $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$. *Then the function* $x \mapsto \sin\left(\frac{1}{x}\right)$ *is a bounded function. Also, we have*

$$\lim_{x \to 0} x^2 = 0.$$

Thus

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

The next theorem is another useful variant on these themes. Here an unknown function is sandwiched between two functions whose limit behavior is known, allowing us to conclude that a limit exists. This theorem is often taught as **the squeeze theorem**.

Theorem 3.3.2

Let E be a subset of $\mathbb{R}$ and f, g, h three functions such that $f, g, h : E \to \mathbb{R}$ and consider a real number x_0 a point of accumulation of the common domain E. Suppose that the limits

$$\lim_{x \to x_0} g(x) = \ell \text{ and } \lim_{x \to x_0} h(x) = \ell.$$

exist and that

$$g(x) \leq f(x) \leq h(x)$$

for all $x \in E$ except perhaps at x_0. Then

$$\lim_{x \to x_0} f(x) = \ell.$$

Example 3.3.10. *Let us prove that the limit*

$$\lim_{x \to 0} x^3 \cos\left(\frac{1}{x^2}\right) = 0.$$

Certainly we notice that the expression $\cos\left(\frac{1}{x^2}\right)$ *satisfies the inequalities*

$$-|x^3| \leq x^3 \cos\left(\frac{1}{x^2}\right) \leq |x^3|$$

are valid for all x *(except* $x = 0$ *where the function is undefined). Since*

$$\lim_{x \to 0}(-|x^3|) = \lim_{x \to 0} |x^3| = 0,$$

we obtain that $\lim_{x \to 0} x^3 \cos\left(\frac{1}{x^2}\right) = 0$.

3.4 Continuous functions

3.4.1 Continuity of a function at a point

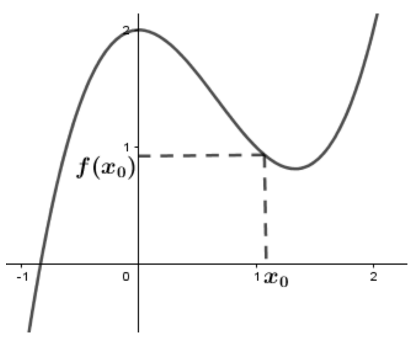
Definition 3.4.1

Let x_0 be a real number and f be a function defined in a neighborhood of x_0. The function f is continuous at x_0 provided

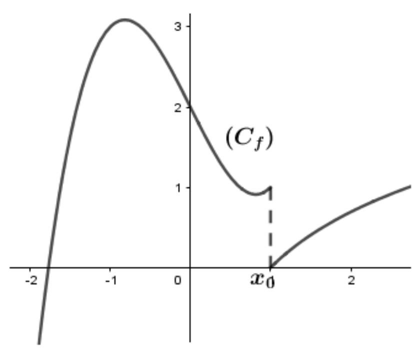
$$\lim_{x \to x_0} f(x) = f(x_0).$$

Remark 3.4.1

- f is continuous on I if it is continuous at every $x_0 \in I$.
- If f is not continuous at x_0 i.e $\lim_{x \to x_0} f(x) \neq f(x_0)$ we say that the function f is discontinuous at x_0.



f is continuous at x0



f is discontinuous at x0

Example 3.4.1.

(a) *Every polynomial is continuous on* $\mathbb{R}$.

(b) *Every rational function is continuous at each point in its domain (i.e., at each* $x \in \mathbb{R}$ *at which the denominator does not vanish).*

(c) *The function* $x \mapsto \ln(x)$ *is continuous on* $]0, +\infty[$.

(d) *The function* $x \mapsto \ln(x)$ *is continuous on* $]0, +\infty[$.

(e) *The function* $x \mapsto \sqrt{x}$ *is continuous on* $[0, +\infty[$.

(f) *The functions* $x \mapsto exp(x),\ x \mapsto \cos(x)$ *and* $x \mapsto \sin(x)$ *are continuous on* $\mathbb{R}$.

Definition 3.4.2

Let x_0 be a real number and f be a function defined in a neighborhood of x_0. The function f is continuous at x_0 provided

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0).$$

Example 3.4.2. *The function*

$$f(x) = \begin{cases} e^{x-2} & \text{if } x < 2, \\ 1 & \text{if } x = 2, \\ \dfrac{x+2}{2x} & \text{if } x > 2, \end{cases}$$

is continuous at $x_0 = 2$. *Indeed, we have*

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} e^{x-2} = e^0 = 1.$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x+2}{2x} = 1$$

and

$$f(2) = 1.$$

Then, f *is continuous at* 2.

3.4.2 Continuous extension at a point

Proposition 3.4.1

Let x_0 be a real number and $f : E/\{x_0\} \to \mathbb{R}$ be a function. If the limit

$$\lim_{x \to x_0} f(x) = \ell$$

exists then f admits an extension by continuity at x_0 and the function $\widetilde{f}$ defined by

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in E/\{x_0\}, \\ \ell & \text{if } x = x_0. \end{cases}$$

is continuous at x_0.

Example 3.4.3. *The function* $x \mapsto \dfrac{sin(x)}{x}$ *admits an extension by continuity at* $x_0 = 0$. *Indeed, we have*

$$\mathcal{D}f = \mathbb{R}^*$$

and the limit

$$\lim_{x\to 0}\frac{sin(x)}{x}=1$$

exists, thus the function $x\mapsto\dfrac{sin(x)}{x}$ *admits an extension by continuity at* $x_0=0$ *and the function*

$$\widetilde{f}(x)=\begin{cases}\dfrac{sin(x)}{x} & \textit{if}\ \ x\in\mathbb{R}^*,\\ 1 & \textit{if}\ \ x=0.\end{cases}$$

is continuous at $x_0=0$.

Example 3.4.4. *Are the following functions admit an extension by continuity at* x_0*?*

$$f_1(x)=\left(\sqrt{x}-1\right)^{\frac{2}{x-4}},\ x_0=4,\ f_2(x)=\ln(|x|),\ x_0=0,$$
$$f_3(x)=\frac{|x-2|}{x-2},\ x_0=2,\ f_4(x)=\cos\left(\frac{1}{x}\right),\ x_0=0.$$

(a) The extension by continuity of f_1 *at* $x_0=4$:

$$\lim_{x\to 4}f_1(x)=\lim_{x\to 4}\left(\sqrt{x}-1\right)^{\frac{2}{x-4}}\ \ \textit{is an indeterminate form of the type}\ 1^{\infty}.$$

Thus $\lim_{x\to 4}(\sqrt{x}-1)^{\frac{2}{x-4}}=e^{\lambda}$ *with*

$$\begin{aligned}\lambda&=\lim_{x\to 4}\left(\sqrt{x}-2\right)\frac{2}{x-4}\\&=2\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}\\&=\frac{1}{2}.\end{aligned}$$

So $\lim_{x\to 4}(\sqrt{x}-1)^{\frac{2}{x-4}}=e^{1/2}$.
Then f_1 *admits an extension by continuity at* $x_0=4$ *and the function*

$$\widetilde{f_1}(x)=\begin{cases}(\sqrt{x}-1)^{\frac{2}{x-4}} & \textit{if}\ \ x\neq 2,\\ e^{1/2} & \textit{if}\ x=2,\end{cases}$$

is continuous at $x_0=2$.
(b) The extension by continuity of f_2 *at* $x_0=0$: *we have*

$$\lim_{x\to 0}f_2(x)=\lim_{x\to 0}\ln(|x|)=-\infty.$$

So, f_2 *cannot be extended by continuity at* $x_0=0$.
(c) The extension by continuity of f_3 *at* $x_0=2$:

$$\lim_{x\to 2}f_3(x)=\lim_{x\to 2}\frac{|x-2|}{x-2}=\begin{cases}1 & \textit{if}\ \ x\to 2^+,\\ -1 & \textit{if}\ \ x\to 2^-.\end{cases}$$

Thus, f_3 cannot be extended by continuity at $x_0 = 2$.
(d) The extension by continuity of f_4 at $x_0 = 0$.

$$\lim_{x\to 0} f_4(x) = \lim_{x\to 0} \cos\left(\frac{1}{x}\right) \ (\textit{does not exist}).$$

Thus, f_4 cannot be extended by continuity at $x_0 = 0$

3.4.3 Intermediate value theorem

Theorem 3.4.1

Let $a, b, k \in \mathbb{R}$ and f be a continuous function on $[a, b]$. If

$$(f(a) < k < f(b)) \ \vee \ (f(b) < k < f(a)),$$

then,

$$\exists c \in]a, b[: \ f(c) = k,$$

or, the equation $f(x) = k$ admits at least one solution $c \in]a, b[$.

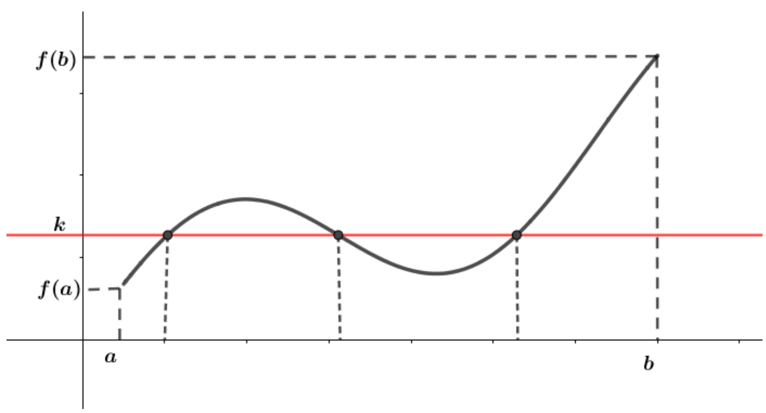


Figure 3.4.1: A continuous function where I.V.T holds for multiple values in $]a, b[$.

Theorem 3.4.2

Let $a, b, k \in \mathbb{R}$ and f be a continuous and monotone function on $[a, b]$. If

$$(f(a) < k < f(b)) \ \vee \ (f(b) < k < f(a)),$$

then,

$$\exists! c \in]a, b[: \ f(c) = k,$$

or, the equation $f(x) = k$ admits one solution $c \in]a, b[$.

Example 3.4.5. *Let $f(x) = x^2 - 3$ and $g(x) = \ln(x) + 2$ be two functions.*

(a) Prove that the equation $f(x) = -2$ admits at least one solution in $]0, 2[$. We have

- *f is continuous on $\mathbb{R}$, so, f is continuous on $[0, 2]$.*
- *$f(0) = -3$ and $f(2) = 4$ so $f(0) < -2 < f(2)$.*

Then,

$$\exists c \in]0, 2[: \ f(c) = -3.$$

(b) Prove that the equation $g(x) = 2$ admits one solution between e^{-1} and e. We have

- *g is continuous on $]0, +\infty[$ so, g is continuous on $[e^{-1}, e]$.*
- *$g(e^{-1}) = 1$ and $f(e) = 3$ so $g(e^{-1}) < 2 < g(e)$.*
- *$\forall x \in]0, +\infty[$, we have $g'(x) = \frac{1}{x} > 0$ thus g is increasing (monotone) on $[e^{-1}, e]$.*

Then,

$$\exists! c \in]e^{-1}, e[: \ g(c) = 2.$$

Corollary 3.4.1

Let f be a continuous function on $[a, b]$.

(a) If $f(a) \times f(b) < 0$ then, the equation $f(x) = 0$ admits at least one solution in $]a, b[$.

(b) If $f(a) \times f(b) < 0$ and f is monotone on $[a, b]$ then, the equation $f(x) = 0$ admits one solution in $]a, b[$.

Example 3.4.6. *Let $f(x) = cos(2x)$ and $g(x) = x - e^{-x}$ be two functions.*

(a) Prove that the equation $f(x) = 0$ admits at least one solution in $]-\frac{\pi}{2}, \pi[$. We have

- *f is continuous on $\mathbb{R}$, so, f is continuous on $[-\frac{\pi}{2}, \pi]$.*
- *$f\left(-\frac{\pi}{2}\right) = -1$ and $f(\pi) = 1$ thus, $f\left(-\frac{\pi}{2}\right) \cdot f(\pi) < 0$.*

Then,

$$\exists c \in]-\frac{\pi}{2}, \pi[: \ f(c) = 0.$$

(b) Prove that the equation $g(x) = 0$ admits one solution between 0 and 1. We have

- *g is continuous on $\mathbb{R}$ so, g is continuous on $[0, 1]$.*
- *$g(0) = -1$ and $g(1) = 0.633$ thus, $g(0) \cdot g(1) < 0$.*

- *$\forall x \in \mathbb{R}$, we have $g'(x) = 1 + e^{-x} > 0$ thus g is increasing (monotone) on $[0, 1]$.*

Then,

$$\exists! c \in]0,1[: \ g(c) = 0.$$

Corollary 3.4.2

Let $a, b \in \mathbb{R}$ and f a function defined and continuous on $[a, b]$.
The direct image of the interval $[a, b]$ i.e $f([a, b])$ is an interval. Moreover,

(i) if f is increasing on $[a, b]$, then $f([a, b]) = [f(a), f(b)]$.

(ii) if f is decreasing on $[a, b]$, then $f([a, b]) = [f(b), f(a)]$.

Example 3.4.7. .
• *The function $f(x) = ln(x)$ is continuous and increasing on $]0, +\infty[$, thus f is continuous and increasing on $]0, e^2]$. Then*

$$f(]0, e^2]) =]\lim_{x\to 0} f(x), f(e^2)] =]-\infty, 2].$$

• *The function $f(x) = \dfrac{1}{x}$ is continuous and decreasing on $]-\infty, 0[$, thus f is continuous and decreasing on $]-\infty, -1]$. Then*

$$f(]-\infty, -1]) = [f(-1), \lim_{x\to -\infty} f(x)[= [-1, 0[.$$

3.4.4 Inverse functions

Suppose that a function $f : I \to J$ has an inverse. This simply means that there is a function g (called the inverse of f) that reverses the mapping: If $f(a) = b$ then $g(b) = a$ where $a \in I$ and $b \in J$. We can assume that I and J are intervals. Thus f maps the interval I onto the interval J and the inverse function g then maps J back to I. Not all functions have an inverse, the following theorem give us when a function admits an inverse function.

Theorem 3.4.3

Let $I \subseteq \mathbb{R}$ be an interval and f be a continuous and monotone function on I. Then,

(i) f is bijective from I to $f(I)$.

(ii) f admits an inverse function f^{-1} defined from $f(I)$ to I. Moreover, f^{-1} has the same variation of f.

Remark 3.4.2

The connection between a function $f : I \to J$ and its inverse $f^{-1} : J \to I$ is given by

$$f(f^{-1}(x)) = x, \ \forall x \in J,$$

or

$$f^{-1}(f(x)) = x, \ \forall x \in I.$$

Example 3.4.8. *Let* $f(x) = e^{\sqrt{x}}$ *be a function defined on* $[0, +\infty[$.
1) f *is bijective from* $[0; +\infty[$ *to* $f([0, +\infty[)$. *Indeed*

- f *is continuous on* $[0, +\infty[$ *since* f *is a composition of two continuous functions.*
- f *is increasing on* $[0, +\infty[$ *since* f *is a composition of two increasing functions.*
- $f([0, +\infty[) = [f(0), \lim\limits_{x \to +\infty} f(x)[= [1, +\infty[$. *Thus,* f *is bijective from* $[0, +\infty[$ *to* $[1, +\infty[$.

2) Determine f^{-1} *the inverse function of* f.

- $f^{-1} : [1; +\infty[\to [0, +\infty[$. *Moreover, if* $y \in [1, +\infty[$ *we have*

$$\begin{aligned} f(x) = y &\implies e^{\sqrt{x}} = y \\ &\implies \sqrt{x} = \ln(y) \\ &\implies x = \ln^2(y) \end{aligned}$$

- *Then, the inverse function* f^{-1} *is defined as follows* $f^{-1} : \begin{array}{cl} [1; +\infty[& \to [0, +\infty[\\ x & \mapsto \ln^2(x). \end{array}$

3.5 DIFFERENTIATION

3.5.1 Differentiability of a function at a point

Definition 3.5.1

Let f be a real valued function defined on an interval I and let $x_0 \in I$. The derivative of f at x_0, denoted by $f'(x_0)$ and defined as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

provided either that this limit exists. Moreover,If $f'(x_0)$ is finite we say that f is differentiable at x_0. If f is differentiable at every point of a set $E \subset I$, we say that f is differentiable on E.

Definition 3.5.2

Let f be a real valued function defined on $\mathcal{D}_f \subset \mathbb{R}$. The domain of differentiability $\mathcal{D}_f^d$ of f is the set of $\mathbb{R}$ when f is differentiable at every point of it.

Example 3.5.1. .

1. *Prove that the function $f(x) = 2e^x$ is differentiable at $x_0 = 0$.*
   We have $\mathcal{D}_f = \mathbb{R}$ and $0 \in \mathbb{R}$, also

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)-f(x_0)}{x-0} &= \lim_{x\to 0}\frac{2e^x-2e^0}{x}\\ &= 2\lim_{x\to 0}\frac{e^x-1}{x}\\ &= 2 \textit{ (does exist).}\end{aligned}$$

   Then f is differentiable at $x_0 = 0$ and $f'(0) = 2$.

2. *Prove that the function $g(x) = |x-1|$ does not Differentiable at $x_0 = 1$.*
   We have $\mathcal{D}_g = \mathbb{R}$ and $1 \in \mathbb{R}$ also,

$$\begin{aligned}\lim_{x\to 1}\frac{g(x)-g(1)}{x-1} &= \lim_{x\to 1}\frac{|x-1|-|1-1|}{x-1}\\ &= \lim_{x\to 1}\frac{|x-1|}{x-1}\\ &= \begin{cases}1 & \textit{if } x\to 1^+\\ -1 & \textit{if } x\to 1^-.\end{cases}\end{aligned}$$

   Then g does not differentiable at $x_0 = 1$.

Example 3.5.2. *Prove that the function $h(x) = \sqrt{x}$ does not differentiable at $x_0 = 0$.*
We have $\mathcal{D}_h = [0, +\infty[$ and $0 \in [0, +\infty[$, also

$$\begin{aligned}\lim_{x\to 0}\frac{h(x)-h(0)}{x-0} &= \lim_{x\to 0}\frac{\sqrt{x}}{x}\\ &= \lim_{x\to 0}\frac{1}{\sqrt{x}}\\ &= +\infty.\end{aligned}$$

Then h does not differentiable at $x_0 = 0$.

3.5.2 Right-hand an left-hand derivative

Definition 3.5.3

We say that a real function f is left differentiable at x_0 if

$$\lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}=\ell_1$$

exists. The limit ℓ_1 is called the left differentiable number of f at x_0 and denoted by $f'_L(x_0)$.

Definition 3.5.4

We say that a real function f is right differentiable at x_0 if

$$\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}=\ell_2$$

exists. The limit ℓ_2 is called the right differentiable number of f at x_0 and denoted by $f'_R(x_0)$.

Corollary 3.5.1

Let f be a function defined on $\mathcal{D}_f$ and $x_0\in\mathcal{D}_f$.
f is called Differentiable at x_0 if

$$\boxed{f'_L(x_0)=f'_R(x_0).}$$

Theorem 3.5.1

Let f be defined in a neighborhood I of x_0. If f is differentiable at x_0, then f is continuous at x_0.

Definition 3.5.5

Let f be a function defined and Differentiable at x_0. Then, the equation of tangent line of f at x_0 is

$$\boxed{y=f'(x_0)(x-x_0)+f(x_0).}$$

Remark 3.5.1

A function f is not differentiable at x_0 if

$$\boxed{\left(\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\pm\infty\right)\vee(f\text{ is not continuous at }x_0)\vee(f'_L(x_0)\neq f'_R(x_0)).}$$

Example 3.5.3. *Study the differentiability of the following functions:*

$$f(x) = \begin{cases} e^x & si\ x \leq 0, \\ x+1 & si\ x > 0, \end{cases}, x_0 = 0, \quad g(x) = \begin{cases} x^2+1 & si\ x \leq 0, \\ e^x & si\ x > 0, \end{cases}, \ x_0 = 0,$$

$$h(x) = \begin{cases} \ln(x) & si\ x > 1, \\ x+1 & si\ x \leq 1. \end{cases}, \ x_0 = 1.$$

1. $f(x) = \begin{cases} e^x & si\ x \leq 0, \\ x+1 & si\ x > 0, \end{cases}, x_0 = 0$. *We have*

   - $f(0) = e^0 = 1.$
   - $\lim\limits_{x \to 0^+} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0^+} \dfrac{x+1-1}{x} = 1$. *Thus,* $f'_R(0) = 1.$
   - $\lim\limits_{x \to 0^-} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0^-} \dfrac{e^x - 1}{x} = 1$. *Thus,* $f'_L(0) = 1.$

   Therefore, $f'_L(0) = f'_R(0)$. *Then* f *is differentiable at* $x_0 = 0$ *and* $f'(0) = 1.$

2. $g(x) = \begin{cases} x^2+1 & si\ x \leq 0, \\ e^x & si\ x > 0, \end{cases}$ *at* $x_0 = 0$. *We have*

   - $g(0) = 0^2 + 1 = 1.$
   - $\lim\limits_{x \to 0^+} \dfrac{g(x) - g(0)}{x - 0} = \lim\limits_{x \to 0^+} \dfrac{e^x - 1}{x} = 1$. *Thus,* $g'_R(0) = 1.$
   - $\lim\limits_{x \to 0^-} \dfrac{g(x) - g(0)}{x - 0} = \lim\limits_{x \to 0^-} \dfrac{x^2+1-1}{x} = 0$. *Thus,* $g'_L(0) = 1.$

   Therefore, $g'_L(0) \neq g'_R(0)$. *Then* g *is not differentiable at* $x_0 = 0.$

3. $h(x) = \begin{cases} \ln(x) & si\ x > 1, \\ x+1 & si\ x \leq 1. \end{cases}$ *at* $x_0 = 1$. *We have*

   $$h(1) = 1 + 1 = 2, \ \lim_{x \to 1^+} h(x) = \lim_{x \to 1^+} (x+1) = 2 \textit{ and } \lim_{x \to 1^+} h(x) = \lim_{x \to 1^-} e^x = e.$$

   Therefore, $\lim\limits_{x \to 1^+} h(x) \neq \lim\limits_{x \to 1^-} h(x)$. *Then* h *is not continuous at* $x_0 = 1$ *which implies that* h *is not differentiable at* $x_0 = 1.$

3.5.3 Computations of derivatives

Definition 3.5.6

A function f is said to be differentiable in an open interval $]a, b[$ if it is differentiable at every point x_0 in this interval. The differentiability of f is denoted by f'.

Differentiability of usual functions: The following table is a summary of the main formulas to know, where x is a variable.

Function	Differentiability
x^n	$nx^{n-1} \quad (n \in \mathbb{R})$
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\sqrt{x}$	$\frac{1}{2}\frac{1}{\sqrt{x}}$
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$1 + \tan^2 x \; or \; \frac{1}{\cos^2 x}$

Proposition 3.5.1

If f, g are differentiable functions and λ is any constant, then

(a) $(f + g)'(x) = f'(x) + g'(x)$,

(b) $(\lambda f)'(x) = \lambda f'(x)$,

(c) $(f \times g)'(x) = f'(x)g(x) + f(x)g'(x)$,

(d) $\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{f(x)^2}$ (si $f(x) \neq 0$),

(e) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$ (if $g(x) \neq 0$).

Theorem 3.5.2

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function $x \mapsto (f \circ g)(x)$ is given by

$$(f \circ g)'(x) = g'(x) \cdot f'(g(x)).$$

Result: By the preceding theorem we can deduce the derivative of useful functions as follows

Function	Differentiability
f^n	$nf'f^{n-1} \quad (n \in \mathbb{R})$
$\frac{1}{f}$	$-\frac{f'}{f^2}$
$\sqrt{f}$	$\frac{1}{2}\frac{f'}{\sqrt{f}}$
e^f	$f'e^f$
$\ln f$	$\frac{f'}{f}$
$\cos f$	$-f'\sin f$
$\sin f$	$f'\cos f$
$\tan f$	$f'(1+\tan^2 f) = \frac{f'}{\cos^2 f}$

Example 3.5.4. *Determine the domain, the domain of differentiability of the following functions and then calculate their derivatives*

$$f_1(x) = \ln\left(\frac{x}{2+x}\right),\ f_2(x) = \sqrt{\frac{e^x-1}{x+1}} \text{ and } f_3(x) = \frac{1}{\cos(2x-3)}.$$

Solution:

1. *The function f_1 such that $f_1(x) = \ln\left(\frac{x}{2+x}\right)$.*

   - *Determine the domain of f_1. We have*

   $$\mathcal{D}_{f_1} = \left\{x \in \mathbb{R}, \frac{x}{x+2} > 0 \text{ and } x+2 \neq 0\right\}$$
   $$=]-\infty,-2[\cup]0,+\infty[. \text{ (by using the table of signe).}$$

   - *Determine the domain of differentiability f_1. We have*

   $$\mathcal{D}^d_{f_1} =]-\infty,-2[\cup]0,+\infty[.$$

   - *Calculate the derivative of f_1. $\forall x \in]-\infty,-2[\cup]0,+\infty[$ we have*

   $$f_1'(x) = \frac{\left(\frac{x}{2+x}\right)'}{\frac{x}{2+x}} = \frac{2}{x(2+x).}$$

2. *The function f_2 such that $f_2(x) = \sqrt{\frac{e^x-1}{x+1}}$.*

- *Determine the domain of f_2.*

$$\mathcal{D}_{f_2} = \left\{x \in \mathbb{R}, \frac{e^x - 1}{x+1} \geq 0 \ \textit{ and } x + 1 \neq 0\right\}$$
$$=]-\infty, -1[\cup[0, +\infty[. \ \textit{(by using table of signe).}$$

- *Determine the domain differentiability of f_2.*

$$\mathcal{D}^d_{f_2} = \left\{x \in \mathbb{R}, \frac{e^x - 1}{x+1} > 0 \ \textit{ and } x + 1 \neq 0\right\}$$
$$=]-\infty, -1[\cup]0, +\infty[.$$

- *Calculate the derivative of f_2.* $\forall x \in]-\infty, -1[\cup]0, +\infty[$ *we have*

$$f_2'(x) = \frac{\left(\dfrac{e^x - 1}{1+x}\right)'}{2\sqrt{\dfrac{e^x - 1}{1+x}}}$$
$$= \frac{xe^x + 1}{2(1+x)^2\sqrt{\dfrac{e^x - 1}{1+x}}}.$$

3. *The function f_3 such that* $f_3(x) = \dfrac{1}{\cos(2x-3)}$.

- *Determine the domain of f_3.*

$$\mathcal{D}^d_{f_3} = \{x \in \mathbb{R}, \ \cos(2x-3) \neq 0\}$$
$$= \left\{x \in \mathbb{R}, \ 2x - 3 \neq \frac{(2k+1)\pi}{2}, \ k \in \mathbb{Z}\right\}$$
$$= \mathbb{R} - \left\{\frac{(2k+1)\pi + 6}{4}, \ k \in \ \mathbb{Z}\right\}.$$

- *Determine the domain of differentiability of f_3.*

$$\mathcal{D}^d_{f_3} = \mathbb{R} - \left\{\frac{(2k+1)\pi + 6}{4}, \ k \in \ \mathbb{Z}\right\}.$$

- *Calculate the derivative of f_3.* $\forall x \in \mathbb{R} - \left\{\dfrac{(2k+1)\pi + 6}{4}, \ k \in \ \mathbb{Z}\right\}$ *we have*

$$f_3'(x) = \frac{-(\cos(2x-3))'}{\cos^2(2x-3)}$$
$$= \frac{2\sin(2x-3)}{\cos^2(2x-3)}.$$

3.5.4 Rolle's theorem

Theorem 3.5.3

Let f be a function defined on $\mathcal{D}_f$ and let $[a,b] \subseteq \mathcal{D}_f$ be an interval. If f is continuous on $[a,b]$, differentiable on $]a,b[$ and $f(a) = f(b)$, then

$$\boxed{\exists c \in]a,b[,\ f'(c) = 0}.$$

Otherwise, the equation $f'(x) = 0$ has at least one solution in $]a,b[$.

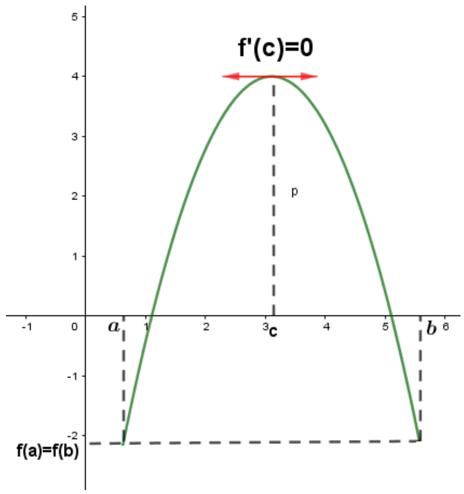


Figure 3.5.1: Geometric interpolation of Rolle's theorem.

Example 3.5.5. *Let $f(x) = 3x^4 - 11x^3 + 12x^2 - 4x + 2$ be a function. Using Rolle's theorem and prove that the equation $f'(x) = 0$ has at least one solution in $]0,1[$.*
Solution:
• *f is continuous on $\mathbb{R}$, thus f is continuous on $[0,1]$.*
• *f is differentiable on $\mathbb{R}$, thus f is differentiable on $]0,1[$.*
• *$f(0) = 2$ and $f(1) = 2$.*
Then, by Rolle's theorem, we get that the equation $f'(x) = 0$ has at least one solution in $]0,1[$.

3.5.5 Mean value theorem

Theorem 3.5.4

Let f be a function defined on $\mathcal{D}_f$ and $[a,b] \subseteq \mathcal{D}_f$ an interval. If f is continuous on $[a,b]$, differentiable on $]a,b[$, then,

$$\boxed{\exists c \in]a,b[,\ (f(b) - f(a)) = (b-a)f'(c)}.$$

Example 3.5.6. *Using mean value theorem and prove that:*

$$\forall x > 0,\ 2x < e^{2x} - 1 < 2xe^{2x}.$$

Solution:

(a). Let $]a,b[=]0,x[$ *and* $f(t)=e^{2t}$.

(b). f *is continuous and differentiable on* $\mathbb{R}$ *thus* f *is continuous on* $[0,x]$ *and differentiable on* $]0,x[$ *and*

$$\forall t\in]0,x[,\ f'(t)=2te^{2t}.$$

(c). According to mean value theorem $\exists c\in]0,x[:\ f(x)-f(0)=(x-0)f'(c)$. *i.e*

$$\exists c\in]0,x[:\ e^{2x}-1=2xe^{2c}....(*)$$

(d). We have

$$\begin{aligned} c\in]0,x[&\implies 0<c<x\\ &\implies 1<e^{2c}<e^{2x}\\ &\implies 2x<2xe^{2c}<2xe^{2x}\ \ (\textit{since } x>0)\end{aligned}$$

(e). From $(*)$ *we conclude that* $2x<e^{2x}-1<2xe^{2x}$.

Then, for all positive number x *we have*

$$2x<e^{2x}-1<2xe^{2x}.$$

3.5.6 Differentiability of inverse function

Proposition 3.5.2

Let I be an open interval and $f:I\to J$ a differentiable and bijective function and $f^{-1}:J\to I$ its inverse function. Then, if $\forall x\in I,\ f'(x)\neq 0$ we have f^{-1} is differentiable and $\forall x\in J$:

$$\boxed{\left(f^{-1}\right)'(x)=\frac{1}{f'\left(f^{-1}(x)\right)}}.$$

Proof. Suppose that $\forall x\in I,\ f'(x)\neq 0$, we know that the connection between a function f and its inverse f^{-1} is given by

$$f(f^{-1}(x))=x,\ \forall x\in J.$$

So, using the derivative of a composite functions we get

$$(f(f^{-1}(x)))'=(x)',$$

i.e

$$(f^{-1}(x))'f'(f^{-1}(x))=1,$$

then, $\forall x\in J$,we have the result

$$\left(f^{-1}\right)'(x)=\frac{1}{f'\left(f^{-1}(x)\right)}.$$

□

Example 3.5.7. *Let $f(x) = x^3 + 1$. f is a bijective function from $]0, +\infty[$ to $]1, +\infty[$ (is continuous and monotone function).*
The inverse of the function $f(x) = x^3 + 1$ with $x \in]0, \infty[$ is $f^{-1}(x) = \sqrt[3]{x-1}$ with $x \in]1, \infty[$.
Since $\forall x \in]0, +\infty[,\ f'(x) = 2x^2 \neq 0$ then

$$\boxed{\left(f^{-1}\right)'(x) = \frac{1}{f'\big(f^{-1}(x)\big)} = \frac{1}{2[\sqrt[3]{x-1}]^2} = \frac{1}{2(x-1)^{2/3}},\ \forall x > 1.}$$

3.5.7 Indeterminate forms and L'Hôpital's rule

Theorem 3.5.5

Let f and g are differentiable functions in a deleted neighborhood $\mathcal{V}$ of $x = x_0$. Suppose that

- $\lim\limits_{x \to x_0} f(x) = 0,$
- $\lim\limits_{x \to x_0} g(x) = 0,$
- for every $x \in \mathcal{V},\ g'(x) \neq 0,$
- $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)}$ exists.

Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Example 3.5.8. *Use L'Hôpital's rule to evaluate*

$$\lim_{x \to 0} \frac{e^x - 1}{x} \quad \textit{and} \quad \lim_{x \to 1^+} \frac{\sqrt{x-1}}{x-1}.$$

• $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x}$ *is an indeterminate form of the type* $\dfrac{0}{0}$.
Let $f(x) = e^x - 1$ and $g(x) = x$. Then

$$\lim_{x \to 0} f(x) = 0,\ \lim_{x \to 0} g(x) = 0,\ f'(x) = e^x \ \textit{ and } g'(x) = 1.$$

The conditions of L'Hôpital's Rule are satisfied. Thus,

$$\begin{aligned} \lim_{x \to 0} \frac{e^x - 1}{x} &= \lim_{x \to 0} \frac{(e^x - 1)'}{x'} \\ &= \lim_{x \to 0} \frac{e^x}{1} = 1 \end{aligned}$$

• $\lim\limits_{x \to 1} \lim\limits_{x \to 1^+} \dfrac{\sqrt{x-1}}{x-1}$ *is an indeterminate form of the type* $\dfrac{0}{0}$.
Let $F(x) = \sqrt{x-1}$ and $G(x) = x - 1$. Then

$$\lim_{x \to 1} F(x) = 0,\ \lim_{x \to 1} G(x) = 0,\ f'(x) = \frac{1}{2\sqrt{x-1}} \ \textit{ and } G'(x) = 1.$$

The conditions of L'Hôpital's Rule are satisfied. Thus,

$$\lim_{x\to 1^+} \frac{\sqrt{x-1}}{x-1} = \lim_{x\to 1} \lim_{x\to 1^+} \frac{(\sqrt{x-1})'}{(x-1)'}$$
$$= \lim_{x\to 1} \frac{\frac{1}{2\sqrt{x-1}}}{1} = +\infty.$$

Theorem 3.5.6

Let a be a real number and f and g be two differentiable functions over $]-\infty, a[$. Suppose that

- $\lim_{x\to-\infty} f(x) = 0,$
- $\lim_{x\to-\infty} g(x) = 0,$
- for every $x \in]-\infty, a[,\ g'(x) \neq 0,$
- $\lim_{x\to-\infty} \frac{f(x)}{g(x)}$ exists.

Then
$$\lim_{x\to-\infty} \frac{f(x)}{g(x)} = \lim_{x\to-\infty} \frac{f'(x)}{g'(x)}.$$
A similar result holds when we replace $+\infty$ by $-\infty$ in the hypotheses.

Proof. Let $x = \frac{-1}{t}$, then as $x \to -\infty$ we get $t \to 0^+$ and vice-versa. Define two functions ϕ and ψ by

$$\phi(t) = f\left(\frac{-1}{t}\right) \quad \text{and } \psi(t) = g\left(\frac{-1}{t}\right).$$

Both functions ϕ and ψ are defined on some interval $]0, b[$ and we have

$$\lim_{t\to 0^+} \phi(t) = \lim_{t\to 0^+} \psi(t) = 0.$$

By Theorem 3.5.7, we have

$$\lim_{t\to 0^+} \frac{\phi(t)}{\psi(t)} = \lim_{t\to 0^+} \frac{\phi'(t)}{\psi'(t)}. \tag{3.2}$$

On the other hand,

$$\lim_{t\to 0^+} \frac{\phi'(t)}{\psi'(t)} = \lim_{t\to 0^+} \frac{(1/t^2) f'\left(\frac{-1}{t}\right)}{(1/t^2) g'\left(\frac{-1}{t}\right)}$$
$$= \lim_{x\to-\infty} \frac{f'(x)}{g'(x)}. \tag{3.3}$$

Using (3.2), (3.3) we get

$$\lim_{x\to-\infty}\frac{f(x)}{g(x)} = \lim_{t\to0^+}\frac{\phi(t)}{\psi(t)} = \lim_{x\to-\infty}\frac{f'(x)}{g'(x)}.$$

To prove the case when $x\to+\infty$ we follows the similar steps by supposing $x=\dfrac{1}{t}$.. □

Example 3.5.9. *Use L'Hôpital's rule to evaluate*

$$\lim_{x\to+\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\ln\left(1-\frac{1}{x}\right)}.$$

$\lim_{x\to+\infty}\dfrac{\ln\left(1+\frac{1}{x}\right)}{\ln\left(1-\frac{1}{x}\right)}$ *is an indeterminate form of the type* $\dfrac{0}{0}$.
Let $f(x)=\ln\left(1+\dfrac{1}{x}\right)$ *and* $g(x)=\ln\left(1-\dfrac{1}{x}\right)$. *Then*

$$\lim_{x\to+\infty}f(x)=0,\ \lim_{x\to+\infty}g(x)=0,\ f'(x)=\frac{-1/x^2}{1+1/x}\ \text{and}\ g'(x)=\frac{1/x^2}{1-1/x}.$$

The conditions of L'Hôpital's Rule are satisfied. Thus,

$$\lim_{x\to+\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\ln\left(1-\frac{1}{x}\right)} = \lim_{x\to+\infty}\frac{\frac{-1/x^2}{1+1/x}}{\frac{1/x^2}{1-1/x}} = \lim_{x\to+\infty}\frac{-(1-1/x)}{(1+1/x)} = -1.$$

Theorem 3.5.7

Let f and g are differentiable functions in a deleted neighborhood $\mathcal{V}$ of $x=x_0$. Suppose that

- $\lim_{x\to x_0} g(x)=+\infty$,
- for every $x\in\mathcal{V},\ g'(x)\neq0$,
- $\lim_{x\to x_0}\dfrac{f(x)}{g(x)}$ exists.

Then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

The analogous statements are valid if $x_0 = \pm\infty$ or if $\lim_{x\to x_0} g(x) = -\infty$.

Example 3.5.10. *Use L'Hôpital's rule to evaluate*

$$\lim_{x\to+\infty} \frac{x^2}{ln(x)}$$

• $\lim_{x\to+\infty} \frac{x^2}{ln(x)}$ *is an indeterminate form of the type* $\frac{\infty}{\infty}$.
Let $f(x) = x^2$ *and* $g(x) = ln(x)$*. Then*

$$\lim_{x\to+\infty} g(x) = +\infty,\ f'(x) = 2x \ \text{ and } g'(x) = \frac{1}{x}.$$

The conditions of L'Hôpital's Rule are satisfied. Thus,

$$\begin{aligned}\lim_{x\to+\infty} \frac{x^2}{ln(x)} &= \lim_{x\to+\infty} \frac{(x^2)'}{(ln(x))'}\\ &= \lim_{x\to+\infty} \frac{2x}{1/x}\\ &= \lim_{x\to+\infty} 2x^2 = +\infty.\end{aligned}$$

3.6 Practice Exercises

Exercise 3.1

Determine the domain of the following functions:

$$f_1(x) = \sqrt{e^{3x} - 2},\ f_2(x) = \ln\left(\frac{x-3}{x+1}\right),\ \ f_3(x) = \left(\frac{x+1}{x}\right)^{\frac{1}{x-1}},\ f_4(x) = \frac{\sqrt{-x^2+4x-3}}{\lfloor x \rfloor}.$$

Exercise 3.2

Determine the period of the following periodic-functions:

$$f(x) = \sin(2x-5),\ \ g(x) = \cos\left(\frac{x+3}{2}\right),\ \ h(x) = \tan(4-2x).$$

Exercise 3.3

Study the parity of the following functions.

$$f(x) = \ln\left(\frac{3-x}{3+x}\right), \quad g(x) = \frac{x^2}{|x|+7}, \quad h(x) = \sqrt{(x+x^3)(x^2+x^4)}.$$

Exercise 3.4

Compute the following limits:

$$\lim_{x\to+\infty}\left(\sqrt{x^2+1}-\sqrt{x}\right), \quad \lim_{x\to 0^+}\ln(x)\left(\cos^2(x)-1\right), \quad \lim_{x\to 1}\frac{\sin(\pi x)}{x-1}$$

$$\lim_{x\to+\infty}\frac{\sqrt[3]{x}}{\sqrt{x}-1}, \lim_{x\to-1}\left[(x+1)E(\frac{x-2}{x+1})\right].$$

Exercise 3.5

Determine the domain of the following functions:

$$f_1(x) = \left(x^2+2x-3\right)^{\frac{1}{2-|x|}},$$

$$f_2(x) = \left(\sqrt{x}-1\right)^{\frac{1}{sin(x)}}, \quad f_3(x) = \left(\frac{x}{x+3}\right)^{\sqrt{-x^2-3x+4}}.$$

Exercise 3.6

Compute the following limits:

(a) $\lim_{x\to 1^+}\left(\frac{x-1}{x}\right)^{x^2-1}, \quad \lim_{x\to 2^-}\left(\sqrt{2-x}\right)^{x-2}, \quad \lim_{x\to+\infty}\left(\frac{e^x}{x}\right)^{e^{-x}}.$

(b) $\lim_{x\to+\infty}\left(\frac{x-1}{x}\right)^{x^2-1}, \quad \lim_{x\to 0}(\cos(x))^{\frac{1}{e^x-1}}, \quad \lim_{x\to\pi}\left(\sin\left(\frac{x}{2}\right)\right)^{\frac{1}{x-\pi}}, \quad \lim_{x\to 0^+}\left(\sqrt{x}+1\right)^{\ln(x)}.$

Exercise 3.7

Study the continuity of the following functions at the indicated point x_0:

$$f(x) = \frac{x+3}{|x|+1}\ x_0 = 0, \quad g(x) = \begin{cases} \dfrac{e.e^x-1}{\ln(x+2)} & \text{if } x > -2 \wedge x \neq -1, \\ 1 & \text{if } x = -1, \end{cases} \quad x_0 = -1,$$

$$h(x) = \begin{cases} \dfrac{\sqrt[3]{x}-1}{\sqrt{x}-1} & \text{if } x \geq 0 \wedge x \neq 1, \\ 0 & \text{if } x = 1, \end{cases} \quad x_0 = 1.$$

Exercise 3.8

Study the continuity of the following functions on their domains:

$$f(x) = \begin{cases} e^{\frac{-x}{2}} & \text{if } 0 < x \leq 1, \\ (2-\sqrt{x})^{\frac{1}{x-1}} & \text{si } 1 < x \leq 2, \end{cases} \qquad g(x) = \begin{cases} ln(1+x^2) & \text{if } x \leq 0, \\ sin(x+\frac{\pi}{2}) & \text{si } x > 0. \end{cases}$$

Exercise 3.9

1) Are the following functions admit an extension by continuity at x_0? If yes, define $\widetilde{f}$.

$$f_1(x) = \frac{\cos(\pi x)+1}{x-1}, \ x_0 = 1, \ f_2(x) = \left(\sqrt{2x}-1\right)^{\frac{1}{(x-2)^2}}, \ x_0 = 2, \ f_3(x) = \frac{x}{2|x|}, \ x_0 = 0,$$

$$f_4(x) = \sin(x)\cos\left(\frac{1}{x}\right), \ x_0 = 0.$$

2) Consider a function f defined on $\mathbb{R} - \{2\}$ by:

$$f(x) = \begin{cases} \dfrac{1-cos(\pi x)}{(x-2)^2} & \text{if } x < 2, \\ \alpha\sqrt{2+x} + \dfrac{\sqrt{x}-\sqrt{2}}{\sqrt{x-2}} & \text{if } x > 2. \end{cases} \quad \text{where } \alpha \in \mathbb{R},$$

For which values of α, is the function f extendable by continuity at $x = 2$? Define $\widetilde{f}$.

Exercise 3.10

Use the intermediate value theorem to show that:

(a) The equation $3\sin(x) - x^2 = 1$ has at least one solution in $]0, \frac{\pi}{6}[$.

(b) The equation $e^x - x = \dfrac{3}{2}$ has an unique solution in $]0, \ln(3)[$.

(c) The equation $\ln(x) = \dfrac{x}{3}$ has an unique solution between $e+1$ and $2e$.

Exercise 3.11

Let h be a function defined as follows:

$$h(x) = \begin{cases} \dfrac{1}{x} ln\left(\dfrac{1+e^{2x}}{1+e^{x}}\right) & \text{if } x > 0, \\ \dfrac{1}{2} & \text{if } x = 0, \\ \dfrac{1-cos(x)}{x^2} + x & \text{if } x < 0. \end{cases}$$

a/ Show that h is continuous at the point $x = 0$.

b/ Check that the equation $h(x) = 0$ has at least a solution in the interval $\left]\dfrac{-\pi}{2}, 0\right[$.

Exercise 3.12

Study the differentiability of the following functions at x_0 :

$$f_1(x) = 3x^2 - 4x - 5,\ x_0 = 3,\quad f_2(x) = |x^2 - 1|,\ x_0 = 1,\quad f_3(x) = e^{\sqrt{x}},\ x_0 = 0.$$

Exercise 3.13

Let a, b two real number and f be a function defined by

$$f(x) = \begin{cases} \sqrt[3]{x} & \text{si } x \leq -1, \\ ax^2 + bx + 1 & \text{si } x > -1, \end{cases} .$$

Determine a and b so that f be differentiable at $x_0 = -1$.

Exercise 3.14

Determine the domain of differentiability of the following functions and then calculate their derivatives:

$$f_1(x) = \ln\left(\frac{x+2}{2x}\right),\ f_2(x) = \sqrt{\frac{x^2-1}{x^2+1}},\ f_3(x) = e^{\sin(2x-3)},\ f_4(x) = (x-1)^{\sqrt{x}}.$$

Exercise 3.15

Use L'Hôpital's Rule and compute the following limits:

$$\lim_{x\to 0} \frac{\sqrt{x^2+1} - \sqrt{x+1}}{x},\ \lim_{x\to \frac{1}{2}} \frac{\ln(2x)}{\cos(\pi x)},\ \lim_{x\to 0} \frac{e^{3x} - e^{5x}}{sin(x)},$$

$$\lim_{x\to 0}\frac{e^{ax}-ax-1}{ax^2},\ a\in\mathbb{R}^*,\quad \lim_{x\to 0^+}\frac{\sin(\sqrt{x})-\sqrt{x}}{x^2+x},$$
$$\lim_{x\to\frac{\pi}{2}}\frac{\cos^3(x)}{x-\frac{\pi}{2}},\quad \lim_{x\to\pi}\frac{\tan(x)}{ln\left(\frac{x}{\pi}\right)}.$$

Exercise 3.16

Are the conditions of Rolle's theorem hold for the following functions?

$$f_1(x)=x^2-4x \text{ on }]0,4[,\ f_2(x)=\begin{cases} x & \text{if} 0\le x\le 1\\ 2-x & \text{if } 1<x\le 2.\end{cases} \text{ on }]0,2[,\ f_3(x)=x+\frac{1}{x},\quad \text{on }]\frac{1}{4},4[.$$

Exercise 3.17

Define a function f on $[-1,+\infty[$ as follows:

$$f(x)=\begin{cases}\sqrt{1+x} & \text{if } -1\le x\le 0,\\ \\ \cos(\pi x)+ax+b & \text{if } x>0.\end{cases}\qquad a,b\in\mathbb{R}.$$

1. Determine a and b such that f is continuous and differentiable at $x_0=0$.
2. Put $a=\frac{1}{2}$ and $b=0$. Using Rolle's theorem and show that $f'(x)=0$ has at least a solution in $\left]\frac{-15}{16},\frac{1}{2}\right[$.

Exercise 3.18

Use the mean value theorem and show that the following inequalities are hold.

1. $\forall x>0,\ x+1<e^x<xe^x+1$
2. $\forall x<2,\ \frac{2-x}{3-x}<\ln(3-x)<2-x$ then conclude the limit: $\lim_{x\to 2^-}\frac{\ln(3-x)}{x-2}$.
3. $\forall x>1,\ \frac{\sqrt{x}-1}{2\sqrt{x}}<\sqrt{x}-1<\frac{x-1}{2}$.

Exercise 3.19

Let f be a function defined on $\mathbb{R}$ by

$$f(x) = \frac{1}{10}x^5 + 2x^2 - 1$$

1. Calculate f' the derivative of f over its domain of differentiability .
2. Set up the table of variations related to f over its domain.
3. Conclude that f is a bijective function from $[0, +\infty[$ to $[-1, +\infty[$.
4. Determine the Domain of f^{-1} and then conclude the variation of f^{-1}.
5. Calculate $f(1),\ f'(1)$ and then conclude $f^{-1}\left(\frac{11}{10}\right)$.
6. Deduce $(f^{-1})'\left(\frac{11}{10}\right)$ after verifying the differentiability of f^{-1} at $x_0 = \frac{11}{10}$.

Exercise 3.20

Let $f : \mathbb{R} - \{\frac{1}{3}\} \to \mathbb{R}^*$ be a function defined by $f(x) = \frac{-4}{3x-1}$.

1) Set up the table of variations related to f over its domain.
2) Conclude that f is a bijective function from $\mathbb{R} - \{\frac{1}{3}\}$ to $\mathbb{R}^*$.
3) Determine the inverse f^{-1} (the starting set, the arrival set and the expression).
4) Conclude the variation of f^{-1} (without using the table of variations).
5) Show that f^{-1} is differentiable and then calculate $(f^{-1})'$ by two methods.

Chapitre 4
Elementary functions and applications

4.1 Exponential and logarithmic functions

4.1.1 Exponential function

Definition 4.1.1

We call exponential function the unique differentiable function on $\mathbb{R}$ such that $f' = f$ and $f(0) = 1$. We denote this function by

$$\begin{array}{rcl} exp : \mathbb{R} & \to &]0, +\infty[\\ x & \to & e^x. \end{array}$$

Properties:

1.
   - The exponential function is continuous and differentiable on $\mathbb{R}$ and $(e^x)' = e^x$.
   - The exponential function is strictly increasing on $\mathbb{R}$.
   - $\forall x \in \mathbb{R},\ e^x > 0$.
2. For all x, y in $\mathbb{R}$ and for all n in $\mathbb{Z}$, we have
   - $e^x \cdot e^y = e^{x+y}$,
   - $\dfrac{e^x}{e^y} = e^{x-y}$,
   - $(e^x)^n = e^{nx}$.
   - $\lim\limits_{x\to-\infty} e^x = 0$ and $\lim\limits_{x\to+\infty} e^x = +\infty$.

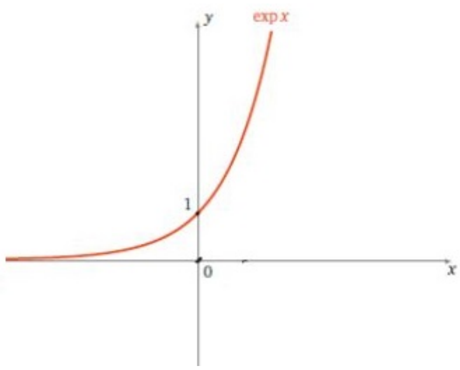


Figure 4.1.1: The graph of exponential function.

4.1.2 Logarithm function

Definition 4.1.2

The function $x \mapsto e^x$ is continuous and strictly increasing from $\mathbb{R}$ to $]0, +\infty[$, therefore achieves a bijection. Thus, it admits an inverse function called logarithm function and defined as

$$\begin{array}{rcl} ln :]0, +\infty[& \to & \mathbb{R} \\ x & \to & ln(x). \end{array}$$

Properties:

1.
   - The logarithm function is continuous and differentiable on $]0, +\infty[$ and $(\ln(x))' = \dfrac{1}{x}$.
   - The logarithm function is strictly increasing on $]0, +\infty[$.
   - $\ln(1) = 0,\ \forall x \in]0, 1[,\ \ln(x) < 0$ and $\forall x > 1,\ ln(x) > 0$.

2. For all x, y in $]0, +\infty[$ and for all n in $\mathbb{Z}$, we have
   - $\ln(x \cdot y) = \ln(x) + \ln(y)$,
   - $\ln\left(\dfrac{x}{y}\right) = \ln(x) - \ln(y)$,
   - $\ln(x^n) = n\ln(x)$.
   - $\lim\limits_{x\to 0^+} \ln(x) = -\infty$ and $\lim\limits_{x\to+\infty} \ln(x) = +\infty$.

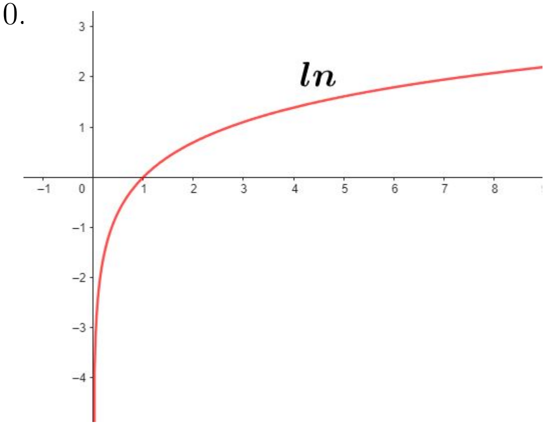


Figure 4.1.1: The graph of logarithm function.

4.1.3 Exponentials and Logarithms of base a

4.1.3.1 Exponential of base a

Definition 4.1.3

For any positive number a such that $a \neq 1$. An exponential function f of base a is defined as

$$f: \begin{array}{ccc} \mathbb{R} & \to &]0, +\infty[\\ x & \to & a^x = e^{x\ln(a)}. \end{array}.$$

Remark 4.1.1

If $a = e$ the exponential function of base a is exactly the exponential function.

Properties: The function $x \mapsto a^x$ is continuous, differentiable and satisfies:

- $(a^x)' = \ln(a) \cdot a^x$,
- if $a > 0$ the function $x \mapsto a^x$ is strictly increasing and $\lim\limits_{x\to+\infty} a^x = +\infty$,
- if $0 < a < 1$ the function $x \mapsto a^x$ is strictly decreasing and $\lim\limits_{x\to+\infty} a^x = 0$.

4.1.3.2 Logarithm of base a

Definition 4.1.4

For any positive number a such that $a \neq 1$. An logarithm function of base a is denoted $\log_a$ and defined as

$$\begin{aligned} \log_a :]0, +\infty[&\rightarrow \mathbb{R} \\ x &\rightarrow \log_a(x) = \frac{\ln(x)}{\ln(a)}. \end{aligned}.$$

Properties: The function $x \mapsto \log_a(x)$ from $]0, +\infty[$ to $\mathbb{R}$ is continuous, differentiable and satisfies:

- $(\log_a(x))' = \dfrac{1}{x\ln(a)}$.
- if $a > 1$ the function $x \mapsto \log_a(x)$ is strictly increasing.
- if $0 < a < 1$ the function $x \mapsto \log_a(x)$ is strictly decreasing.

4.2 Trigonometric functions and their inverses

4.2.1 The function sinus and its inverse.

4.2.1.1 The function sinus

Definition 4.2.1

The function sine is the function defined by $\sin : \mathbb{R} \rightarrow [-1, 1]$ which, to any real x, associates the real $\sin(x)$.

The representative graphs of $x \mapsto \sin(x)$ over $\mathbb{R}$ is the following:

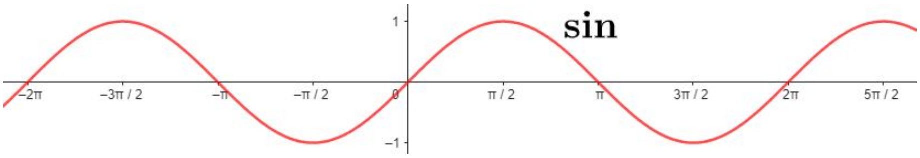


Figure 4.2.1.1: The graph of the function sine.

Example 4.2.1.

$$\sin\left(\frac{\pi}{2}\right) = 1, \sin(0) = 0, \ \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \ \textit{and} \ \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

4.2.1.2 The function arcsinus

Definition 4.2.2

The function $x \mapsto \sin(x)$ is continuous and strictly increasing from $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to $[-1, 1]$, therefore achieves a bijection. Thus, it admits an inverse function defined as

$$\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

The arcsinus function also denoted $\sin^{-1}$.

Example 4.2.2.

$$\arcsin(1) = \frac{\pi}{2}, \arcsin(0) = 0, \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}, \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

The representative graphs of $x \mapsto \sin(x)$ and $x \mapsto \arcsin(x)$ are as follows:

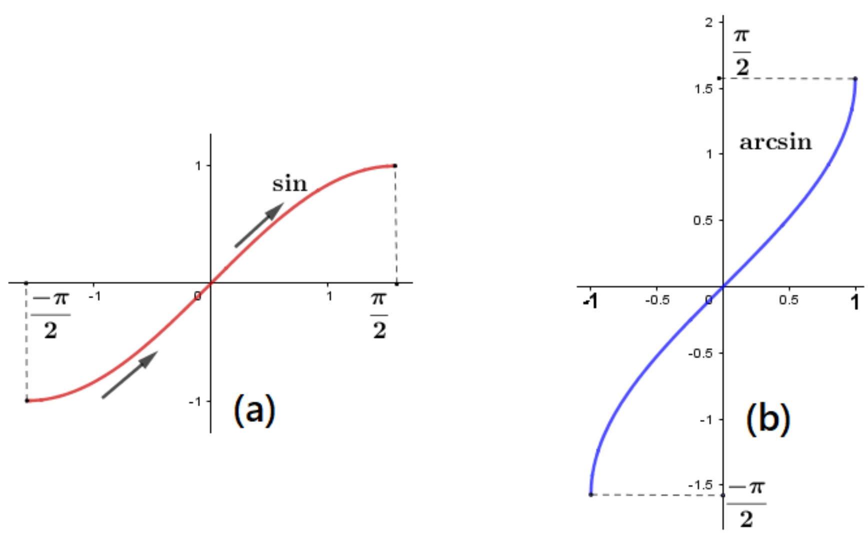


Figure 4.2.1.2: (a) represents the graph of sin function on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and (b) represents the graph of arcsin function on $[-1, 1]$.

Remark 4.2.1

Here, the functions $x \mapsto arcsin(x)$ and $x \mapsto \sin(x)$ are symmetrical with respect to the line $y = x$.

Proposition 4.2.1

The arcsin function satisfies the following properties:

- arcsin function is odd, continuous and increasing on $[-1,1]$.
- arcsin function is differentiable on $]-1,1[$ and $\forall x \in]-1,1[$, $\arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$.
- if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ we have $\sin(x) = y \iff x = \arcsin y$.
- $\forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ we have $\arcsin\big(\sin(x)\big) = x$.
- $\forall x \in [-1,1]$ we have
$$\sin\big(\arcsin(x)\big) = x$$
and
$$\cos(\arcsin(x)) = \sqrt{1-x^2} \ (\text{ according to the equation } \cos^2(\arcsin(x)) + \underbrace{\sin^2(\arcsin(x))}_{=x^2} = 1).$$

Example 4.2.3. .

- $\lim\limits_{x\to+\infty} \arcsin(\dfrac{x-1}{x+1}) = \arcsin(1) = \dfrac{\pi}{2}$.
- $\lim\limits_{x\to 2} \sin^{-1}\left(\dfrac{\sqrt{x+1}}{x}\right) = \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{3}$.
- $\lim\limits_{x\to 0} \dfrac{\sin^{-1}(x)}{x}$ *is an indeterminate form of type* $\dfrac{0}{0}$.

Put $y = \sin^{-1}(x)$ *we get* $x = \sin(y)$. *If* $x \to 0$ *we have* $y \to 0$. *Therefore,*

$$\begin{aligned}\lim_{x\to 0} \frac{\sin^{-1}(x)}{x} &= \lim_{y\to 0} \frac{y}{\sin(y)} \\ &= \lim_{y\to 0} \frac{1}{\frac{\sin(y)}{y}} = 1.\end{aligned}$$

- $\lim\limits_{x\to 0} \dfrac{cos(arcsin(x)) - 1}{x^2}$ *is an indeterminate form of type* $\dfrac{0}{0}$. *We have*

$$\forall x \in [-1,1],\ \cos(\arcsin(x)) = \sqrt{1-x^2}.$$

Then

$$\begin{aligned}\lim_{x\to 0} \frac{cos(arcsin(x)) - 1}{x^2} &= \lim_{x\to 0} \frac{\sqrt{1-x^2}-1}{x^2} \\ &= \lim_{x\to 0} \frac{-x^2}{x^2(\sqrt{1-x^2}+1)} \\ &= \frac{-1}{2}.\end{aligned}$$

4.2.1.3 Some results of the composition $arcsin \circ U$

In the following we talking about the composition between the arcsinus function and an other function U defined on $\mathcal{D}_U \subseteq \mathbb{R}$. Let f be a function defined by

$$f(x) = arcsin(U(x)).$$

a) The domain of f.

Definition 4.2.3

The domain of f is

$$\boxed{\mathcal{D}_f = \mathcal{D}_U \cap \{x \in \mathbb{R} : -1 \leq U(x) \leq 1\}.}$$

Remark 4.2.2

if $\mathcal{D}_U = \mathbb{R}$ we write

$$\boxed{\mathcal{D}_f = \{x \in \mathbb{R} : -1 \leq U(x) \leq 1\}.}$$

Example 4.2.4. *Determine the domain of the following functions*

$$f_1(x) = \arcsin(\sqrt{x} - 2) \ \ \textit{and} \ f_2(x) = \arcsin\left(2\ln(x) - 1\right).$$

• *The domain of f_1 is*

$$\mathcal{D}_f = [0, +\infty[\cap\{x \in \mathbb{R} : -1 \leq \sqrt{x} - 2 \leq 1\}.$$

We have $-1 \leq \sqrt{x} - 2 \leq 1 \implies 1 \leq x \leq 9$. *Thus,*

$$\mathcal{D}_f = [0, +\infty[\cap[1, 9] = [1, 9].$$

• *The domain of f_2 is*

$$D_{f_2} =]0, +\infty[\cap\{x \in \mathbb{R} : \ -1 \leq 2\ln(x) - 1 \leq 1\}.$$

We have $-1 \leq 2\ln(x) - 1 \leq 1 \implies 1 \leq x \leq e$. *Thus,*

$$\mathcal{D}_{f_2} = [1, e].$$

$b)$ **Continuity of f**

Corollary 4.2.1

If U is a continuous function then, the function f is continuous too on $\mathcal{D}_f$.

Example 4.2.5. *The function f_3 such that $f_3(x) = \arcsin(e^x - 1)$ is defined and continuous on $]-\infty, \ln(2)]$ since we have*

$$\begin{aligned} D_{f_3} &= \{x \in \mathbb{R} : \ -1 \leq e^x - 1 \leq 1\} \\ &=]-\infty, \ln(2)]. \end{aligned}$$

and the function $x \mapsto e^x - 1$ is continuous on $\mathbb{R}$.

$c)$ **Differentiability**

Corollary 4.2.2

If $x \mapsto U(x)$ is a differentiable function and $-1<U(x)<1$, then, f is differentiable and

$$\boxed{\forall x \in \mathcal{D}_f^d,\ f'(x) = \frac{U'(x)}{\sqrt{1 - U^2(x)}}.}$$

Example 4.2.6. *Determine the domain and the domain of differentiability of the following functions and then calculate their derivatives.*

$$g(x) = \arcsin(e^x - 2) \text{ and } h(x) = \arcsin(\sqrt{x+4} - 3).$$

1. *The function g :*

   - *Determine the domain of g :*

   $$\begin{aligned} \mathcal{D}_g &= \{x \in \mathbb{R},\ -1 \leq e^x - 2 \leq 1\} \\ &= [0, \ln(3)]. \end{aligned}$$

   - *Determine the domain of differentiability of g :*

   $$\begin{aligned} \mathcal{D}_g &= \{x \in \mathbb{R},\ -1 < e^x - 2 < 1\} \\ &=]0, \ln(3)[. \end{aligned}$$

   - *Calculate the derivative of g. For all x in $]0, \ln(3)[$ we have*

   $$\begin{aligned} g'(x) &= \frac{(e^x - 2)'}{\sqrt{1 - (e^x - 2)^2}} \\ &= \frac{e^x}{\sqrt{1 - (e^{2x} - 4e^x + 4)}} \\ &= \frac{e^x}{\sqrt{(1 - e^x)(e^x - 3)}}. \end{aligned}$$

2. *The function h :*

- *Determine the domain of* h :

$$\begin{aligned}\mathcal{D}_h &= [-4, +\infty[\cup \{x \in \mathbb{R},\ -1 \leq \sqrt{x+4} - 3 \leq 1\} \\ &= [0, 12].\end{aligned}$$

- *Determine the domain of differentiability of* h :

$$\begin{aligned}\mathcal{D}_h^d &=]-4, +\infty[\cup \{x \in \mathbb{R},\ -1 < \sqrt{x+4} - 3 < 1\} \\ &=]0, 12[.\end{aligned}$$

- *Calculate the derivative of* h*. For all* x *in* $]0, 12[$ *we have*

$$\begin{aligned}h(x) &= \frac{(\sqrt{x+4} - 3)'}{\sqrt{1 - (\sqrt{x+4} - 3)^2}} \\ &= \frac{\frac{1}{2\sqrt{x+4}}}{\sqrt{6\sqrt{x+4} - x - 12}} \\ &= \frac{1}{2\sqrt{x+4}\sqrt{6\sqrt{x+4} - x - 12}}.\end{aligned}$$

4.2.2 The function Cosinus and its inverse.

4.2.2.1 The function cosinus

Definition 4.2.4

The function cosine is the function defined by $\cos : \mathbb{R} \to [-1, 1]$ which, to any real x, associates the real $\cos(x)$.

The representative graphs of $x \mapsto \cos(x)$ over $\mathbb{R}$ is the following:

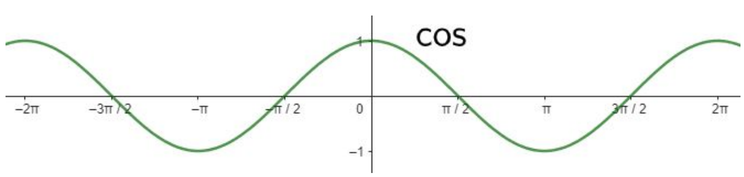


Figure 4.2.2.1: The graph of the function cosine.

Example 4.2.7.

$$\cos(0) = 1,\ \cos(\pi) = -1,\ \cos(\frac{\pi}{3}) = \frac{1}{2} \textit{ and } \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}.$$

4.2.2.2 The function arccos

Definition 4.2.5

The function $x \mapsto \cos(x)$ is continuous and strictly decreasing from $[0, \pi]$ to $[-1, 1]$, therefore achieves a bijection. Thus, it admits an inverse function defined as

$$\arccos : [-1, 1] \to [0, \pi] .$$

The arccosinus function also denoted $\cos^{-1}$.

Example 4.2.8.

$$\arccos(1) = 0 \;\; \arccos(-1) = \pi \;\; \arccos(\frac{1}{2}) = \frac{\pi}{3} \;\; \textit{and} \;\; \arccos(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}.$$

The representative graphs of $x \mapsto \cos(x)$ and $x \mapsto \arccos(x)$ are as follows:

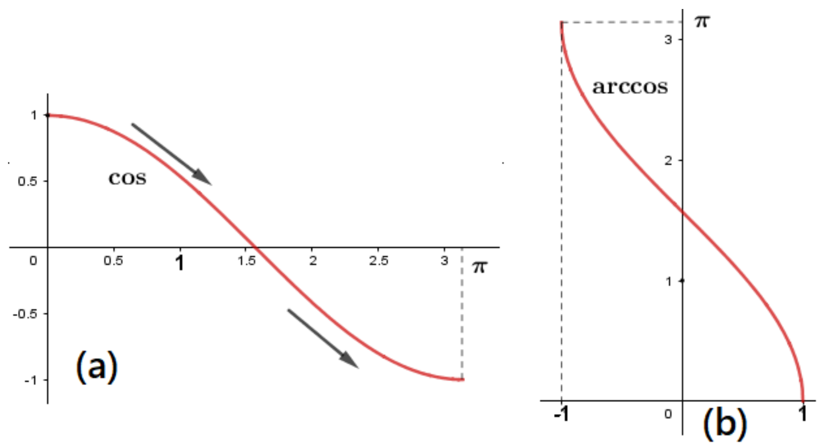


Figure 4.2.2.2: (a) represents the graph of cos function on $[0, \pi]$ and (b) represents the graph of arccos function on $[-1, 1]$.

Remark 4.2.3

Also, the functions $x \mapsto arccos(x)$ and $x \mapsto \cos(x)$ are symmetrical with respect to the line $y = x$.

Proposition 4.2.2

The arccosinus function satisfies the following properties:

- arccos function is continuous and decreasing on $[-1, 1]$.
- arccos function is differentiable on $]-1, 1[$ and $\forall x \in]-1, 1[$, $\arccos'(x) = \dfrac{-1}{\sqrt{1 - x^2}}$.
- if $x \in [0, \pi]$ we have $\cos(x) = y \iff x = \arccos y$.

- $\forall x \in [0, \pi]$ we have $\arccos\big(\cos(x)\big) = x$.
- $\forall x \in [-1, 1]$ we have
$$\cos\big(\arccos(x)\big) = x$$
and
$$\sin(\arccos(x)) = \sqrt{1 - x^2}.$$

4.2.2.3 Some results of the composition $arccos \circ U$

As the preceding we have some results about the composition between the arccosinus function and an other function U defined on $\mathcal{D}_U \subseteq \mathbb{R}$. Let g be a function defined by

$$g(x) = arccos(U(x)).$$

a) The domain of g.

Definition 4.2.6

The domain of g is

$$\boxed{\mathcal{D}_g = \mathcal{D}_U \cap \{x \in \mathbb{R} : -1 \leq U(x) \leq 1\}.}$$

Remark 4.2.4

if $\mathcal{D}_U = \mathbb{R}$ we write

$$\boxed{\mathcal{D}_g = \{x \in \mathbb{R} : -1 \leq U(x) \leq 1\}.}$$

$b)$ **Continuity of** g

Corollary 4.2.3

If U is a continuous function then, the function g is continuous too on $\mathcal{D}_g$.

Example 4.2.9. *Determine the domain of the following functions*

$$g_1(x) = \arccos(x^2 - 8).$$

The domain of g_1 *is*

$$\mathcal{D}_{g_1} = \{x \in \mathbb{R} : -1 \leq x^2 - 8 \leq 1\}.$$

We have

$$\begin{aligned} -1 \leq x^2 - 8 \leq 1 &\implies \sqrt{7} \leq |x| \leq \sqrt{9} \\ &\implies (-\sqrt{9} \leq x \leq -\sqrt{7}) \vee (\sqrt{7} \leq x \leq \sqrt{9}). \end{aligned}$$

Thus,

$$\mathcal{D}_{g_1} = [-\sqrt{9}, -\sqrt{7}] \cup [\sqrt{7}, \sqrt{9}].$$

Moreover, g_1 *is continuous on* $[-\sqrt{9}, -\sqrt{7}] \cup [\sqrt{7}, \sqrt{9}]$ *since the function* $x \mapsto x^2 - 8$ *is continuous.*

$c)$ **Differentiability of** g

Corollary 4.2.4

If $x \mapsto U(x)$ is a differentiable function and $-1<U(x)<1$, then, g is differentiable and

$$\forall x \in \mathcal{D}_g^d,\ g'(x) = \frac{-U'(x)}{\sqrt{1-U^2(x)}}.$$

Example 4.2.10. *Determine the domain and the domain of differentiability of the following function and then calculate its derivative.*

$$g_2(x) = \arccos(2 - e^x).$$

- *Determine the domain of* g_2 :

$$\begin{aligned}\mathcal{D}_{g_2} &= \{x \in \mathbb{R},\ -1 \leq 2 - e^x \leq 1\} \\ &= [0, \ln(3)].\end{aligned}$$

- *Determine the domain of differentiability of* g_2 :

$$\begin{aligned}\mathcal{D}_{g_2^d} &= \{x \in \mathbb{R},\ -1 < 2 - e^x < 1\} \\ &=]0, \ln(3)[.\end{aligned}$$

- *Calculate the derivative of* g_2*. For all* x *in* $]0, \ln(3)[$ *we have*

$$\begin{aligned}g_2'(x) &= \frac{-(2-e^x)'}{\sqrt{1-(2-e^x)^2}} \\ &= \frac{e^x}{\sqrt{1-(e^{2x}-4e^x+4)}} \\ &= \frac{e^x}{\sqrt{(1-e^x)(e^x-3)}}.\end{aligned}$$

4.2.3 The Tangent function and its inverse.

4.2.3.1 The tangent function

Definition 4.2.7

The tangent function is the function defined by $\tan : \mathbb{R}\backslash\left\{\frac{(2k+1)\pi}{2},\ k \in \mathbb{Z}\right\} \to \mathbb{R}$ which, to any real x, associates the real $\tan(x)$ such that $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

The representative graphs of $x \mapsto \tan(x)$ is the following:

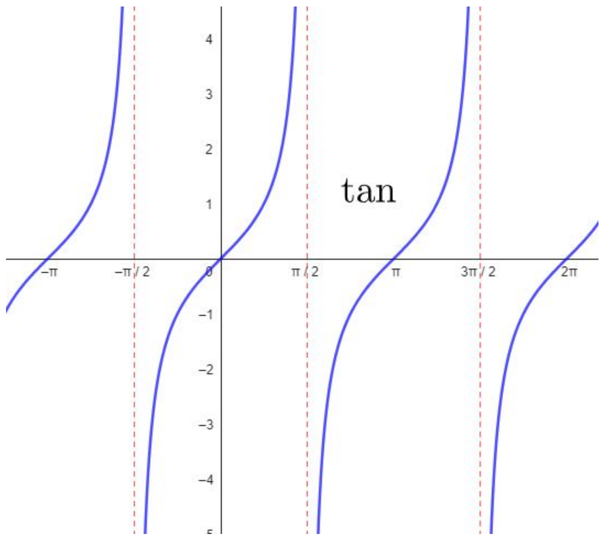


Figure 4.2.3.1: The graph of tangent function.

Example 4.2.11.

$$\tan(0) = 0,\ \tan\left(\frac{\pi}{3}\right) = \sqrt{3} \text{ and } \tan\left(\frac{\pi}{4}\right) = 1.$$

4.2.3.2 The function arctan

Definition 4.2.8

The function $x \mapsto \tan(x)$ is continuous and strictly increasing from $]-\frac{\pi}{2}, +\frac{\pi}{2}[$ to $\mathbb{R}$, therefore achieves a bijection. Thus, it admits an inverse function defined as

$$\arctan : \mathbb{R} \to]-\frac{\pi}{2}, +\frac{\pi}{2}[.$$

The arctan function also denoted $\tan^{-1}$ or $arctg$.

Example 4.2.12.

$$\arctan(0) = 0\ \arctan(\sqrt{3}) = \frac{\pi}{3} \text{ and } \arctan(1) = \frac{\pi}{4}.$$

The representative graphs of $x \mapsto \tan(x)$ and $x \mapsto \arctan(x)$ are as follows:

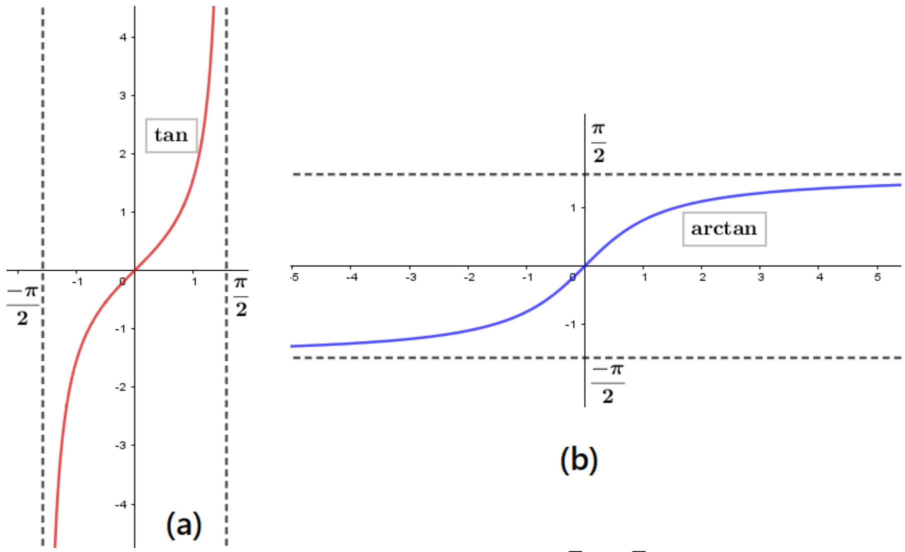


Figure 4.2.3.2: (a) represents the graph of tan function on $]-\frac{\pi}{2},+\frac{\pi}{2}[$ and (b) represents the graph of arctan function on $\mathbb{R}$.

Remark 4.2.5

The functions $x \mapsto arctan(x)$ and $x \mapsto \tan(x)$ are symmetrical with respect to the line $y = x$. Moreover

- $\lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}$ and $\lim_{x\to+\infty} \arctan(x) = \frac{\pi}{2}$
- the graph of the arctan function admits an horizontal asymptotes $y = \frac{-\pi}{2}$ over $-\infty$ and $y = \frac{\pi}{2}$ over $+\infty$.

Proposition 4.2.3

The arctangent function satisfies the following properties:

- arctan function is odd, continuous and increasing on $\mathbb{R}$.
- arctan function is differentiable on $\mathbb{R}$ and $\forall x \in \mathbb{R}$, $\arctan'(x) = \frac{1}{1+x^2}$.
- if $x \in]-\frac{\pi}{2},+\frac{\pi}{2}[$ we have $\tan(x) = y \iff x = \arctan y$.
- $\forall x \in]-\frac{\pi}{2},+\frac{\pi}{2}[$ we have $\arctan\big(\tan(x)\big) = x$.
- $\forall x \in \mathbb{R}$ we have $\tan\big(\arctan(x)\big) = x$.
- $\forall x < 0,\ \arctan(x) + \arctan(\frac{1}{x}) = -\frac{\pi}{2}$
- $\forall x > 0,\ \arctan(x) + \arctan(\frac{1}{x}) = \frac{\pi}{2}$.

4.2.3.3 Some results of the composition $arctan \circ U$

As the preceding, in the following we talking about some results concerning the composition between the arctangent function and an other function U defined on $\mathcal{D}_U \subseteq \mathbb{R}$. Let h be a function defined by

$$h(x) = arctan(U(x)).$$

a) The domain of h.

Definition 4.2.9

The domain of h is exactly the domain of U which means

$$\boxed{\mathcal{D}_h = \mathcal{D}_U.}$$

$b)$ **Continuity of** h

Corollary 4.2.5

If U is a continuous function then, the function h is continuous too on $\mathcal{D}_h$.

Example 4.2.13. *Determine the domain of the following functions*

$$h_1(x) = \arctan(\sqrt{2-x}).$$

The domain of h_1 *is*

$$\begin{aligned}\mathcal{D}_{h_1} &= \{x \in \mathbb{R} : 2 - x \geq 0\} \\ &=]-\infty, 2].\end{aligned}$$

Moreover, h_1 *is continuous on* $]-\infty, 2]$ *since the function* $x \mapsto \sqrt{2-x}$ *is continuous.*

$c)$ **Differentiability of** h

Corollary 4.2.6

If $x \mapsto U(x)$ is a differentiable function then, h is differentiable and

$$\forall x \in \mathcal{D}_h^d, \ h'(x) = \frac{U'(x)}{1 + U^2(x)}.$$

Example 4.2.14. *Determine the domain and the domain of differentiability of the following function and then calculate its derivative.*

$$h_2(x) = \arctan\left(\frac{\sqrt{1-x}}{x}\right).$$

- *Determine the domain of* h_2 :

$$\begin{aligned}\mathcal{D}_{h_2} &= \{x \in \mathbb{R},\ 1-x \geq 0\ \text{ and } x \neq 0\}\\ &=]-\infty, 1] \cap \mathbb{R}^*\\ &=]-\infty, 0[\cup]0, 1].\end{aligned}$$

- *Determine the domain of differentiability of* h_2 :

$$\begin{aligned}\mathcal{D}^d_{h_2} &= \{x \in \mathbb{R},\ 1-x > 0\ \text{ and } x \neq 0\}\\ &=]-\infty, 1[\cap\mathbb{R}^*\\ &=]-\infty, 0[\cup]0, 1[.\end{aligned}$$

- *Calculate the derivative of* h_2. *For all* x *in* $]-\infty, 0[\cup]0, 1[$ *we have*

$$\begin{aligned}h_2'(x) &= \frac{\left(\dfrac{\sqrt{1-x}}{x}\right)'}{1+\left(\dfrac{\sqrt{1-x}}{x}\right)^2}\\ &= \frac{\left(\dfrac{-x}{2\sqrt{1-x}} - \sqrt{1-x}\right)/x^2}{1+\dfrac{1-x}{x^2}}\\ &= \frac{-1}{2x^2\sqrt{1-x}(x^2-x+1)}.\end{aligned}$$

4.2.4 The cotangent function and its inverse.

4.2.4.1 The cotangent function

Definition 4.2.10

The cotangent function is the function defined by $\cot : \mathbb{R}\backslash\{k\pi,\ k \in \mathbb{Z}\} \to \mathbb{R}$ which, to any real x, associates the real $\cot(x)$ such that $\cot(x) = \dfrac{1}{\tan(x)}$.
The cotangent function also denoted ctg.

Remark 4.2.6

Also, can write the cotangent function as follows

$$\cot(x) = \frac{\cos(x)}{\sin(x)} \quad \text{or} \quad \cot(x) = \tan\left(\frac{\pi}{2} - x\right).$$

The representative graphs of $x \mapsto \cot(x)$ is the following:

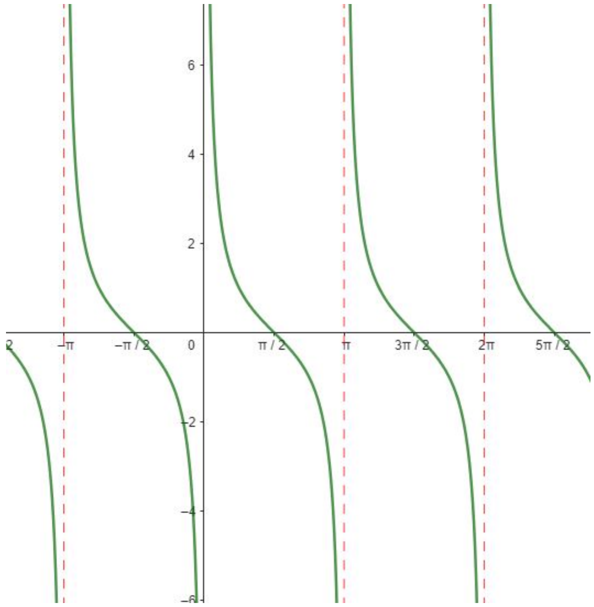


Figure 4.2.4.1: The graph of cotangent function.

Example 4.2.15.

$$\cot\left(\frac{\pi}{2}\right) = 0,\ ctg\left(\frac{\pi}{6}\right) = \sqrt{3} \textit{ and } ctg\left(\frac{\pi}{4}\right) = 1.$$

4.2.4.2 The function arccotangent

Definition 4.2.11

The function $x \mapsto \cot(x)$ is continuous and strictly decreasing from $]0, \pi[$ to $\mathbb{R}$, therefore achieves a bijection. Thus, it admits an inverse function defined as

$$\text{arccot} : \mathbb{R} \to]0, \pi[.$$

The arccotangent function also denoted $\cot^{-1}$ or $arcctg$.

Remark 4.2.7

Since we have for any real number x, $cot(x) = \tan\left(\frac{\pi}{2} - x\right)$, we can defined the the inverse function arccot as follows:

$$\forall x \in \mathbb{R},\ \text{arccot}(x) = \frac{\pi}{2} - \arctan(x).$$

Example 4.2.16.

$$\text{arccot}\,(0) = \frac{\pi}{2}\ \text{arccot}(\sqrt{3}) = \frac{\pi}{6} \textit{ and } arccgt(1) = \frac{\pi}{4}.$$

The representative graphs of $x \mapsto \cot(x)$ and $x \mapsto \text{arccot}(x)$ are as follows:

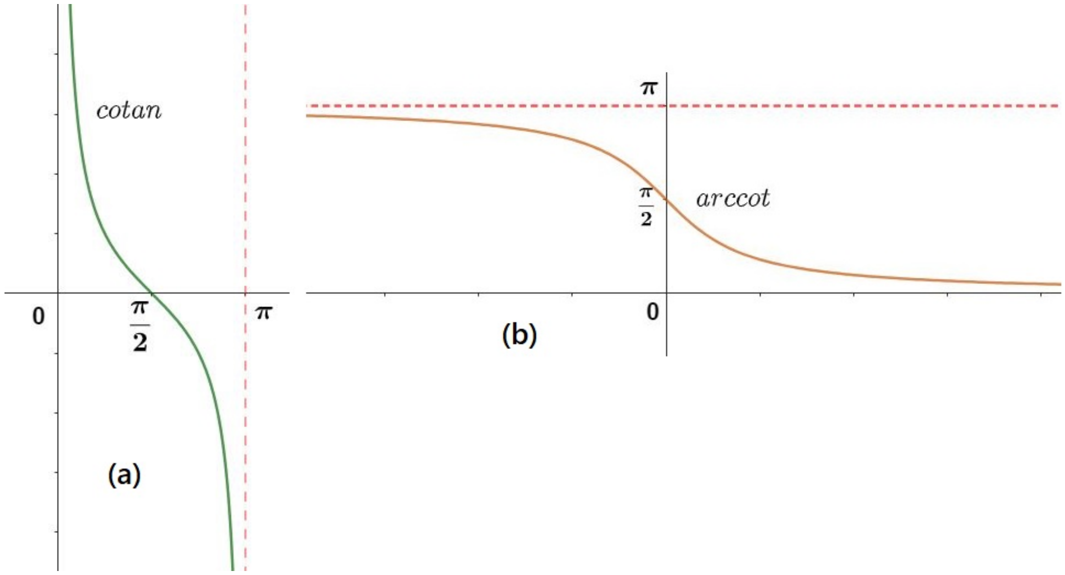


Figure 4.2.4.2: (a) represents the graph of cotangent function on $]0, \pi[$ and (b) represents the graph of arccot function on $\mathbb{R}$.

Remark 4.2.8

The functions $x \mapsto arccot(x)$ and $x \mapsto \cot(x)$ are symmetrical with respect to the line $y = x$. Moreover

- $\lim_{x \to -\infty} \operatorname{arccot}(x) = \pi$ and $\lim_{x \to +\infty} \operatorname{arccot}(x) = 0$
- the graph of the arccot function admits an horizontal asymptotes $y = \pi$ as x approaches $-\infty$ and $y = 0$ as x approaches $+\infty$.

Proposition 4.2.4

The arctangent function satisfies the following properties:

- arccot function neither even nor odd, continuous and decreasing on $\mathbb{R}$.
- arccot function is differentiable on $\mathbb{R}$ and $\forall x \in \mathbb{R}$, $\operatorname{arccot}'(x) = \dfrac{-1}{1+x^2}$.
- if $x \in]0, \pi[$ we have $\cot(x) = y \iff x = \operatorname{arccot} y$.
- $\forall x \in]0, \pi[$ we have $\operatorname{arccot}\big(\cot(x)\big) = x$.
- $\forall x \in \mathbb{R}$ we have $\cot\big(\operatorname{arccot}(x)\big) = x$.

4.2.4.3 Some results of the composition $arccot \circ U$

As the preceding, in the following we talking about some results concerning the composition between the arccotangent function and an other function U defined on $\mathcal{D}_U \subseteq \mathbb{R}$. Let h be a function defined by

$$k(x) = arccot(U(x)).$$

a) The domain of k.

Definition 4.2.12

The domain of k is exactly the domain of U which means

$$\boxed{\mathcal{D}_k = \mathcal{D}_U.}$$

$b)$ **Continuity of k**

Corollary 4.2.7

If U is a continuous function then, the function k is continuous too on $\mathcal{D}_k$.

$c)$ **Differentiability of k**

Corollary 4.2.8

If $x \mapsto U(x)$ is a differentiable function then, k is differentiable and

$$\forall x \in \mathcal{D}_k^d,\ k'(x) = \frac{-U'(x)}{1 + U^2(x)}.$$

Example 4.2.17. *Determine the domain and the domain of differentiability of the following function and then calculate its derivative.*

$$k(x) = \operatorname{arccot}\left(\sqrt{\ln(x)}\right).$$

- *Determine the domain of k :*

$$\begin{aligned}\mathcal{D}_k &= \{x \in \mathbb{R},\ \ln(x) \geq 0\}\\ &= \{x \in \mathbb{R},\ x \geq 1\}\\ &= [1, +\infty[.\end{aligned}$$

- *Determine the domain of differentiability of k :*

$$\begin{aligned}\mathcal{D}_k &= \{x \in \mathbb{R},\ \ln(x) > 0\}\\ &= \{x \in \mathbb{R},\ x > 1\}\\ &=]1, +\infty[.\end{aligned}$$

- *Calculate the derivative of k. For all x in $]1,+\infty[$ we have*

$$k'(x) = \frac{-\left(\sqrt{\ln(x)}\right)'}{1+\left(\sqrt{\ln(x)}\right)^2} = \frac{-\dfrac{1/x}{2\sqrt{\ln(x)}}}{1+\ln(x)} = \frac{-1}{2\sqrt{\ln(x)}(1+\ln(x))}.$$

4.3 Hyperbolic functions and their inverses

4.3.1 The functions sinus hyperbolic and cosinus hyperbolic

Definition 4.3.1

- The sinus hyperbolic function denoted sh and defined from $\mathbb{R}$ to $\mathbb{R}$ as follows

$$\boxed{sh(x) = \frac{e^x - e^{-x}}{2}}$$

- The cosinus hyperbolic function denoted ch and defined from $\mathbb{R}$ to $[1,+\infty[$ as follows

$$\boxed{ch(x) = \frac{e^x + e^{-x}}{2}}$$

The representative graphs of the function $x \mapsto ch(x)$ and $x \mapsto sh(x)$ are as follows

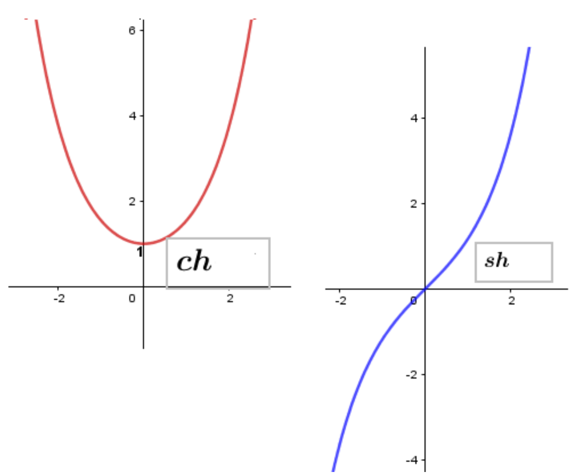


Figure 4.3.1: The graph of sh and ch functions.

Proposition 4.3.1

1. For all $x \in \mathbb{R}$, we have
   - the function $x \mapsto ch(x)$ is even and $x \mapsto sh(x)$ is odd.
   - the functions $x \mapsto ch(x)$ and $x \mapsto sh(x)$ are continuous
   - the functions $x \mapsto ch(x)$ and $x \mapsto sh(x)$ are differentiable and for all $x \in \mathbb{R}$,
   $$ch'(x) = sh(x) \text{ and } sh'(x) = ch(x).$$
2. $ch(0) = 1$, $\lim_{x \to -\infty} ch(x) = +\infty$ and $\lim_{x \to +\infty} ch(x) = +\infty$
3. $sh(0) = 0$, $\lim_{x \to -\infty} sh(x) = -\infty$ and $\lim_{x \to +\infty} sh(x) = +\infty$
4. for all $x \in \mathbb{R}$,
   $$ch^2(x) - sh^2(x) = 1.$$

Example 4.3.1. *Compute the following limits:*

$$\lim_{x \to -\infty} (sh(x) + ch(x)) \textit{ and } \lim_{x \to 0} \frac{sh(2x)}{x}.$$

Solution.

• $\lim_{x \to -\infty} (sh(x) + ch(x))$ *is an indeterminate form of type* $\infty - \infty$*, thus*

$$\begin{aligned} \lim_{x \to -\infty} (sh(x) + ch(x)) &= \lim_{x \to -\infty} \left(\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} \right) \\ &= \lim_{x \to -\infty} \left(\frac{2e^x}{2} \right) \\ &= \lim_{x \to -\infty} e^x = 0. \end{aligned}$$

• $\lim_{x \to 0} \frac{sh(2x)}{x}$ *is an indeterminate form of type* $\frac{0}{0}$.
Thus, we can use L'Hôpital's rule to calculate this limit. We have:

$$\lim_{x \to 0} \frac{sh(2x)}{x} = \lim_{x \to 0} \frac{2ch(2x)}{1} = 2.$$

4.3.2 The function tangent hyperbolic

Definition 4.3.2

The function tangent hyperbolic denoted th and defined from $\mathbb{R}$ to $]-1,1[$ as follows

$$\boxed{th(x) = \frac{sh(x)}{ch(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}}.$$

The representative graph of the function tangent hyperbolic is

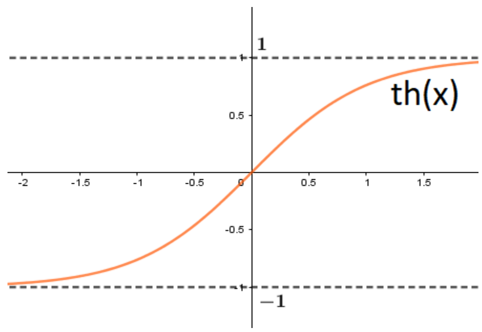


Figure 4.3.2: The graph of tangent hyperbolic function

Remark 4.3.1

The graph of tangent hyperbolic function admits an horizontal asymptotes $y = -1$ as x approaches $-\infty$ and other $y = 1$ as x approaches $+\infty$ i.e

$$\lim_{x \to -\infty} th(x) = -1 \text{ and } \lim_{x \to +\infty} th(x) = 1.$$

Proposition 4.3.2

- The function tangent hyperbolic is odd and continuous on $\mathbb{R}$.
- For all real number x the function $x \mapsto th(x)$ is differentiable and we have
$$th'(x) = \frac{1}{ch^2(x)}.$$
- $th(0) = 0$ and for all real number x we have $-1 < th(x) < 1$.

Example 4.3.2. *Consider two functions f and g defined by*

$$f(x) = \frac{1 - th(x)}{1 + th(x)} \quad \textit{and} \quad g(x) = \ln\left(\sqrt{f(x)}\right).$$

• *Domain of f.*

$$\mathcal{D}_f = \{x \in \mathbb{R} : \ 1 + th(x) \neq 0\} = \mathbb{R}.$$

• *Show that g is well defined. $\forall x \in \mathbb{R}$, we have*

$$\begin{aligned} -1 < th(x) < 1 &\implies (0 < 1 - th(x)) \ \textit{and}\ (0 < 1 + th(x)) \\ &\implies \frac{1 - th(x)}{1 + th(x)} > 0 \\ &\implies f(x) > 0 \end{aligned}$$

Thus g is well defined.
• *Check that $\forall x \in \mathbb{R}, \ g(x) = -x$. We have*

$$\begin{aligned} g(x) = \ln\left(\sqrt{f(x)}\right) &= \frac{1}{2}\ln(f(x)) \\ &= \frac{1}{2}\ln\left(\frac{1 - th(x)}{1 + th(x)}\right) \\ &= \frac{1}{2}\ln\left(\frac{1 - \dfrac{e^x - e^{-x}}{e^x + e^{-x}}}{1 + \dfrac{e^x - e^{-x}}{e^x + e^{-x}}}\right) \\ &= \frac{1}{2}\ln\left(\frac{2e^{-x}}{2e^x}\right) \\ &= -x. \end{aligned}$$

Notation 4.3.1

The hyperbolic cotangent function, denoted coth and defined on $\mathbb{R}^*$ by:

$$coth(x) = \frac{1}{th(x)}.$$

4.4 The inverse hyperbolic functions

4.4.1 The argument sinus hyperbolic function

Definition 4.4.1

The function $x \mapsto sh(x)$ is continuous and increasing from $\mathbb{R}$ to $\mathbb{R}$. Then, it's a bijective function and its inverse function denoted argsh or sh^{-1}.

The representative graphs of the function $x \mapsto sh(x)$ and $x \mapsto argsh(x)$ are as follows

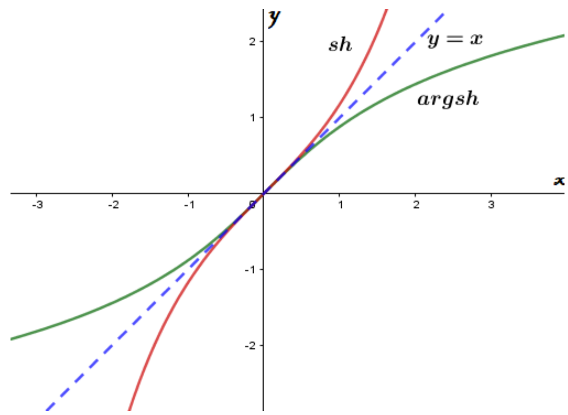


Figure 4.4.1: The graph of argument sinus hyperbolic function and its inverse

Remark 4.4.1

The functions $x \mapsto sh(x)$ and $x \mapsto argsh(x)$ are symmetrical with respect to the line $y = x$.

Proposition 4.4.1

- The function $x \mapsto argsh(x)$ is defined and continuous from $\mathbb{R}$ to $\mathbb{R}$.
- The function $x \mapsto argsh(x)$ is differentiable on $\mathbb{R}$ and $\forall x \in \mathbb{R}$, we have

$$(argsh)'(x) = \frac{1}{\sqrt{x^2+1}}.$$

- Since the function $x \mapsto sh(x)$ has a exponential expression we guess that its inverse $x \mapsto argsh(x)$ has a logarithmic expression
- By follows the steps to determine an inverse function we obtain that

$$argsh(x) = \ln(x + \sqrt{1+x^2}).$$

Example 4.4.1.

$$argsh(0) = 0 \textit{ since } sh(0) = 0$$
$$argsh\left(\frac{3}{4}\right) = \ln(2) \;\; \textit{since } sh(\ln(2)) = \frac{3}{4}$$

otherwise,

$$argsh(0) = \ln(0 + \sqrt{1+0^2}) = \ln(1) = 0,$$
$$argsh\left(\frac{3}{4}\right) = \ln\left(\frac{3}{4} + \sqrt{1 + \left(\frac{3}{4}\right)^2}\right) = \ln(2).$$

4.4.2 The argument cosinus hyperbolic function

Definition 4.4.2

The function $x \mapsto ch(x)$ is continuous and increasing from $[0, +\infty[$ to $[1, +\infty[$. Then, it's a bijective function and its inverse function denoted argch or ch^{-1}.

The representative graphs of the function $x \mapsto ch(x)$ and $x \mapsto argch(x)$ are as follows

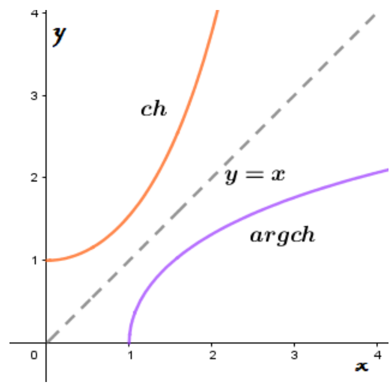


Figure 4.4.1: The graph of argument cosinus hyperbolic function and its inverse

Remark 4.4.2

The functions $x \mapsto ch(x)$ and $x \mapsto argch(x)$ are symmetrical with respect to the line $y = x$.

Proposition 4.4.2

- The function $x \mapsto argch(x)$ is defined and continuous from $[1, +\infty[$ to $[0, +\infty[$
- The function $x \mapsto argch(x)$ is differentiable on $]1, +\infty[$ and $\forall x \in]1, +\infty[$, we have

$$(argch)'(x) = \frac{1}{\sqrt{x^2 - 1}}.$$

- The logarithmic expression of the function $x \mapsto argch(x)$ is as follows

$$argch(x) = \ln(x + \sqrt{x^2 - 1}).$$

Example 4.4.2.

$$argch(1) = 0 \textit{ since } ch(0) = 1$$

$$argch\left(\frac{5}{4}\right) = \ln(2) \textit{ since } ch(\ln(2)) = \frac{5}{4}$$

otherwise

$$argch(1) = \ln(1+\sqrt{1^2-1}) = \ln(1) = 0,$$
$$argch\left(\frac{5}{4}\right) = \ln\left(\frac{5}{4}+\sqrt{\left(\frac{5}{4}\right)^2-1}\right) = \ln(2).$$

4.4.3 The argument tangent hyperbolic function

Definition 4.4.3

The function $x \mapsto th(x)$ is continuous and increasing from $\mathbb{R}$ to $]-1,1[$. Then, it's a bijective function and its inverse function is denoted argth or th^{-1}.

Proposition 4.4.3

- The argument tangent hyperbolic function is defined and continuous from $]-1,1[$ to $\mathbb{R}$.
- The function $x \mapsto argth(x)$ is differentiable on $]-1,1[$ and $\forall x \in]-1,1[$,

$$(Argth)'(x) = \frac{1}{1-x^2}.$$

- The logarithmic expression of the function $x \mapsto argth(x)$ is

$$argth(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

The representative graphs of the function $x \mapsto th(x)$ and $x \mapsto argth(x)$ are as follows

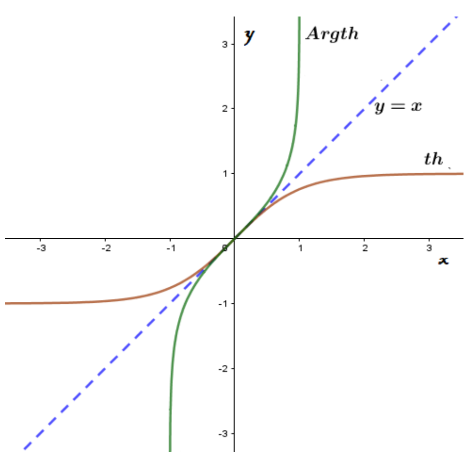


Figure 4.4.3: The graph of the argument tangent hyperbolic function and its inverse.

Example 4.4.3.

$$argth(0) = 0 \textit{ since } th(0) = 1,$$
$$argth\left(\frac{3}{5}\right) = \ln(2) \ \textit{ since } th(\ln(2)) = \frac{3}{5}.$$

otherwise,

$$argth(1) = \frac{1}{2}\ln\left(\frac{1+0}{1-0}\right) = \ln(1) = 0,$$
$$argsh\left(\frac{3}{5}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{3}{5}}{1-\frac{3}{5}}\right) = \frac{1}{2}\ln(4) = \ln(2).$$

4.5 Practice Exercises

Exercise 4.1

Determine the Domain and the domain of differentiability of the following functions and calculate their derivatives.

$$f_1(x) = arccos\left(\frac{1}{2+x}\right), \ f_2(x) = arcsin\left(\ln(x) - 1\right), \ f_3(x) = arctan\left(\sqrt{x^2-1}\right),$$

$$f_4(x) = \arcsin\left(e^x - 1\right), \ \ f_5(x) = \arccos\left(\frac{x-1}{x+1}\right), \ \ f_6(x) = \arctan\left(\sqrt{1-\ln(x)}\right)$$

Exercise 4.2

Using the L'Hôpital's Rule and compute the following limits:

$$\lim_{x\to e}\frac{arcsin\left(\ln(x)-1\right)}{x-e}, \quad \lim_{x\to+\infty}\frac{2arctan(\sqrt{x^2-1})-\pi}{1-th(x)}, \quad \lim_{x\to 0^+}\frac{3arccos\left(\frac{1}{2+x}\right)-\pi}{argsh(x^2)}$$

Exercise 4.3

Consider a function g defined by

$$g(x) = arcsin(3x - 4x^3)$$

1. Show that $3x - 4x^3 - 1 = -4(x - \frac{1}{2})^2(x+1)$ and $3x - 4x^3 + 1 = 4(x + \frac{1}{2})^2(1-x)$.
2. Determine the Domain of g.
3. Determine the domain of differentiability of g and then calculate g' the derivative of g.
4. Let h be a function defined by $h(x) = g(x) - 3arcsin(x)$.

(a) Determine the domain of differentiability of h and then calculate h' the derivative of h.

(b) Conclude that h in a constant function and then determine his value.

Exercise 4.4

Let $f(x) = \arcsin(\ln(\sqrt{x}))$.
1) Determine the Domain and the Domain differentiability of the function f
2) Calculate f' the differentiability of f.
2) Prove that f is bijective from $\mathcal{D}_f$ to K and determine K.
3) Determine f^{-1} (starting set, arrival set and the expression).

Exercise 4.5

Determine the Domain and the domain of differentiability of the following functions and then calculate their derivatives.

$$f_1(x) = sh\left(x^{1-x}\right),\ f_2(x) = ch\left(\frac{2}{1+x^2}\right), f_3(x) = th(x - e^x),$$

$$f_4(x) = argsh(x^2), f_5(x) = argch(sh(x)),\ f_6(x) = argth\left(\frac{e^{1-x} - 2}{2}\right).$$

Exercise 4.6

Let f be a function defined by

$$f(x) = arctan(ch(x))$$

1. Determine the Domain of f.
2. Show that f is continuous and even on $\mathcal{D}_f$.
3. Determine the domain of differentiability of f and then calculate f' the derivative of f .
4. Set up the table of variations related to f over it's Domain.
5. Using the intermediate value theorem (I.V.T) and show that:

   The equation $f(x) - 7x^3 = 0$ has at least a solution in $]0; \ln(10)[$.

6. Calculate the limit $\lim\limits_{x\to+\infty} \dfrac{2f(x) - \pi}{e^{-x}}$.

Exercise 4.7

1. Prove that $\forall x \in \mathbb{R}, \ \ sh(2x) = 2sh(x)ch(x)$.
2. Find all values of $x \in \mathbb{R}$ that satisfy $sh(2x) - 3th(x) - sh(x) = 0$.

Exercise 4.8

Let f be a function defined by $f(x) = argth\left(\dfrac{2}{x+4}\right)$.

(a) Determine the Domain of f.

(b) Determine the domain of differentiability of f and then calculate f' the differentiability of f.

(c) Set up the table of variations related to f over it's Domain.

(d) Give the expression logarithmic of f.

Exercise 4.9

Let f be a function defined by $f(x) = argch(\sqrt{x})$.

(a) Determine the Domain of f.

(b) Show that f is continuous on $\mathcal{D}_f$.

(c) Determine the domain of differentiability of f and then calculate f' the differentiability of f .

(d) Set up the table of variations related to f over it's Domain.

(e) Give the expression logarithmic of f.

(f) Using the mean value theorem (M.V.T) and show that

$$\forall x > 1, \ \frac{x-1}{2\sqrt{x}\sqrt{x-1}} < Argch(\sqrt{x}).$$

Conclude the limit $\lim\limits_{x \to 1} \dfrac{Argch(\sqrt{x})}{x-1}$.

Chapitre 5

TAYLOR Expansions

5.1 Negligible function

Definition 5.1.1

Let f and g be two functions defined on a set D and $x_0 \in \mathbb{R}$ or equals $\pm\infty$. We say that f is negligible with respect to g in the neighborhood of x_0 if there exists a function $\varepsilon : D \to \mathbb{R}$ such that $\forall x \in D$,

$$f(x) = g(x)\,\varepsilon(x) \qquad \text{with} \qquad \lim_{x\to x_0} \varepsilon(x) = 0.$$

We denote $f = o(g)$ as $x \to x_0$ or $f(x) = o(g(x))$ as $x \to x_0$ or simply $f(x) = o(g(x))$.

Note (in practice). If g does not vanish near x_0 (except possibly at x_0), saying that f is negligible with respect to g near x_0 is equivalent to saying

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0.$$

Example 5.1.1.

- $\sin(x) - x = o(x)$ *when* $x \to 0$ *since* $\lim_{x\to 0} \dfrac{\sin(x) - x}{x} = 0.$
- $\ln(x) - x + 1 = o((x-1))$ *when* $x \to 1$ *since* $\lim_{x\to 1} \dfrac{\ln(x) - x + 1}{x - 1} = 0.$
- *Let* $f(x) = x$ *and* $g(x) = x^2$. *Then*

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{x}{x^2} = \lim_{x\to+\infty} \frac{1}{x} = 0,$$

so $f = o(g)$ *as* $x \to +\infty$, *i.e.*

$$x = o(x^2).$$

More generally, one can show similarly that $x^n = o(x^p)$ *as* $x \to +\infty$ *whenever* $n < p$.

Example 5.1.2. *Show that* $\cos x - 1 + \frac{x^2}{2} = o(x^2)$ *when* $x \to 0$.

If $\lim_{x\to 0} \dfrac{\cos x - 1 + \dfrac{x^2}{2}}{x^2} = 0$ we have $\cos x - 1 + \dfrac{x^2}{2} = o(x^2)$ when $x \to 0$ So

$$\frac{\cos x - 1 + \frac{x^2}{2}}{x^2} = \frac{\cos x - 1}{x^2} + \frac{\frac{x^2}{2}}{x^2} = \frac{\cos x - 1}{x^2} + \frac{1}{2},$$

and

$$\frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{\cos^2(x) - 1}{x^2(1+\cos(x))} = \frac{-1}{(1+\cos(x))}\frac{\sin^2(x)}{x^2}.$$

Since $\lim_{t\to 0} \frac{\sin t}{t} = 1$, we obtain

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^2} = -\frac{1}{2}\cdot 1^2 + \frac{1}{2} = 0.$$

Therefore $\cos x - 1 + \frac{x^2}{2} = o(x^2)$ as $x \to 0$.

Properties

Proposition 5.1.1

Let $f, g, h, f_1, f_2, g_1, g_2$ be functions defined on the same set D and $x_0 \in D$ (or on the boundary).

1. If $f = o(g)$ as $x \to x_0$ and $g = o(h)$ as $x \to x_0$, then $f = o(h)_{x\to x_0}$ (transitivity)
2. If $f_1 = o(g_1)$ as $x \to x_0$ and $f_2 = o(g_2)$ as $x \to x_0$, then $f_1 \cdot f_2 = o(g_1 \cdot g_2)_{x\to x_0}$ (product allowed)
3. If $f_1 = o(g)_{x\to x_0}$ and $f_2 = o(g)$ as $x \to x_0$, then $f_1 + f_2 = o(g)_{x\to x_0}$ (sum allowed... or almost)
4. If $f = o(g)$ as $x \to x_0$ and f, g do not vanish near x_0, then $\frac{1}{g} = o\left(\frac{1}{f}\right)$ as $x \to x_0$

Proof. Everything is proved by taking the quotient or playing with the definition. For example, for 3.: If $f_1 = o(g)$ and $f_2 = o(g)$, then $f_1(x) = \varepsilon_1(x)g(x)$ and $f_2(x) = \varepsilon_2(x)g(x)$, thus

$$(f_1 + f_2)(x) = (\varepsilon_1(x) + \varepsilon_2(x))\, g(x) =: \varepsilon(x)\, g(x) \quad \text{with } \varepsilon(x) \to 0 \text{ as } x \to x_0,$$

so $f_1 + f_2 = o(g)$ as $x \to x_0$. □

Definition 5.1.2

Let f and g be two functions defined on a set $I \subset \mathbb{R}$, and let x_0 be a limit point of I. We say that f is *equivalent* to g as $x \to x_0$, and we write

$$f(x) \sim g(x) \quad \text{as } x \to x_0,$$

if and only if

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1,$$

provided that $g(x) \neq 0$ near x_0.

Remark 5.1.1

The relation $f(x) \sim g(x)$ is equivalent to saying that

$$f(x) = g(x) + o(g(x)) \quad \text{as } x \to x_0.$$

Example 5.1.3. *When $x \to 0$, we have the following equivalences:*

$$\sin x \sim x, \qquad 1 - \cos x \sim \frac{x^2}{2}, \qquad e^x \sim 1 + x, \qquad \ln(1 + x^2) \sim x^2.$$

5.2 Finite Expansions

5.2.1 Definitions

Definition 5.2.1

Let I be an interval and $f : I \to \mathbb{R}$ a function. Let x_0 be an element of I or one of its endpoints. Let n be a natural number. We say that f admits a *finite expansion of order* n at x_0 if there exist real numbers $a_0, a_1, \ldots, a_n$ and a function $\varepsilon : I \to \mathbb{R}$ such that

$$f(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + (x - x_0)^n \varepsilon(x) \quad \text{and} \quad \lim_{x \to x_0} \varepsilon(x) = 0.$$

- The polynomial $P_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n$ is called the main part of finite expansion at zero.
- $(x - x_0)^n \varepsilon(x)$ is called the rest of finite expansion at zero.

Example 5.2.1. *Since $1 - x^{n+1} = (1 - x)(1 + x + \cdots + x^n)$, we have*

$$\frac{1 - x^{n+1}}{1 - x} = 1 + x + \cdots + x^n,$$

thus

$$\frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^n + \frac{x^{n+1}}{1 - x}.$$

Therefore the function

$$f(x) = \frac{1}{1 - x}$$

admits a finite expansion at 0 of order n, with in this case

$$\varepsilon(x) = \frac{x}{1 - x}.$$

Properties

(1) Uniqueness: If f admits a finite expansion at x_0, then it's unique. That is, if

$$f(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + (x - x_0)^n\varepsilon_1(x)$$

and

$$f(x) = b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)^n + (x - x_0)^n\varepsilon_2(x),$$

with $\lim_{x\to x_0} \varepsilon_1(x) = \lim_{x\to x_0} \varepsilon_2(x) = 0$, then $a_0 = b_0,\ a_1 = b_1,\ \ldots,\ a_n = b_n$.

(2) Truncation: If f admits a finite expansion of order n at x_0,

$$f(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + (x - x_0)^n\varepsilon(x),$$

then for any $p < n$, f also admits a finite expansion of order p obtained by truncation:

$$f(x) = a_0 + a_1(x - x_0) + \cdots + a_p(x - x_0)^p + (x - x_0)^p\varepsilon_p(x),$$

where $\varepsilon_p(x) = (x - x_0)^{n-p}\varepsilon(x)$.

(3) Necessary condition. If f admits a finite expansion at x_0, then $\lim_{x\to x_0} f(x)$ exists and equals a_0. This provides a criterion to show that a function does *not* admit a finite expansion.

(4) Differentiability. If f admits a finite expansion of order $n \geq 1$ at x_0,

$$f(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + (x - x_0)^n\varepsilon(x),$$

then f is differentiable at x_0 and its derivative is given by $f'(x_0) = a_1$.

(5) Polynomial case. If f is a polynomial of degree n, then its finite expansion at any point x_0 is itself.

(6) Parity. Let f admit a finite expansion of order n at the origin (at zero):

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + x^n\varepsilon(x), \qquad \varepsilon(x) \xrightarrow[x\to 0]{} 0.$$

Then:

1. If f is **even**, all coefficients of odd degree vanish:

   $$a_1 = a_3 = a_5 = \cdots = 0,$$

   and the expansion contains only even powers of x:

   $$f(x) = a_0 + a_2x^2 + a_4x^4 + \cdots + a_{2p}x^{2p} + x^{2p}\varepsilon(x).$$

2. If f is **odd**, all coefficients of even degree vanish:

   $$a_0 = a_2 = a_4 = \cdots = 0,$$

   and the expansion contains only odd powers of x:

   $$f(x) = a_1x + a_3x^3 + a_5x^5 + \cdots + a_{2p+1}x^{2p+1} + x^{2p+1}\varepsilon(x).$$

5.2.2 Functions of class $\mathcal{C}^n$

Definition 5.2.2

Let $I \subset \mathbb{R}$ be an open interval. For $n \in \mathbb{N}^*$, we say that:
• $f : I \to \mathbb{R}$ is a function of class $\mathcal{C}^n$ on I if f is n times differentiable on I and $f^{(n)}$ is continuous. Here, $f^{(n)}$ is the n-th derivative of f i.e

$$f^{(0)} = f,\ f^{(1)} = f',\ f^{(2)} = f'',\ \cdots$$

• f is of class $\mathcal{C}^0$ on I if f is continuous on I.
• f is of class $\mathcal{C}^\infty$ if f is infinitely differentiable on I.

Example 5.2.2. *Let $f(x) = x^5 + 2x^4 - x^3 + 2x - 1$ be a function. We have*

$$\begin{aligned}
f^{(1)}(x) &= 5x^4 + 8x^3 - 3x^2 + 2,\\
f^{(2)}(x) &= 20x^3 + 24x^2 - 6x,\\
f^{(3)}(x) &= 60x^2 + 48x - 6,\\
f^{(4)}(x) &= 120x + 48,\\
f^{(5)}(x) &= 120,\\
f^{(6)}(x) &= 0,\\
&\vdots\\
f^{(n)}(x) &= 0,\ \forall n \geq 6.
\end{aligned}$$

Then f is a function of $\mathcal{C}^\infty$ on $\mathbb{R}$.

5.2.3 Taylor series

Theorem 5.2.1

Let f be a function of class $\mathcal{C}^n$ on I and $x_0 \in I$. Then, the Taylor-Young expansion of up to degree n associated with the function f at x_0 is

$$f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n),$$

where o is a function defined on I such that

$$\lim_{x \to x_0} o((x - x_0)^n) = 0.$$

Example 5.2.3. *The Taylor-Young expansion up to degree n associated with the function $x \mapsto e^x$ at $x_0 = 1$ is*

$$e^x = e^1 + e^1(x - 1) + \frac{e^1}{2}(x - 1)^2 + \cdots + \frac{e^1}{n!}(x - 1)^n + o((x - 1)^n).$$

Since

$$f'(x) = e^x,\ f''(x) = e^x \text{ and } f^{(n)}(x) = e^x,\ \forall n \in \mathbb{N}.$$

5.3 Finite expansion at the origin (at zero)

Definition 5.3.1

Let f be a real valued function. The function f is represented by a finite expansion at zero if there exist real numbers $c_0, c_1, \cdots, c_n$ and a real valued function ε such that

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + x^n \varepsilon(x).$$

where $\lim\limits_{x \to x_0} \varepsilon(x) = 0.$

Remark 5.3.1

Note that $x^n \varepsilon(x) = o(x^n)$.

Example 5.3.1. *Consider the following finite expansion* $1 + 2x - 4x^2 + 4x^3 + x^3\varepsilon(x)$.
• *The main part of this finite expansion is* $1 + 2x - 4x^2 + 4x^3$.
• *The rest of this finite expansion is* $x^3\varepsilon(x)$.

5.3.1 Maclaurin series

5.3.1.1 Definitions

Definition 5.3.2

The Maclaurin series is a special case of Taylor series expansion of a function centered at $x = 0$. If a function f is infinitely differentiable at $x = 0$, then its Maclaurin series is

$$f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 \cdots + \frac{f^{(n)}(0)}{n!}x^n + o(x^n), \tag{5.1}$$

where $\lim\limits_{x \to 0} o(x^n) = 0.$

Example 5.3.2. *Consider the function* $f(x) = \cos(x)$. *We want to find the Maclaurin series of this function. The first derivatives of* f *are*

$$f^{(1)}(x) = -\sin(x),\ f^{(2)}(x) = -\cos(x),\ f^{(3)}(x) = \sin(x), \cdots$$

Since $\sin(0) = 0$, *then, the Maclaurin polynomial of degree* n *associated with* f *is*

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^N \frac{x^{2N}}{(2N)!} + o(x^n)$$

The Maclaurin series of degree 2 *is given by* $\cos(x) = 1 - \frac{x^2}{2!} + o(x^2)$

The Maclaurin series of degree 5 *is given by* $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5)$

5.3.1.2 Basic Maclaurin expansion

Here, are Maclaurin series expansions, which can be found by using formula (5.1) for some commonly used functions.

- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n)$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + o(x^n)$
- $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} + o(x^n)$
- $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + o(x^{2n+1})$
- $\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$
- $\tan(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots + o(x^n)$
- $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + o(x^n)$
- $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + o(x^n)$
- $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + o(x^n)$
- $\sqrt{1+x} = \quad 1 + \frac{x}{2} - \frac{1}{8}x^2 + \dfrac{x^3}{16}\cdots + o(x^n)$
- $\sqrt{1-x} = \quad 1 - \frac{x}{2} - \frac{1}{8}x^2 - \dfrac{x^3}{16}\cdots + o(x^n)$
- $ch(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{2n!} + o(x^{2n+1})$
- $sh(x) = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$
- $th(x) = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots + o(x^n)$
- $Arcsin(x) = x + \frac{1}{2}\frac{x^3}{3} + \frac{3}{8}\frac{x^5}{5} + \cdots + \dfrac{1 \times 3 \times \cdots (2n-1)}{2 \times 4 \times \cdots \times 2n}\frac{x^{2n+1}}{2n+1} + o(x^{2n+2})$
- $Arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + o(x^{2n+2})$

5.3.2 Algebraic combinations of finite expansions

Let $a_0, a_1, \cdots, a_n, b_0, b_1, \cdots, b_n$ are real numbers and f, g be two functions such that

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + o(x^n)$$

and

$$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots + b_nx^n + o(x^n).$$

5.3.2.1 Sum and difference

Definition 5.3.3

- The sum $f + g$ or the difference $f - g$ has a finite expansion of the form:

$$(f \pm g)(x) = (a_0 \pm b_0) + (a_1 \pm b_1)x + \cdots + (a_n \pm b_n)x^n + o(x^n).$$

- Let λ be a real number. The constant multiple λf has a finite expansion of the form:

$$\lambda f(x) = \lambda a_0 + \lambda a_1 x + \cdots + \lambda a_n x^n + o(x^n).$$

Example 5.3.3. *Let $f(x) = e^x$ and $g(x) = \sqrt{x+1}$ be two functions.*
The finite expansions of f and g of degree 2 at zero are as follows:

$$f(x) = 1 + x + \frac{x^2}{2} + o(x^2) \ \text{ and } \ g(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + o(x^2).$$

Thus, the finite expansion of degree 2 at zero of $f - 2g$ is

$$\left[1 + x + \frac{x^2}{2}\right] - 2\left[1 + \frac{x}{2} - \frac{x^2}{8}\right] + o(x^2) = -1 + \frac{3x^2}{4} + o(x^2)$$

5.3.2.2 Product

Definition 5.3.4

The finite expansion at zero of $f \times g$ is obtained by keeping only the monomials of degree n or less, that means the product $f(x)g(x)$ admits a finite expansion:

$$f(x)g(x) = c_0 + c_1x_1 + c_2x^2 + \cdots + c_mx^m + o(x^n) \qquad \text{as } x \to 0,$$

where the coefficients c_m are given by the convolution

$$c_m = \sum_{i=0}^{m} a_i\, b_{m-i} \qquad (0 \leq m \leq n).$$

Example 5.3.4. *Let $f(x) = \cos(x)$ and $g(x) = \sin(x)$.*
We have

$$\cos(x) = 1 - \frac{x^2}{2} + o(x^3),$$

and

$$\sin(x) = x - \frac{2x^3}{3} + o(x^3).$$

Then the finite expansion of degree 3 at zero of $f \times g$ is

$$\left[1 - \frac{x^2}{2}\right] \times \left[x - \frac{x^3}{6}\right] + o(x^3) = x - \frac{2x^3}{3} + o(x^3).$$

5.3.2.3 Quotient

Definition 5.3.5

The quotient $\dfrac{f}{g}$ where $g(0) \neq 0$ has a finite expansion at zero of degree n and obtained by the euclidean division according to the increasing degrees to order n of the polynomial $(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + c_nx^n)$ by the polynomial $(b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots + b_nx^n)$.

Example 5.3.5. *Compute the finite expansion at zero of degree 3 of the function f such that $f(x) = \dfrac{x^3 + x + 4}{ch(x)}$.*
We have

$$x^3 + x + 4 = 4 + x + x^3 + o(x^3) \quad \text{and} \quad ch(x) = 1 + \frac{x^2}{2} + o(x^3).$$

Thus,

$$\begin{array}{rrrrr|l}
4 & +x & & & +x^3 & 1 + \frac{x^2}{2} \\
-4 & -2x^2 & & & & 4 + x - 2x^2 + \frac{x^3}{2} \\
\hline
& x & -2x^2 + x^3 & & & \\
& -x & & -\frac{x^3}{2} & & \\
\hline
& -2x^2 & & +\frac{x^3}{2} & & \\
& & 2x^2 & & +x^4 & \\
\hline
& & & +\frac{x^3}{2} & +x^4 &
\end{array}$$

Then, $f(x) = 4 + x - 2x^2 + \frac{x^3}{2} + o(x^3)$.

5.3.2.4 Composite

Proposition 5.3.1

Let $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$ be neighbourhoods of 0. Let

$$f : I \to J, \qquad h : J \to \mathbb{R}$$

be functions such that $\lim_{x\to 0} h(x) = 0$ and $0 \in I \cap I$. Assume there exists an integer $n \geq 1$ for which the following finite expansions hold:

$$h(x) = b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_n x^n + o(x^n), \qquad \text{as } x \to 0,$$
$$f(y) = a_0 + a_1 y + a_2 y^2 + \cdots + a_n y^n + o(y^n), \qquad \text{as } y \to 0,$$

where $a_i, b_j \in \mathbb{R}$. Then the composite $f \circ h : I \to \mathbb{R}$ admits a finite expansion at 0 of order n. More precisely,

$$f(h(x)) = c_0 + c_1 x^1 + c_2 x^2 + \cdots c_m x^m + o(x^n) \qquad \text{if } x \to 0,$$

where each coefficient c_m is obtained by substituting the polynomial part of h into the polynomial part of f and collecting terms of degree m (i.e. perform the formal substitution and keep only monomials of degree $\leq n$).

Example 5.3.6. *Consider the function $f(x) = e^{\sin(x)}$.*
Give the finite expansion of degree 3 at zero of f.
We have $\sin(x) = x - \frac{x^3}{6} + o(x^3)$ and $\sin(0) = 0$. Put $U = \sin(x)$ we get

$$e^{\sin(x)} = e^U = 1 + U + \frac{U^2}{2} + \frac{U^3}{6} + O(U^3)...(*)$$

Calculating U^2 and U^3:

$$U^2 = U \times U = \left(x - \frac{x^3}{6}\right)\left(x - \frac{x^3}{6}\right) + o(x^3) = x^2 + o(x^3).$$
$$U^3 = U^2 \times U = x^2 \times \left(x - \frac{x^3}{6}\right) + o(x^3) = x^3 + o(x^3).$$

Replacing these results in $(*)$ we get

$$e^{\sin(x)} = 1 + \left(x - \frac{x^3}{6}\right) + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$$
$$= 1 + x + \frac{x^2}{2} + o(x^3).$$

5.4 Derivation and integration of finite expansions

5.4.1 Derivative of a finite expansion

Let f be a function admitting a finite expansion of order n at x_0:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + (x - x_0)^n \varepsilon(x),$$

where $\varepsilon(x) \xrightarrow[x \to x_0]{} 0$. Then f is differentiable near x_0, and its derivative $f'(x)$ admits the finite expansion of order $n - 1$:

$$f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + na_n(x - x_0)^{n-1} + (x - x_0)^{n-1}\varepsilon_1(x),$$

where $\varepsilon_1(x) \to 0$ as $x \to x_0$.

Remark 5.4.1

The derivative of a finite expansion is obtained by term by term differentiation of the polynomial part, and the order decreases by one.

Example 5.4.1. *Consider a function f such that $f(x) = e^{2x}$. The finite expansion of f near 0 of order 4 is:*

$$f(x) = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + o(x^4),$$

i.e.

$$f(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + o(x^4). \tag{5.2}$$

Compute the derivative: we have $\forall x \in \mathbb{R}$:

$$f'(x) = 2e^{2x}.$$

The finite expansion of f' near 0 of order 3 is:

$$\begin{aligned} f'(x) &= 2\left(1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + o(x^3),\right) \\ &= 2 + 4x + 4x^2 + \frac{8}{3}x^3 + o(x^3) \end{aligned}$$

Differentiating the finite expansion in (5.2) with respect to x and we get

$$f'(x) = 2 + 4x + 4x^2 + \frac{8}{3}x^3 + o(x^3)$$

which is exactly the finite expansion of order 3 of the derivative near 0.

5.4.2 Integration of a finite expansion

If f admits a finite expansion of order n at x_0:

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + (x-x_0)^n\varepsilon_1(x),$$

then its primitive F defined by

$$F(x) = \int_{x_0}^{x} f(t)\,dt$$

admits the finite expansion of order $n+1$:

$$F(x) = a_0(x-x_0) + \frac{a_1}{2}(x-x_0)^2 + \frac{a_2}{3}(x-x_0)^3 + \cdots + \frac{a_n}{n+1}(x-x_0)^{n+1} + (x-x_0)^{n+1}\varepsilon_2(x),$$

where $\varepsilon_2(x) \to 0$ as $x \to x_0$.

Remark 5.4.2

Integrating a finite expansion increases the order by one, and each coefficient is divided by its new exponent.

Example 5.4.2. *Consider a function f such that $f(x) = \dfrac{1}{1+x^2}$. The finite expansion of f near 0 of order 4 is*

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + o(x^4). \tag{5.3}$$

We know that any primitive of the function f is

$$F(x) = \arctan x + C,$$

for some constant C. Thus, by integrating term by term (5.3) we get

$$\int \frac{1}{1+x^2}\,dx = x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^5).$$

Therefore, we deduce the finite expansion of $\arctan x$ *near 0 of order 5:*

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^5).$$

5.4.3 Finite expansions at a point $x_0 \neq 0$

A function f admits a finite expansion of order n at x_0 if and only if the function

$$g(h) = f(x_0 + t)$$

admits a finite expansion of order n at 0. More precisely, if

$$g(h) = a_0 + a_1 t + \cdots + a_n t^n + t^n \varepsilon(t)$$

is the finite expansion of g at 0, then

$$f(x) = a_0 + a_1(x-x_0) + \cdots + a_n(x-x_0)^n + (x-x_0)^n\varepsilon(x-x_0)$$

is the finite expansion of f at x_0.

In practice. If we know how to compute the finite expansion of f at 0, then the finite expansion of f at any point x_0 can be deduced from that of $f(x_0+h)$ at 0.

Definition 5.4.1

A function f can be represented by a finite expansion at point $x_0 \neq 0$ if the function $t \mapsto f(t+x_0)$ with $t = x - x_0$ can be represented by finite expansion at zero. Moreover, the finite expansion of f at x_0 is:

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + ... + a_n(x-x_0)^n + o((x-x_0)^n)$$

Example 5.4.3. *Let* $f(x) = \dfrac{2}{2+x}$. *Give the finite expansion of f of degree 3 at* $x_0 = -1$.
Put $t = x - (-1) = x + 1$. *If* $x = -1$ *then,* $t = 0$. *Moreover*

$$f(x) = \frac{2}{2+x} = \frac{2}{2+(t-1)} = 2 \cdot \frac{1}{1+t}.$$

The finite expansion of $\dfrac{1}{1+t}$ *is* $1 - t + t^2 - t^3 + o(t^3)$. *Thus,*

$$2 \cdot \frac{1}{1+t} = 2 - 2t + 2t^2 - 2t^3 + o(t^3).$$

So,

$$f(x) = 2 - 2(x+1) + 2(x+1)^2 - 2(x+1)^3 + o((x+1)^3).$$

Yields

$$f(x) = -2x + 2(x+1)^2 - 2(x+1)^3 + o((x+1)^3).$$

Example 5.4.4. *Compute the finite expansion of the function* $f(x) = \cos x$ *at the point* $\frac{\pi}{3}$.

We consider the function $g(h) = \cos(\frac{\pi}{3} + t)$ *and look for its finite expansion at* 0 *up to order* 3.

We know that

$$\cos\left(\frac{\pi}{3} + t\right) = \cos\frac{\pi}{3}\cos t - \sin\frac{\pi}{3}\sin t.$$

Using known expansions near 0:

$$\cos t = 1 - \frac{t^2}{2} + o(t^2), \qquad \sin t = t - \frac{t^3}{6} + o(t^3),$$

we obtain

$$g(t) = \frac{1}{2}\left(1 - \frac{t^2}{2}\right) - \frac{\sqrt{3}}{2}\left(t - \frac{t^3}{6}\right) + o(t^3),$$

that is,

$$g(t) = \frac{1}{2} - \frac{\sqrt{3}}{2}t - \frac{1}{4}t^2 + \frac{\sqrt{3}}{12}t^3 + o(t^3).$$

Now replacing $t = x - \frac{\pi}{3}$, *we get*

$$f(x) = \cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + o\big((x - \tfrac{\pi}{3})^3\big).$$

Since the computation of the finite expansion at any point x_0 reduces to that at 0, we shall henceforth only study the case of finite expansions at the origin.

5.4.4 Finite expansions near infinity

Definition 5.4.2

A function f can be represented by a finite expansion near infinity ($\pm\infty$) if the function $t \mapsto f(\frac{1}{t})$ with $t = \dfrac{1}{x}$ can be represented by finite expansion at zero and

$$f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + ... + \frac{a_n}{x^n} + o(\frac{1}{x^n}).$$

Example 5.4.5. *Let $f(x) = \dfrac{x}{x-1}$. Giving the finite expansion of f of degree 3 near $+\infty$.*
Put $t = \dfrac{1}{x}$. Thus, if $x \to +\infty$ then $t \to 0$ and we have

$$f(x) = \frac{x}{x-1} = \frac{1/t}{1/t-1} = \frac{1}{1-t} \tag{5.4}$$

The finite expansion of $\dfrac{1}{1-t}$ is $1 + t + t^2 + t^3 + o(t^3)$.
From (5.4) we deduce that

$$f(x) = \frac{x}{x-1} = 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + o(\frac{1}{x^3}) \qquad \textit{with } x \textit{ near } +\infty$$

5.5 Application of finite expansion

5.5.1 Compute limits

Finite expansion is very effective in removing indeterminate forms for calculating limits. Generally we use the finite expansion of degree 2.

Example 5.5.1. $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+3x} - \sqrt{1-3x}}{\ln(1+x)}$ *is an indeterminate form of the type $\dfrac{0}{0}$.*
We have

- $\sqrt{1+3x} = 1 + \dfrac{3x}{2} - \dfrac{9x^2}{8} + x^2\varepsilon(x)$.
- $\sqrt{1-3x} = 1 - \dfrac{3x}{2} - \dfrac{9x^2}{8} + x^2\varepsilon(x)$.
- $\ln(1+x) = x - \dfrac{x^2}{2} + x^2\varepsilon(x)$,

Thus,

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{1+3x}-\sqrt{1-3x}}{\ln(1+3x)} &= \lim_{x\to 0}\frac{\left(1+\frac{3x}{2}-\frac{9x^2}{8}\right)-\left(1-\frac{3x}{2}-\frac{9x^2}{8}\right)+x^2\varepsilon(x)}{x-\frac{x^2}{2}+x^2\varepsilon(x)}\\ &= \lim_{x\to 0}\frac{3x+x^2\varepsilon(x)}{x-\frac{x^2}{2}+x^2\varepsilon(x)}\\ &= \lim_{x\to 0}\frac{3+x\varepsilon(x)}{1-\frac{x}{2}+x\varepsilon(x)} = 3.\end{aligned}$$

Example 5.5.2. $\lim_{x\to -1}\frac{1-e^{x+1}}{\sin(x+1)}$ *is an indeterminate form of the type* $\frac{0}{0}$.
Put $t = x+1$*. Thus, if* x *goes to* -1 *then* t *goes to* 0 *and*

$$\lim_{x\to -1}\frac{1-e^{x+1}}{\sin(x+1)} = \lim_{t\to 0}\frac{1-e^t}{\sin(t)}.$$

Moreover, we have the following finite expansions:

- $1-e^t = 1-(1+t+\frac{t^2}{2})+t^2\varepsilon(t) = -t-\frac{t^2}{2}+t^2\varepsilon(t).$
- $\sin(t) = t+t^2\varepsilon(t).$

Then,

$$\begin{aligned}\lim_{t\to 0}\frac{1-e^t}{\sin(t)} &= \lim_{t\to 0}\frac{-t-\frac{t^2}{2}+t^2\varepsilon(t)}{t+t^2\varepsilon(t)}\\ &= \lim_{t\to 0}\frac{-1-\frac{t}{2}+t\varepsilon(t)}{1+\varepsilon(t)} = -1\end{aligned}$$

Example 5.5.3. $\lim_{x\to -\infty}\left(x-x^2\ln\left(1+\frac{1}{x}\right)\right)$ *is an indeterminate form of the type* $0\cdot\infty$).
Put $y = \frac{1}{x}$*. Thus if* x *goes to* $-\infty$ *then* t *goes to* 0 *and we have*

$$\begin{aligned}x-x^2\ln\left(1+\frac{1}{x}\right) &= \frac{1}{y}-\frac{1}{y^2}\ln(1+y)\\ &= \frac{1}{y}-\frac{1}{y^2}\left[y-\frac{y^2}{2}+y^2\varepsilon(y)\right]\\ &= \frac{1}{2}-\varepsilon(y).\end{aligned}$$

Therefore,

$$\lim_{x\to-\infty}\left(x - x^2\ln\left(1+\frac{1}{x}\right)\right) = \lim_{y\to 0}\left[\frac{1}{2} - \varepsilon(y)\right]$$
$$= \frac{1}{2}.$$

5.5.2 Tangent and position

Definition 5.5.1

Let f be a function has a finite expansion at x_0 of degree k with $k \geq 2$ such that

$$f(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots + c_k(x-x_0)^k + o((x-x_0)^k).$$

where $c_i,\ i = 0, \cdots, k$ are real numbers. Then, the graph representing the function f admits at the point $(x_0, f(x_0))$ a tangent of equation:

$$y = c_0 + c_1(x - x_0)$$

which is the finite expansion of degree 1. The position of the graph relative to its tangent is given by the sign of the principal part of $f(x) - y$.

Example 5.5.4. *Let* $f(x) = \sqrt{1+2x}$.

- *Use the finite expansion and determine the equation of the tangent of a function f defined by at the point* $(0, f(0))$.
- *Determine the position of the graph relative to this tangent near* $(0, f(0))$.

Solution:
• *We have*

$$\sqrt{1+2x} = 1 + x - \frac{x^2}{2} + o(x^2).$$

Thus, the tangent equation near $(0, f(0))$ *is*

$$y = 1 + x.$$

• *We take the principal part of the difference* $f(x) - y$ *we get* $-\frac{x^2}{2} < 0$. *We conclude that the graph* (C_f) *is below its tangent near* $(0, f(0))$.

Example 5.5.5. *Consider a function g defined by* $g(x) = \sin(x-1)$.

- *Use the finite expansion and determine the equation of the tangent of the function at the point* $(1, g(1))$.
- *Determine the position of the graph relative to this tangent near* $(1, g(1))$.

Solution:

• *Put* $t = x - 1$. *Since* x *approaches* 1 *so,* t *approaches* 0 *and*

$$\sin(x-1) = \sin(t) = t - \frac{t^3}{6} + o(t^3).$$

Thus,

$$\sin(x-1) = (x-1) - \frac{(x-1)^3}{6} + o((x-1)^3).$$

Then, the tangent equation near $(1, g(1))$ *is*

$$y = x - 1.$$

• *We take the principal part of the difference* $g(x) - y$ *we get* $g(x) - y = \dfrac{(x-1)^3}{6} \begin{cases} < 0 & \text{if } x < 1, \\ > 0 & \text{if } x > 1. \end{cases}$.

- *If* $x < 1$, *the graph* (C_g) *is below its tangent near* $(1, g(1))$
- *If* $x > 1$, *the graph* (C_g) *is above its tangent near* $(1, g(1))$.

5.5.3 Asymptote and position

Definition 5.5.2

Let f be a function has a finite expansion as $x \to \infty$ of degree k with $k \geq 2$:

$$f(x) = ax + c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_k}{x^k} + o\left(\frac{1}{x^k}\right).$$

where $a, \ c_i, i = 0, \cdots k$ are real numbers. Then, the graph representing the function f admits as $x \to \infty$ an asymptote of equation:

$$y = ax + c_0.$$

The position of the graph relative to its asymptote is given by the sign of the principal part of $f(x) - y$.

Remark 5.5.1

- If $a \neq 0$ we have an oblique asymptote.
- If $a = 0$ we have an horizontal asymptote.

Example 5.5.6. *Let consider two functions* f *and* g *such that*

$$f(x) = \frac{2x}{x-1} \ \text{ and } \ g(x) = \frac{x^2+3}{x-1}.$$

- *Determine the equations of the asymptotes* (Δ_1) *and* (Δ_2) *as* $x \to \pm\infty$ *of* f *and* g *respectively.*

- *Discus the position of the graphs relative to their asymptotes near $\pm\infty$.*

Solution: *Put $t = \dfrac{1}{x}$. Thus, if x goes to $\pm\infty$ then t goes to 0.*

- *We have*
$$f(x) = \frac{2x}{x-1} = \frac{2/t}{1/t-1} = \frac{2}{1-t} \tag{5.5}$$
The finite expansion of $\dfrac{1}{1-t}$ is $1+t+t^2+t^3+o(t^3)$.
From (5.5) we deduce that
$$\begin{aligned} f(x) &= 2+2t+2t^2+2t^3+o(t^3) \\ &= 2+\frac{2}{x}+\frac{2}{x^2}+o\left(\frac{1}{x^2}\right). \end{aligned}$$
Then the equation of (Δ_1) as $x \to \pm\infty$ is
$$(\Delta_1) : y = 2.$$
Moreover, we have $f(x) - y = \dfrac{2}{x}$ (principal part). Then, the graph (C_f) is below its asymptote (Δ_1) as $x \to -\infty$ and above the asymptote (Δ_1) near $+\infty$.

- *Concerning the function g we have*
$$\begin{aligned} g(x) &= \frac{x^2+3}{x-1} \\ &= \frac{1/t^2+3}{1/t-1} \\ &= \frac{1}{t}\frac{1+3t^2}{1-t}. \end{aligned}$$
The finite expansion:
$$\begin{aligned} \frac{1}{t}\frac{1+3t^2}{1-t} =& \frac{1}{t}\frac{1}{1-t}(1+3t^2) \\ &= \frac{1}{t}(1+t+t^2+t^2\varepsilon(t))(1+3t^2+t^2\varepsilon(t)) \\ &= \frac{1}{t}\left(1+t+4t^2+t^2\varepsilon(t)\right) \\ &= \frac{1}{t}+1+4t+t\varepsilon(t). \end{aligned}$$
We deduce that
$$\begin{aligned} g(x) &= \frac{1}{t}+1+4t+o(t) \\ &= x+1+\frac{4}{x}+o\left(\frac{1}{x}\right). \end{aligned}$$

Thus, the equation of (Δ_2) *as* $x \to \pm\infty$ *of* g *is*

$$(\Delta_2) : y = x + 1.$$

Moreover, we have $g(x) - y = \dfrac{4}{x}$ *(principal part). Then, the graph* (C_g) *is below its asymptote* (Δ_2) *as* $x \to -\infty$ *and above the asymptote* (Δ_2) *near* $+\infty$*.*

5.6 Generalized finite expansion by regularization

Let f be a function defined in the neighbourhood of 0, except perhaps at 0. We assume that f does not admit a finite expansion in the neighbourhood of 0, but that the function $x^p\, f(x)$ admits such an expansion. We can then write in the neighbourhood of 0

$$x^p\, f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + o(x^n).$$

Thus

$$f(x) = \frac{1}{x^p}\left[a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + o(x^n)\right].$$

Hence f admits a *generalized finite expansion of order* $n - p$ *at* 0.

Example 5.6.1. *Let* $f(x) = \dfrac{1}{x}$*. Then* f *does not admit a finite expansion at* 0 *(since* $f(x) \to +\infty$ *as* $x \to 0^+$*). But for* $p = 1$ *we get*

$$x^p f(x) = x \cdot \frac{1}{x} = 1,$$

which admits the trivial finite expansion

$$x\, f(x) = 1 + o(1) \qquad \text{as } x \to 0.$$

Thus f *admits a generalized finite expansion of order* 0 *with* $p = 1$*.*

Example 5.6.2. *Consider the function*

$$f(x) = \frac{e^x - 1}{x^2}.$$

Multiplying by x^2 *gives*

$$x^2 f(x) = e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + o(x^n).$$

Hence

$$f(x) = \frac{1}{x} + \frac{1}{2} + \frac{x}{6} + \cdots + o(x^{n-2}),$$

which shows that f *admits a generalized finite expansion of order* $n - 2$ *at* 0*.*

Example 5.6.3. *Let*

$$f(x) = \frac{\sin x}{x^3}.$$

Multiplying by x^3 *we obtain*

$$x^3 f(x) = \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5).$$

Thus

$$f(x) = \frac{1}{x^2} - \frac{1}{6} + \frac{x^2}{120} + o(x^2),$$

which is a generalized finite expansion at 0.

Example 5.6.4. *Take*

$$f(x) = \frac{1 - \cos x}{x^4}.$$

Multiplication by x^4 *gives*

$$x^4 f(x) = 1 - \cos x = \frac{x^2}{2} - \frac{x^4}{24} + o(x^4).$$

Hence

$$f(x) = \frac{1}{2x^2} - \frac{1}{24} + o(1),$$

which is a generalized finite expansion of order 0 *at* 0.

5.7 Practice Exercises

Exercise 5.1

1. Find the Taylor series of degree n for the following functions:

$$f_1(x) = x^4 - 3x^2 + 1,\ x_0 = 1,\ f_2(x) = xe^x,\ x_0 = -1,$$

$$f_3(x) = \frac{1}{x},\ x_0 = 1,\ f_4(x) = \ln(x-1),\ x_0 = 2.$$

2. Find the Taylor series of degree 3 for the function $f(x) = sin(x)$ at $x_0 = \pi$.

3. Find the Maclaurin series of indicated degree for each of the following functions:

$$f(x) = \cos(\pi x),\ n = 4,\ g(x) = \ln(2 + x),\ n = 3,\ h(x) = \sqrt{x+3},\ n = 2.$$

Exercise 5.2

Give the finite expansion at zero of indicated degree for the following functions

$$f_1(x) = 5 + x + th(x) - sin(x) + arctan(x),\ n = 3,\ f_2(x) = \sqrt{1-x} - \sqrt{1+x},\ n = 3,$$

$$f_3(x) = \tan(x) \cdot e^x,\ n = 3,\ f_4(x) = \frac{2 - x^2}{1 - x},\ n = 3,\ f_5(x) = ch(x)\sqrt{1 + 2x},\ n = 2$$

$$f_6(x) = \frac{\ln(1+x)}{1 + sh(x)},\ n = 2,\ f_7(x) = \frac{2 + cos(x)}{x^3 + x + 1},\ n = 3,\ f_8(x) = \frac{e^{2x}}{1 + e^x},\ n = 3,$$

$$f_9(x) = sh(1 - e^x),\ n = 3,\ f_{10}(x) = \ln(2 + th(x)),\ n = 3,\ f_{11}(x) = \frac{(1-x)^3}{1 + 2tan(x)},\ n = 2.$$

Exercise 5.3

1. Determine the finite expansion at x_0 of indicated degree for the following functions

$$f(x) = ch(2x-1),\ x_0 = \frac{1}{2},\ n = 4,\ g(x) = sin(x), x_0 = \frac{\pi}{2},\ n = 4,$$

$$h(x) = \frac{\ln(x)}{1 - sh(1-x)}, x_0 = 1,\ n = 3.$$

2. Determine the finite expansion as $x \to \infty$ of indicated degree for the following functions

$$f(x) = \ln\left(\frac{x-1}{x+1}\right),\ (\text{near } +\infty,\ n=3),\ g(x) = \sqrt{x^2 - xe^{\frac{1}{x}}},\ (\text{near } +\infty,\ n=2),$$

$$h(x) = x + \sqrt{1+x^2},\ (\text{near } -\infty,\ n=4).$$

3. Using the finite expansion and compute the following limits:

$$\lim_{x\to 0} \frac{ch(x)\sqrt{1+2x} - 1}{tan(x)e^x},\ \lim_{x\to 1} \frac{ln(x)}{1+\cos(\pi x)},\ \lim_{x\to -\infty} (x + \sqrt{1+x^2}).$$

Exercise 5.4

Consider the following function $f(x) = \dfrac{e^{3x}}{1+\ln(1+2x)}$.

1. Give the finite expansion of degree 3 at zero for the function f.
2. Check that $4f^{(1)}(0) + 3f^{(2)}(0) + f^{(3)}(0) = 0$.
3. Let g be a function defined by

$$g(x) = xf\left(\frac{1}{x}\right).$$

   Find the finite expansion of degree 2 near $+\infty$ for the function g.

Chapitre 6

Integrals and Primitives

6.1 Primitive functions

Definition 6.1.1

Let I be a non-empty open interval in $\mathbb{R}$ and f be a function defined on I. A primitive (antiderivative) of f is a function F defined and differentiable on I such that:

$$F'(x) = f(x) \text{ for all } x \in I.$$

The derivative of function f denoted f' and generally the primitive of f denoted by F.

Remark 6.1.1

Let a be a real number, f a function defined on I and F its primitive. Then,

i) f has an infinity primitives of the form $F + c$, where $c \in \mathbb{R}$.

ii) The function $x \mapsto F(x) - F(a)$ is the unique primitive of f which cancels out at a.

iii) We say that $G : I \to \mathbb{R}$ is a primitive of f on I if and only if the difference $G - F$ is constant on I.

Theorem 6.1.1

If f is continuous on I, then f has a primitive function F on I.

Definition 6.1.2

Let f be a real function. We introduce the symbol

$$\int f(x)\,dx,$$

called the *indefinite integral* of f with respect to x. If F is an antiderivative (primitive) of f on an interval I, i.e.

$$F'(x) = f(x) \quad \text{for all } x \in I,$$

then we write

$$\int f(x)\,dx = F(x) + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant (constant of integration).

Proposition 6.1.1

If F is a primitive function of f and G is a primitive function of g on an interval I and $\alpha, \beta \in \mathbb{R}$, then

$$\int [\alpha f(x) \pm \beta g(x)] dx = \alpha \int f(x)dx \pm \beta \int g(x)dx$$
$$= \alpha F(x) \pm \beta G(x) + c.$$

The following table lists the primitives of the usual functions:

Function	Primitive	Domain
0	c	$\mathbb{R}$
a	ax+c	$\mathbb{R}$
$x^n,\ n \in \mathbb{N}$	$(1/(n+1))x^{n+1}$+c	$\mathbb{R}$
$cos(x)$	$sin(x) + c$	$\mathbb{R}$
$sin(x)$	$-cos(x) + c$	$\mathbb{R}$
$\frac{1}{x}$	$Ln\lvert x\rvert + c$	$\mathbb{R}^*$
$e^{ax},\ a \in \mathbb{R}^*$	$(1/a)e^{ax}$+c	$\mathbb{R}$
$ch(x)$	$sh(x)$	$\mathbb{R}$
$sh(x)$	$ch(x)$	$\mathbb{R}$
$\frac{1}{1+x^2}$	$arctg(x) + c$	$\mathbb{R}$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}argtan(\frac{x}{a}) + c$	$a \neq 0$ and $x \in \mathbb{R}$.
$\frac{1}{cos^2(x)}$	$tang(x) + c$	$\mathbb{R} - \{(2k+1)\frac{\pi}{2},\ k \in \mathbb{Z}\}$
$\frac{1}{ch^2(x)}$	$th(x) + c$	$\mathbb{R}$
$\frac{1}{\sqrt{1-x^2}}$	$arcsin(x) + c$	$]-1;1[$
$\frac{-1}{\sqrt{1-x^2}}$	$arccos(x) + c$	$]-1;1[$
$\frac{1}{\sqrt{x^2+1}}$	$argsh(x) + c$	$\mathbb{R}$
$\frac{1}{\sqrt{x^2-1}}$	$argch(x) + c$	$]1;+\infty[$

Example 6.1.1.

- $\int (x^2 + 2x)dx = \frac{1}{3}x^3 + x^2 + c$
- $\int (cos(x) + sin(x))dx = sin(x) - cos(x) + c$

Let U be a differentiable function on I. According to the composite of functions, we have the following table:

Function	Primitive	Remarks	Function	Primitive	Remarks
$u' \times u^n,\ n \in \mathbb{N}$	$\frac{1}{n+1}u^{n+1}+c$	/	$\frac{u'}{1+u^2}$	$arctg(u)+c$	/
$\frac{u'}{u^n}, n \in \mathbb{N}, n>1$	$\frac{-1}{(n-1)u^{n-1}}+c,$	$u \neq 0$ on I	$u'ch(u)$	$sh(u)$	/
$u'e^u$	e^u+c	/	$u'sh(u)$	$ch(u)$	/
$\frac{u'}{2\sqrt{u}}$	$\sqrt{u}+c$	$u>0$ on I	$\frac{u'}{ch^2(u)}$	$th(u)+c$	/
$\frac{u'}{u}$	$Ln\|u\|+c$	$u \neq 0$ on I	$\frac{u'}{\sqrt{1-u^2}}$	$arcsin(u)+c$	$-1<u<1$
$u'cos(u)$	$sin(u)+c$	/	$\frac{-u'}{\sqrt{1-u^2}}$	$arccos(u)+c$	$-1<u<1$
$u'sin(u)$	$-cos(u)+c$	/	$\frac{u'}{\sqrt{u^2+1}}$	$argsh(u)+c$	/
$\frac{u'}{\sqrt{u^2-1}}$	$argch(u)+c$	$u>1$	$\frac{u'}{1-u^2}$	$argth(u)+c$	$-1<u<1$

Example 6.1.2.

$$\begin{aligned}\int x^2 e^{x^3}dx &= \frac{1}{3}\int 3x^2 e^{x^2}dx\\ &= \frac{1}{3}e^{x^3}+c,\end{aligned}$$

$$\begin{aligned}\int \frac{xdx}{\sqrt{1+x^2}} &= \int \frac{2xdx}{2\sqrt{1+x^2}}\\ &= \sqrt{1+x^2}+c.\end{aligned}$$

6.2 Definite integral.

6.2.1 Definitions

Definition 6.2.1

Let f be a continuous function on $[a,b]$ and F its primitive. A definite integral is denoted by

$$\int_a^b f(x)dx$$

Here:
- a is called the lower limit of integration,
- b is called the upper limit of integration,
- $f(x)$ is the integrand,
- dx is the differential.

Its value is the difference between the values of F at the end points i.e

$$\int_a^b f(x)dx = F(b) - F(a) = \Big[F(x)\Big]_a^b.$$

Example 6.2.1.

$$\begin{aligned}\int\limits_0^{ln(2)} e^{2x}dx &= \frac{1}{2}\int\limits_0^{ln(2)} 2e^{2x}dx \\ &= \frac{1}{2}\left[e^{2x}\right]_0^{ln(2)} \\ &= \frac{1}{2}(e^{ln(4)} - 1) = \frac{3}{2}.\end{aligned}$$

$$\begin{aligned}\int\limits_0^1 \frac{2x}{3+x^2}dx &= \left[ln|3+x^2|\right]_0^1 \\ &= ln(4) - ln(3) \\ &= ln\left(\frac{4}{3}\right).\end{aligned}$$

Remark 6.2.1

(a) $\int f(x)dx$ denotes a function from I to $\mathbb{R}$ while the integral $\int_a^b f(x)dx$ denotes a real number.

(b) In the notation $\int_a^b f(x)dx$, the variable x can be replaced by any other variable without occurrence in f. We have for example

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(y)dy = ...$$

(c) It immediately follows that $\int_a^a f(x)dx = F(a) - F(a) = 0.$

Proposition 6.2.1

Let f be a continuous function on I and $a \in I$. Then, the function F defined by:

$$F(x) = \int_a^x f(t)dt$$

is the primitive of f which cancels at point a.

Example 6.2.2. *Consider a function f such that*

$$f(x) = \frac{1}{x-4}.$$

Find F the primitive of f such that and $F(1) = 0$.
By definition we have:

$$\begin{aligned} F(x) &= \int_1^x \frac{1}{t-4} dt \\ &= \Big[Ln|t-4| + c \Big]_1^x \\ &= Ln|x-4| - Ln(3). \end{aligned}$$

Proposition 6.2.2

Let f and g be two integrable functions on $[a, b]$ and α, β are real numbers. Then,

i) $\int_a^b [\alpha f(x) \pm \beta g(x)] dx = \alpha \int_a^b f(x)dx \pm \beta \int_a^b g(x)dx$

ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

iii) $\forall c \in [a, b], \int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx.$

iv) If $f(x) \geq 0 \ \forall x \in [a, b]$, then $\int_a^b f(x)dx \geq 0.$

v) If $f(x) \geq g(x) \ \forall x \in [a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx.$

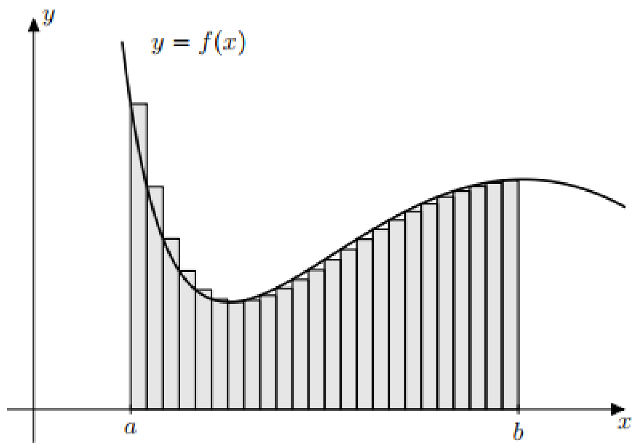
vi) $\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)| \, dx.$

6.2.2 Riemann Sums

Definition 6.2.2

Suppose f is a function on $[a, b]$. Suppose further that $f(x)$ is positive on $[a, b]$. We define

$$\int_a^b f(x)dx \text{ by the area between } f(x) \text{ and the } x-\text{axe for all } x \text{ in } [a, b].$$



An approximation of area by the Riemann integral of a function f over the interval $[a, b]$.

Basic idea:
A Riemann sum is a way of approximating an integral by summing the areas of vertical rectangles and has the form

$$Area = \int_a^b f(x) \approx f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

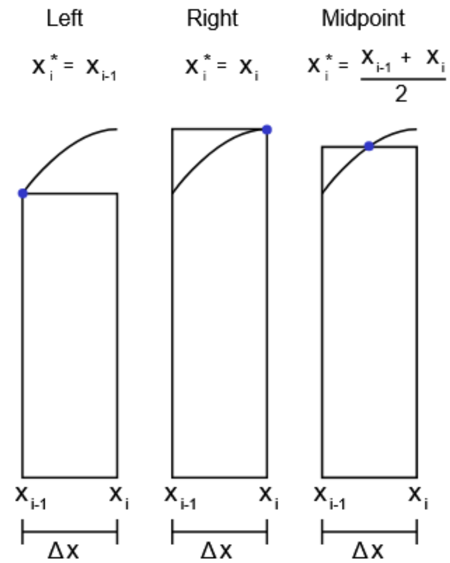
Here Δx represents the width of each rectangle. This is given by the formula

$$\Delta x = \frac{b-a}{n}$$

where n is the number of rectangles.
When n goes to infinity, we hope that we can obtain the exact value of the area

$$A = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

The x-values $x_1, x_2, x_3, \cdots, x_n$ are chosen from the rectangles according to some rule. The three most common rules are:
a. Use the left endpoint of each rectangle.
b. Use the right endpoint of each rectangle.
c. Use the midpoint of each rectangle.



Example 6.2.3. *If* $[a, b] = [1, 3]$ *and* $n = 4$ *then* $\Delta x = \dfrac{3-1}{4} = 0.5$, *so the subintervals would be* $[1; 1.5]$, $[1.5; 2]$, $[2; 2.5]$ *and* $[2.5; 3]$. *In this case:*

a.The right endpoint rule would give

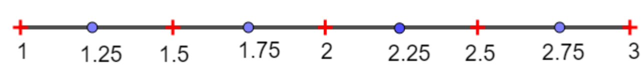
$$f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 + f(3)0.5.$$

b .The left endpoint rule would give

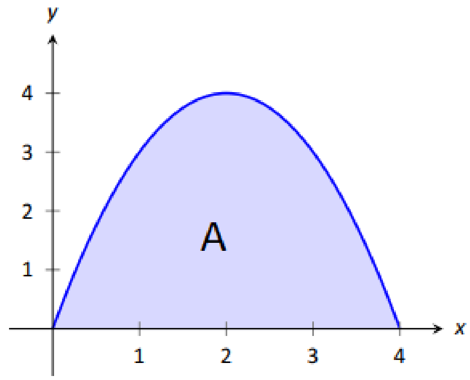
$$f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5.$$

c.The midpoint rule would give

$$f(1.25)0.5 + f(1.75)0.5 + f(2.25)0.5 + f(2.75)0.5.$$



Example 6.2.4. *Let* $f(x) = -x^2 + 4x$ *and* $[a, b] = [0, 4]$. *Approximate the area under* (C_f) *and above the x-axe with* $n = 4$.



The exact value of the area is

$$Area = \int_0^4 f(x)dx = \int_0^4 (-x^2 + 4x)dx$$
$$= \left[\frac{-x^3}{3} + 2x^2\right]_0^4 = \frac{32}{3} \;\; u \cdot a.$$

Approximation by Riemann sum: we have $\Delta x = \dfrac{4-0}{4} = 1.$

a.Left Riemann sum:

$$Area \approx f(0).1 + f(1)1 + f(2).1 + f(3).1 = 10.$$

b .Right Riemann sum:

$$Area \approx f(1).1 + f(2).1 + f(3).1 + f(4).1 = 10.$$

c.The midpoint Riemann sum:

$$Area \approx f(0.5).1 + f(1.5)1 + f(2.5).1 + f(3.5).1 = 11.$$

6.3 Techniques of integration

6.3.1 Integration by parts

Theorem 6.3.1

Let $f, g : [a, b] \to \mathbb{R}$, be two functions with continuous derivatives f' and g' on $[a, b]$. Then,

$$\int_a^b f(x) \cdot g'(x)dx = [f(x) \cdot g(x)]_a^b - \int_a^b f'(x) \cdot g(x)dx \qquad (*)$$

Generally, if U and V be two functions with continuous derivatives dU and dV, we have

$$\int UdV = U \cdot V - \int VdU. \qquad (**)$$

Remark 6.3.1

From $(*)$ and $(**)$ we obtain:

$$\begin{cases} U = f(x) \\ dV = g'(x)dx \end{cases} \quad \text{and} \quad \begin{cases} dU = f'(x)dx \\ V = g(x) \end{cases}$$

Example 6.3.1. *Use the integration by parts and find the following integrals*

$$\int xe^x dx \ \text{ and } \int_1^e x^2 ln(x)dx.$$

In the integral $\int xe^x dx$, *we put* $\begin{cases} U = x \\ dV = e^x dx \end{cases} \implies \begin{cases} dU = 1dx \\ V = e^x \end{cases}$ *Then,*

$$\begin{aligned} \int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + c \\ &= (x-1)e^x + c. \end{aligned}$$

Example 6.3.2. *In the integral* $\int_1^e x^2 ln(x)dx$ *we put* $\begin{cases} U = ln(x) \\ dV = x^2 dx \end{cases} \implies \begin{cases} dU = \frac{dx}{x} \\ V = \frac{x^3}{3} \end{cases}$ *Then,*

$$\begin{aligned} \int_1^e x^2 ln(x)dx &= \left[\frac{x^3}{3} ln(x)\right]_1^e - \int_1^e \frac{x^3}{3}\frac{dx}{x} \\ &= \left[\frac{e^3}{3} ln(e) - \frac{1^3}{3} ln(1)\right] - \frac{1}{3}\int_1^e x^2 dx \\ &= \frac{e^3}{3} ln(e) - \frac{1}{3}\left[\frac{x^3}{3}\right]_1^e = \frac{e^3}{3} ln(e) - \frac{1}{3}\left[\frac{e^3}{3} - \frac{1}{3}\right] \end{aligned}$$

Remark 6.3.2

(a) If the integral is written as the form

$$\int P(x)e^{ax}\, dx, \int P(x)sin(ax)\, dx, \int P(x)cos(ax)\, dx$$

where P is a polynomial and $a \in \mathbb{R}$ in this case we put

$$\begin{cases} U = P(x) \\ dV = e^{ax}dx \text{ or } dV = sin(ax)dx \ \text{ or } dV = cos(ax)dx. \end{cases}$$

(b) If the integral is written as the form

$$\int f(x)ln(x)dx,\ \int f(x)arcsin(x)dx,\ \int f(x)arctan(x)dx \cdots \text{ect}$$

where f is a function has a primitive in this case we put

$$\begin{cases} U = ln(x),\ \arcsin(x),\ \arctan(x) \cdots \text{ect} \\ dV = f(x)dx \end{cases}$$

Example 6.3.3. *Calculate the following integral:*

$$\int 2\arctan(x)dx.$$

We put

$$\begin{cases} U = \arctan(x) \\ dV = 2dx \end{cases} \quad \textit{then} \begin{cases} dU = \dfrac{dx}{1+x^2} \\ V = 2x \end{cases}$$

Thus,

$$\begin{aligned} \int 2\arctan(x)dx &= 2x\arctan(x) - \int \frac{2x}{1+x^2}dx \\ &= 2x\arctan(x) - \ln(1+x^2) + c. \end{aligned}$$

6.3.2 Integration by substitution

Let f and g be two functions such that f and g' are both continuous, then

$$\int g'(x)f(g(x))dx = (F \circ g)(x) + c = F(g(x)) + c(*)$$

where F is the primitive of f. Put $t = g(x)$, we obtain $dt = g'(x)dx$. From $(*)$ we obtain

$$\int g'(x)f(g(x))dx = \int f(t)dt = F(t) + c.$$

Example 6.3.4. *Compute the following integral*

$$\int x^2cos(x^3)dx.$$

Put $t = x^3$. *We have* $dt = 3x^2dx$ *and*

$$\begin{aligned} \int x^2cos(x^3)dx &= \frac{1}{3}\int cos(t)dt \\ &= \frac{1}{3}sin(t) + c \\ &= \frac{1}{3}sin(x^3) + c. \end{aligned}$$

Remark 6.3.3

By using the integration by parts we obtain

$$\int_a^b g'(x)f(g(x))dx = \int_{g(a)}^{g(b)} f(t)dt = [F(t)]_{g(a)}^{g(b)}.$$

where $t = g(x)$.

Example 6.3.5. *Calculate the integral*

$$\int_1^2 \frac{x}{\sqrt{x+1}}dx$$

Put $t = \sqrt{x+1}$. *We have* $x = t^2 - 1$ *and* $dt = \dfrac{dx}{2\sqrt{x+1}}$. *Moreover,*

$$\bullet \text{ if } x = 1 \text{ then } t = \sqrt{2}, \bullet \text{ if } x = 2 \text{ then } t = \sqrt{3}.$$

Thus

$$\begin{aligned}\int_1^2 \frac{x}{\sqrt{x+1}}dx &= \int_{\sqrt{2}}^{\sqrt{3}} 2(t^2-1)dt \\ &= \left[\frac{2t^3}{3} - t^2\right]_{\sqrt{2}}^{\sqrt{3}} \\ &= 2\sqrt{3} - \frac{4\sqrt{2}}{3} - 1.\end{aligned}$$

6.4 Integration by partial fractions

6.4.1 Factorisation of polynomial

Definition 6.4.1

Let Q be a polynomial with real coefficients and $x_0 \in \mathbb{R}$. The real x_0 is said to be a root of the polynomial Q if $Q(x_0) = 0$.

Example 6.4.1. *The real* $x_0 = 1$ *and* $x_1 = -3$ *are roots of the polynomial* $Q(x) = 2x^2 + 4x - 6$ *(since* $Q(1) = 0$ *and* $Q(-3) = 0$*).*

Definition 6.4.2

Let Q be a polynomial with real coefficients and $x_0 \in \mathbb{R}$. The real x_0 is said to be a root by multiplicity k of the polynomial Q if

$$Q(x_0) = 0, Q'(x_0) = 0, Q''(x_0) = 0, \cdots, Q^{(k)}(x_0) \neq 0.$$

Example 6.4.2. *The real number $x_0 = 1$ is a root by multiplicity 3 of the polynomial*

$$Q(x) = x^4 - 3x^3 + 3x^2 - x$$

since

$$\begin{aligned} Q(1) &= 0 \\ Q'(x) &= 4x^3 - 9x^2 + 6x - 1 \implies Q'(1) = 0 \\ Q''(x) &= 12x^2 - 18x + 6 \implies Q''(1) = 0 \\ Q^{(3)} &= 24x - 18 \implies Q^{(3)}(1) \neq 0. \end{aligned}$$

Definition 6.4.3

Let Q be a polynomial with real coefficients. If Q has a roots $x_1, x_2, \cdots, x_n$ by multiplicity $m_1, m_2, \cdots, m_n$ then Q is written as the form

$$Q(x) = a(x - x_1)^{m_1}(x - x_2)^{m_2} \cdots (x - x_n)^{m_n}.$$

Remark 6.4.1

If the polynomial Q has a complex roots then, it's written as

$$Q(x) = a(x^2 + \alpha_1 x + \beta_1)^{m_1}(x^2 + \alpha_2 x + \beta_2)^{m_2} \cdots (x^2 + \alpha_n x + \beta_n)^{m_n}$$

where $x^2 + \alpha_i x + \beta_i$, $\forall i$, has no real roots i.e $\Delta = \alpha_i^2 - 4\beta_i^2 < 0$.

Example 6.4.3.

$$Q_1(x) = (\underbrace{x^2 + x + 1}_{\Delta<0})(\underbrace{x^2 + 1}_{\Delta<0}), \quad Q_2(x) = 2(\underbrace{x^2 + 2}_{\Delta<0})^3(\underbrace{x^2 + 4x + 7}_{\Delta<0}).$$

6.4.2 Integration of rational functions

Let P and Q are tow polynomials and $f(x) = \dfrac{P(x)}{Q(x)}$ be a fraction function.

6.4.2.1 FACT 1

Definition 6.4.4

If $deg(P) \geq deq(Q)$ then, there are polynomials $A(x)$ (*the quotient*) and $B(x)$ (*the remainder*) with $deg(B) < deg(Q)$ such that $P(x) = A(x)Q(x) + B(x)$, or

$$f(x) = A(x) + \frac{B(x)}{Q(x)} \text{ and } \int f(x)dx = \int A(x)dx + \int \frac{B(x)}{Q(x)}dx,$$

where A and B can be found by Euclidean division.

$$\begin{array}{r|l} P(x) & Q(x) \\ \hline B(x) & A(x) \end{array}$$

Example 6.4.4. *Compute the following integrals*

$$\int \frac{2x^3 + 2x + 1}{x^2 + 1}dx \text{ and } \int \frac{2x^2 + 2x - 8}{x^2 + 2x - 3}dx$$

(a). The integral $\int \frac{2x^3 + 2x + 1}{x^2 + 1}dx$. *Looking that* $d^o(2x^3 + 2x + 1) \geq d^o(x^2 + 1)$. *So, using the euclidean division we get*

$$\begin{array}{r|l} 2x^3 + 2x + 1 & x^2 + 1 \\ \cline{2-2} -2x^3 - 2x & 2x \\ \hline 1 & \end{array}$$

Thus,

$$\frac{2x^3 + 2x + 1}{x^2 + 1} = 2x + \frac{1}{x^2 + 1}.$$

We conclude that

$$\int \frac{2x^3 + 2x + 1}{x^2 + 1}dx = \int 2xdx + \int \frac{dx}{x^2 + 1}$$
$$= x^2 + \arctan(x) + c.$$

(b). The integral $\int \frac{2x^2 + 2x - 8}{x^2 + 2x - 3}dx$. *Looking that* $d^o(2x^2+2x-8) \geq d^o(x^2+2x-3)$. *So, using the euclidean division we get*

$$\begin{array}{r|l} 2x^2 + 2x - 8 & x^2 + 2x - 3 \\ \cline{2-2} -2x^2 - 4x + 6 & 2 \\ \hline -2x - 2 & \end{array}$$

Thus,

$$\frac{2x^2 + 2x - 8}{x^2 + 2x - 3} = 2 + \frac{-2x - 2}{x^2 + 2x - 3}.$$

We conclude that

$$\int \frac{2x^2+2x-8}{x^2+2x-3}dx = \int 2dx + \int \frac{-2x-2}{x^2+2x-3}dx$$
$$= 2x - \ln(x^2+2x-3) + c.$$

6.4.2.2 FACT 2

Definition 6.4.5

If $deg(P) < deg(Q)$ and the polynomial Q can be factored into linear and quadratic terms:

$$Q(x) = a(x-x_1)^{m_1}(x-x_2)^{m_2}\cdots(x-x_n)^{m_n}.$$

Then, to this factor,

$$\begin{aligned} f(x) = &\frac{A_1}{(x-x_1)} + \frac{A_2}{(x-x_1)^2} + \cdots + \frac{A_{m_1}}{(x-x_1)^{m_1}} \\ &+ \frac{B_1}{(x-x_2)} + \frac{B_2}{(x-x_2)^2} + \cdots + \frac{B_{m_2}}{(x-x_2)^{m_2}} \\ &+ \cdots + \frac{C_1}{(x-x_n)} + \frac{C_2}{(x-x_n)^2} + \cdots + \frac{C_{m_n}}{(x-x_n)^{m_n}} \end{aligned}$$

Moreover, integrating f amounts to calculating the following two types of integrals:

$$\int \frac{1}{x-x_0}dx = \; ln|x-x_0| + c, \; c \in \mathbb{R}$$
$$\int \frac{1}{(x-x_0)^n}dx = \frac{-1}{n-1}\frac{1}{(x-x_0)^{n-1}} + c, \; c \in \mathbb{R} \text{ and } n \geq 2.$$

Example 6.4.5. *Compute the integral*

$$\int \frac{2}{x(x+1)^2}dx.$$

We have

$$\frac{2}{x(x+1)^2} = \frac{A_1}{x} + \frac{A_2}{x+1} + \frac{A_3}{(x+1)^2}...(E).$$

Finding $A_1,\ A_2$ and A_3.
a) Find A_1: multiple the equation (E) by x, we obtain

$$\frac{2}{(x+1)^2} = A_1 + \frac{A_2 x}{x+1} + \frac{A_3 x}{(x+1)^2}.$$

Put $x=0$ we get $\boxed{A_1 = 2}$.
b) Find A_3: multiple the equation (E) by $(x+1)^2$ yields,

$$\frac{2}{x} = \frac{2(x+1)^2}{x} + \frac{A_2(x+1)}{x} + A_3$$

Put $x = -1$ *we get* $\boxed{A_3 = -2}$.
c) Find A_2*: multiple the equation* (E) *by* $(x+1)$ *yields,*

$$\frac{2}{x(x+1)} = \frac{2(x+1)}{x} + A_2 - \frac{2}{(x+1)}.$$

if $x \to +\infty$, *we get* $\boxed{A_2 = -2}$.
Then,

$$\begin{aligned}\int \frac{2}{x(x+1)^2}dx &= \int \frac{2}{x}dx - \int \frac{2}{x+1}dx - \int \frac{2}{(x+1)^2}dx.\\ &= 2Ln|x| - 2Ln|x+1| + \frac{2}{x+1} + c.\end{aligned}$$

.

Remark 6.4.2

We can fined the number A_1, A_2 and A_3 in

$$\frac{2}{x(x+1)^2} = \frac{A_1}{x} + \frac{A_2}{x+1} + \frac{A_3}{(x+1)^2}...(E)$$

by an other method. Indeed, multiple the equation (E) by $x(x+1)^2$ yields

$$\begin{aligned}2 &= A_1(x+1)^2 + A_2x(x+1) + A_3x\\ &= A_1x^2 + 2A_1x + A_1 + A_2x^2 + A_2x + A_3x\\ &= (A_1 + A_2)x^2 + (2A_1 + A_2 + A_3)x + A_1.\end{aligned}$$

Equating coefficients gives

$$A_1 + A_2 = 0,\ 2A_1 + A_2 + A_3 = 0,\ A_1 = 2.$$

Thus

$$\boxed{A_1 = 2},\ \boxed{A_2 = -2} \text{ and } \boxed{A_3 = -2}.$$

6.4.2.3 FACT 3

Definition 6.4.6

If $deg(P) < deg(Q)$ and the polynomial Q can be factored as follows

$$Q(x) = a(x^2 + \alpha_1 x + \beta_1)^{m_1}(x^2 + \alpha_2 x + \beta_2)^{m_2} \cdots (x^2 + \alpha_n x + \beta_n)^{m_n}$$

where $\alpha_i^2 - 4\beta_i < 0$. Then, to this factor,

$$f(x) = \frac{a_1x+b_1}{(x^2+\alpha_1x+\beta_1)} + \frac{a_2x+b_2}{(x+\alpha_1x+\beta_1)^2} + \cdots + \frac{a_{m_1}x+b_{m_1}}{(x+\alpha_1x+\beta_1)^{m_1}}$$
$$+ \cdots + \frac{c_1x+d_1}{(x^2+\alpha_nx+\beta_n)} + \frac{c_2x+d_2}{(x^2+\alpha_nx+\beta_n)^2} + \cdots + \frac{c_{m_n}x+d_{m_n}}{(x^2+\alpha_nx+\beta_n)^{m_n}}$$

Moreover, integrating f amounts to calculating the following two types of integrals:

$$\int \frac{ax+b}{x^2+\alpha x+\beta}dx \text{ and } \int \frac{ax+b}{(x^2+\alpha x+\beta)^n}dx, \; n \geq 2$$

and using the following two equalities:

$$x^2+\alpha x+\beta = \left(x+\frac{\alpha}{2}\right)^2 - \frac{\alpha^2}{4} + \beta.$$
$$\int \frac{dx}{(1+x^2)^n} = \frac{2n-3}{2n-2}\int \frac{dx}{(1+x^2)} + \frac{1}{2n-2}\frac{x}{(1+x^2)^{n-1}}.$$

Example 6.4.6. *Calculating*

$$\int \frac{3x+3}{(x^2+1)(x^2+4)}dx$$

We have

$$\frac{3x+3}{(x^2+1)(x^2+4)} = \frac{\alpha_1x+\beta_1}{x^2+1} + \frac{\alpha_2x+\beta_2}{x^2+4}.....(E).$$

Finding the coefficients $\alpha_1, \alpha_2, \beta_1$ and β_2. multiple the equation (E) by $(x^2+1)(x^2+4)$ yields

$$\begin{aligned} 3x+3 &= (\alpha_1x+\beta_1)(x^2+4) + (\alpha_2x+\beta_2)(x^2+1) \\ &= \alpha_1x^3 + 4\alpha_1x + \beta_1x^2 + 4\beta_1 + \alpha_2x^3 + \alpha_2x + \beta_2x^2 + \beta_2 \\ &= (\alpha_1+\alpha_2)x^3 + (\beta_1+\beta_2)x^2 + (4\alpha_1+\alpha_2)x + 4\beta_1 + \beta_2 \end{aligned}$$

Equating coefficients gives

$$\begin{aligned} \alpha_1 + \alpha_2 &= 0 \cdots (1) \\ \beta_1 + \beta_2 &= 0 \cdots (2) \\ 4\alpha_1 + \alpha_2 &= 3 \cdots (3) \\ 4\beta_1 + \beta_2 &= 3 \cdots (4). \end{aligned}$$

The difference $(3)-(1)$ gives $3\alpha_1 = 3$ i.e $\boxed{\alpha_1 = 1}$. The equation (1) gives $\boxed{\alpha_2 = -1}$.
Similarly, according to (2) and (4) we get $\boxed{\beta_1 = 1}$ and $\boxed{\beta_2 = -1}$.
Then,

$$\frac{3x+3}{(x^2+1)(x^2+4)} = \frac{x+1}{x^2+1} - \frac{x+1}{x^2+4}.$$

So,

$$\begin{aligned}\int \frac{3x+3}{(x^2+1)(x^2+4)}dx &= \int \frac{x+1}{x^2+1}dx - \int \frac{x+1}{x^2+4}dx\\ &= \int \frac{x}{x^2+1}dx + \int \frac{dx}{1+x^2} - \int \frac{x}{x^2+4}dx - \int \frac{dx}{x^2+2^2}\\ &= \frac{1}{2}\int \frac{2xdx}{x^2+1} + arctan(x) - \frac{1}{2}\int \frac{2xdx}{x^2+4} - \frac{1}{2}arctan\left(\frac{x}{2}\right) + c\\ &= \frac{1}{2}ln(x^2+1) + arctan(x) - \frac{1}{2}ln(x^2+4) - \frac{1}{2}arctan\left(\frac{x}{2}\right) + c\end{aligned}$$

Thus

$$\int \frac{3x+3}{(x^2+1)(x^2+4)}dx = \frac{1}{2}ln\left(\frac{x^2+1}{x^2+4}\right) + arctan(x) - \frac{1}{2}arctan\left(\frac{x}{2}\right) + c$$

.

6.5 Integration of trigonometric functions

6.5.1 Trigonometric formulas

Proposition 6.5.1

Let a and b are real numbers then

(a). $\cos^2(a) = 1 - \sin^2(a)$ and $\sin^2(a) = 1 - \cos^2(a)$

(b). $\sin^2(a) = \frac{1}{2}(1 - \cos(2a))$ and $\cos^2(a) = \frac{1}{2}(1 + \cos(2a))$

(c). $\cos(a)\cos(b) = \frac{1}{2}(\cos(a-b) + \cos(a+b))$

(d). $\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$

(e). $\sin(a)\cos(b) = \frac{1}{2}(\sin(a-b) + \cos(a+b))$

(f). $2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) = \sin(a) + \sin(b)$

(g). $2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right) = \sin(a) - \sin(b)$

(h). $2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) = \cos(a) + \cos(b)$

(i). $2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right) = \cos(b) - \cos(a)$

Example 6.5.1. *Compute the integral* $\int \sin(3x)\cos(4x)dx$*. According to the trigonometric formula (e) in proposition 6.5.1, we have:*

$$\begin{aligned}\sin(3x)\cos(4x) &= \frac{1}{2}\left(\sin(3x-4x)+\cos(3x+4x)\right)\\ &= \frac{1}{2}\left(\sin(-x)+\cos(7x)\right)\\ &= \frac{1}{2}\left(\cos(7x)-\sin(x)\right)\end{aligned}$$

Thus,

$$\int \sin(3x)\cos(4x)dx = \frac{1}{14}\sin(7x)+\frac{1}{2}\cos(x)+c.$$

Example 6.5.2. *Compute the following integral*

$$\int \cos^2(x)dx.$$

Using the trigonometric formula (b) in proposition 6.5.1 we have

$$\cos^2(x) = \frac{1}{2}\left(1+\cos(2x)\right)$$

Thus,

$$\int \cos^2(x)dx = \frac{x}{2}+\frac{1}{4}\sin(2x)+c.$$

6.5.2 Calculate the integral $\int \sin^m(x)\cos^n(x)dx$ with $m, n \in \mathbb{N}$

Proposition 6.5.2

(a) If m and n are even we use the trigonometric formulas:

$$\sin^2(x) = \frac{1}{2}\left(1-\cos(2x)\right) \text{ or } \cos^2(x) = \frac{1}{2}\left(1+\cos(2x)\right)$$

(b) If one is odd we use the integration by substitution. More precisely:

  (a) If m is odd, we put $t = \cos(x)$.
  (b) If n is odd, we put $t = \sin(x)$.

Example 6.5.3. *Compute the following integrals*

$$\int \sin^2(x)\cos^2(x)dx \quad \textit{and} \quad \int \sin^3(x)\cos^2(x)dx$$

.

$$\begin{aligned}\int \sin^2(x)\cos^2(x)dx &= \frac{1}{4}\int (1-\cos(2x))\,(1+\cos(2x))\,dx \\ &= \frac{1}{4}\int \left(1-\cos^2(2x)\right)dx = \frac{1}{4}\int 1dx - \frac{1}{4}\int \cos^2(2x)dx \\ &= \frac{x}{4} - \frac{1}{8}\int (1+\cos(4x))\,dx \\ &= \frac{x}{4} - \frac{x}{8} - \frac{1}{32}\sin(4x) + c\end{aligned}$$

$$\begin{aligned}\int \sin^3(x)\cos^2(x)dx &= \int \sin^2(x)\cos^2(x)\sin(x)dx \\ &= \int (1-\cos^2(x))\cos^2(x)\sin(x)dx\end{aligned}$$

Put $t=\cos(x)$ *we get* $dt=-\sin(x)dx$. *Thus,*

$$\begin{aligned}\int (1-\cos^2(x))\cos^2(x)\sin(x)dx &= \int (t^2-1)t^2dt = \frac{t^5}{5} - \frac{t^3}{3} + c \\ &= \frac{\cos^5(x)}{5} - \frac{\cos^3(x)}{3} + c.\end{aligned}$$

6.6 Integration of rational trigonometric functions

Proposition 6.6.1

If the integral is a rational function of trigonometric, the substitution $z=tan(\frac{x}{2})$ will reduce the integral to rational function.
Let $z=\tan(\frac{x}{2})$ thus $x=2arctan(z)$ and $dx=\dfrac{2dz}{1+z^2}$

(a) $\sin\left(\frac{x}{2}\right) = \dfrac{z}{\sqrt{1+z^2}}$, $\cos\left(\frac{x}{2}\right) = \dfrac{1}{\sqrt{1+z^2}}$

(b) $\sin(x) = \sin(2(\frac{x}{2})) = 2\cos(\frac{x}{2})\sin(\frac{x}{2}) = \dfrac{2z}{1+z^2}$

(c) $\cos(x) = \cos(2(\frac{x}{2})) = \cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2}) = \dfrac{1-z^2}{1+z^2}$.

(d) $tan(x) = \dfrac{\sin(x)}{\cos(x)} = \dfrac{2z}{1-z^2}$.

Example 6.6.1. *Calculate the following integral*

$$\int \frac{1+\sin(x)}{1-\cos(x)}dx.$$

Let $z = \tan\left(\frac{x}{2}\right)$. *Then*

$$\int \frac{1+\sin(x)}{1-\cos(x)}dx = \int \frac{1+\frac{2z}{1+z^2}}{1-\frac{1-z^2}{1+z^2}}\frac{2dz}{1+z^2}$$
$$= \int \frac{z^2+2z+1}{z^2(1+z^2)}dz$$

On the other hand, we have

$$\frac{z^2+2z+1}{z^2(1+z^2)} = \frac{a}{z}+\frac{b}{z^2}+\frac{cz+d}{1+z^2}$$
$$= \frac{2}{z}+\frac{1}{z^2}-\frac{2z}{1+z^2}$$

So,

$$\int \frac{z^2+2z+1}{z^2(1+z^2)}dz = 2\ln|z| - \frac{1}{z} - \ln(1+z^2)+c.$$

Then

$$\int \frac{1+\sin(x)}{1-\cos(x)}dx = 2\ln\left|\tan\left(\frac{x}{2}\right)\right| - \frac{1}{\tan\left(\frac{x}{2}\right)} - \ln\left|1+\tan^2\left(\frac{x}{2}\right)\right| + c$$

.

6.7 Practice Exercises

Exercise 6.1

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1. Determine the primitives of the following functions:

$$\int\left[\frac{2}{1+x^2}-5x^2\right]dx, \int\left[3ch(x)-\frac{e^x}{2}\right]dx, \int\left[\frac{6}{1+x}+\frac{3}{x^2}\right]dx$$
$$\int 2x(x^2+3)^2dx, \int\frac{1}{2+x^2}dx, \int\frac{e^x}{1+e^{2x}}dx, \int\frac{3}{\sqrt{1-9x^2}}dx.$$
$$\int(1-6x)e^{3x^2-x}dx, \int\frac{2x^3}{\sqrt{1+x^4}}dx, \int\frac{\cos(2x)}{1+\sin(2x)}dx, \int\frac{2}{\sqrt{x}+\sqrt{x^3}}dx.$$

2. Calculate the following definite integrals

$$\int_{-1}^{1}\frac{2x+1}{x^2+x+3}dx, \int_{0}^{1}\frac{arccos(x)}{\sqrt{1-x^2}}dx, \int_{0}^{1}\frac{e^t}{1+e^t}dt,$$

$$\int_1^e \frac{dx}{x + x\ln^2(x)}, \int_0^{\frac{\pi}{6}} \frac{\cos(x)}{1-\sin^2(x)}dx, \int_{-1}^1 \frac{dx}{3+(x+2)^2}.$$

Exercise 6.2

Let (u_n) be sequence defined on $\mathbb{N}$ by

$$u_n = \int_n^{n+1} (3e^x + 2\sqrt{x})dx.$$

(a) Show that $\forall n \in \mathbb{N}$, $u_n > 0$.

(b) Calculate the sum $\mathcal{S}_n = u_0 + u_1 + ... + u_n$.

Exercise 6.3

.

1. Determine the primitive of the following functions:

$$\int xsin(2x)dx, \int 2arctan(x)dx, \int \sin(x)\ln(\cos(x))dx, \int x\sin(x)dx, .$$

$$\int x^2e^{2x}dx, \int x^3\ln(x)dx, \int 3arcsin(x)dx, \int 4xarctan(x)dx.$$

2. Calculate the following integrals

$$\int_0^1 x^2\arctan(x), \int_0^\pi e^x\cos(x)dx, \int_0^\pi (2x-1)\cos(x)dx, \int_1^e x^2\ln(x)dx.$$

Exercise 6.4

Let $g(x) = \dfrac{-8x+20}{x^2+2x-3}$.

(a) Find the partial fractions decomposition of g.

(b) Compute the integral $\int g(x)dx$.

Exercise 6.5

Determine the primitive of the following rational functions

$$f_1(x) = \frac{x+1}{x(x-2)^2},\ f_2(x) = \frac{x^4+1}{x(x+1)},\ f_3(x) = \frac{1}{1-x^2},$$

$$f_4(x) = \frac{x^2}{x^2-4},\ f_5(x) = \frac{x^3}{x^2+4x+4},\ f_6(x) = \frac{2x^2}{x^4-1}dx,$$

$$f_7(x) = \frac{x+1}{(x^2+1)(x^2+3)},\ f_8(x) = \frac{1}{(x-1)(x^2-2x+2)}, f_9(x) = \frac{x^5+1}{x(x^2+1)^2}.$$

$$f_{10}(x) = \frac{x+1}{(x^2+2)(x^2+1)},\ f_{11}(x) = \frac{x-2}{(x-1)^2(x^2-x+1)},\ f_{12}(x) = \frac{x-1}{(x^2+1)^2}dx.$$

Exercise 6.6

Calculate the following integrals

$$\int \cos(x)\cos(3x)dx,\ \int \cos^3(x)\sin(x)dx,\ \int \cos^4(x)dx,\ \int \sin^4(x)dx,\ \int \sin^5(x)\cos^2(x)dx$$

Exercise 6.7

Determine the primitive of the following rational trigonometric functions

$$f_1(x) = \frac{\cos^3(x)}{1+\sin^2(x)},\ f_2(x) = \frac{1}{4-5\sin(x)},\ f_3(x) = \frac{1}{4\sin(x)-3\cos(x)}.$$

$$f_4(x) = \tan(x)\ f_5(x) = \frac{dx}{5-4\cos(x)} \text{ and } f_6(x) = \frac{dx}{3\cos(x)+4\sin(x)}.$$

Exercise 6.8

Let $f(x) = \dfrac{1}{1+\cos^2(x)}$ be a function.

1. Prove that $f(x) = \dfrac{1}{(\tan^2(x)+2)\cos^2(x)}$.
2. Using the substitution $z = \tan(x)$ and calculate $\int f(x)dx$.
3. Deduce the value of the integral $\displaystyle\int_{\pi/6}^{\pi/3} \frac{2\tan(x)}{(1+\cos^2(x))(1+\tan^2(x))}dx$.