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Relativistic problem of scalar particles in deformed space.

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Dahou Zineb

Abstract

In this thesis we have treated the problem of the Klein-Gordon oscillator (KGO) with the generalized uncertainty principle (GUP) in deformed space. In the first case we deal with the problem of the scalar particle in the case of the free Klein-Gordon oscillator ($\varepsilon = 0$), the energy spectrum E_n is represented as function n and the wave function $\phi_n(x)$ is obtaind by the Hermite polynomial $H_n(x)$.

In the 2nd case, we have solved the equation of the Klein-Gordon oscillator in the presence of the external electric field ε in the deformed space, where the energy spectrum E_n is given as a function of power of n due to minimal length effect and the wave function $\phi_n(p)$ is defined in term of the Gegenbaouer plynomial $C_n^{\lambda}(p)$. The borderline cases are deduced and confirmed the results obtained, recently the term probabilities Z, U, F, C, S have been calculated. As conclusion to this work we introduced the path intgral treatment of the Klein-Gordon oscillator in absence of the external electric field ε in the deformed space.

Key words : relativistic quantum mechanics, Klein Gordon equation , regular spaces, deformed spaces, minimal length.

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Chapitre 1

General introduction

For the last several decades, the subject of unification the theory of general relativity (GR) and the quantum mechanics (QM) (or quantum field theory (QFT)) under one model known as ”quantum gravity” have been taking a huge interest in theoretical high energy physics. This unification constitute the cornerstone of the modern physics and it directly leads to the birth of a wide range of new physical ideas and mathematical tools. Most of the differences found in this unification come from the different assumptions of these theories about the functioning of the universe :

Theory of GR ; which today is still regarded as the best theory of gravity just for the fact that is able to predict and describe a large number of physical phenomena in Astrophysics (the foundation for the current understanding of black holes), and Cosmology (the standard Big Bang model) explaining the world in the macro-dimension. The GR theory was introduced in 1916 by Albert Einstein, representing gravitational interactions in terms of the geometry of continuous space-time manifold where the space-time is described by a metric that determines the distances separating nearby points (stars, galaxies, etc.) [1].

QM, on the other hand like GR, has achieved several successes since its foundation with Planck’s hypothesis of quanta of energy ; proposed in 1900 to confirmation of this hypothesis by Einstein’s 1905 paper which explained the photoelectric effect. These represent a early attempts to understand microscopic phenomena the known as old quantum theory. The modern theory was formulated few years later by number of physicists : N.Bohr in 1913 with the presentation of Bohr’s model of the atom, L. de Brôglie in 1924 with the proposition of the wave-matterduality, Dirac in 1928-1930, with the discovery of antimatter particles, E.Schrodinger in 1926 thought his famous wave mechanics, W. Heisenberg in 1925-1927 with developing the so called matrix mechanics and the uncertainty Principle. The formulation of the whole theoretical framework of the theory was done by Dirac and von Neammenn, where he introduced his notation that combine the Heisnberg matrix mechanics with Schrödinger wave mechanics [2].

Since the birth of QM and through many experiments we have noticed many time that our Universe has a quantum nature, but QM even with relativistic quantum mechanics, wasn't enough to describe everything in the Universe. We needed second quantification and more fundamental theory, the quantification of the classical field theory to the quantum version (QFT). QFT formulated the knowen Standard Model (SM) of particle physics. SM or more general QFT represents and describes three of the four fundamental interactions of nature (electromagnetism, weak and strong nuclear forces) ignoring the fourth force the gravitational interaction for being much weaker then the other three.

So far, the attempts to incorporate gravity into QFT run into problems is that the calculation of effective cross sections of diffusion leads to very serious discrepancies more exactly we often deal with type of integrals ($\int d^4k(kkk/k^2k^2)$) in the calculation of radiative corrections associated a loop in a Feynman diagram, this integral diverges [3]. The region of integration that generates the divergence is the area known as ultraviolet region (UV).

Theoretical physics put forward certain number of attempts to address this problem of discrepancies, all of which have met with various degrees of success, let us mention a few of this proposals, which are detailed in the literature [4, 5] :

-String theory (ST) was developed in late of 1960, this theory is an attempt not only to describe quantum gravity but also explain the behavior of strong interacting particles : hardons and other particles which are presented in the standard model of particle physics from it premise where everything is made of tiny strings. The strings may be closed into themselves or have loose ends ; they can vibrate, stretch, join or split. This theory admits the supergravity theories as effective low energy theories.

-Loop quantum gravity (LQG) : on the other hand not like ST, is less interested in the matter and it behavior that occupies the space-time than in the quantum characteristics of the space-time itself. Where the smooth functioning of Einstein's GR is replaced by nodes and links to which those quantum characteristics are attached. In this way, space is built up of discrete pieces. LQG is in large part a study of these pieces. in this context, considering the quantization of gravity and the quantization of the metric tensor are two faces for one coin, the space-time metric in the proposition of canonical gravity should stand as an expectation value of wave functional from Hilbert space in way that should come independently totally for non-perturbative quantum theory. The dynamics of this approach are governed by a Hamiltonian operator given some sort of equation. LQG arises to solve some of the problem by considering this equation is ill defined in general case. This approach has long been thought incompatible with ST.

- A new mathematical tool has been in theoretical physics to take gravity in the framework of quantum mechanics based on Synder quantized space time, a generalized pseudo Riemannian geometry where space and time can be reinterpreted as discrete concept non-commutative differential geometry proposed by Alain Connes known as non-commutative geometry (NCG). On steps of this theory the mathematical physicist Pierre Martinetti introduces us to what is called the non-commutative SM in particle physics, formulation of the electroweak forces.

-Doubly Special Relativity (DSR) was proposed by G. Amelino-Camelia, J. Magueijo, and L. Smolin in 2000-2002. This theory attempts to deform and modify the Einstein's special relativity (SR) in order to describe ultra-high-energy particles by introducing an observer independent length more specifically including an additional postulate in SR in way it considers effects on transformation laws between observers and symmetries of space-time. Which leads to a modified law of energy-momentum conservation, the term "Doubly" came back, to the fact there are two observer independent scales, the speed of light c and Planck mass or Planck length. DSR tries to address some of the problems that QG faces in the process of quantization, does not attempt to formulate the full theory.

A various candidate scenarios of QG theory with the ones mentioned above predicts the existence of a minimum measurable length scale [6] where physics is inaccessible. This minimal length is supposed to be near the Planck length ($l_p \approx 10^{-35}m$) and this concept is old as quantum physics itself, it goes back to 1930, with the advent QFT it was considered to solve the problem of divergence and many other problems of elementary particles where few physicists of that time sew the necessity to believe in fundamental length. Heisenberg tried to express minimal length to control the infinities [7] thought supposing that position operators non- commuting $[x^\mu, x^\nu] \neq 0$ which means to abandoned the continuum space time and replacing it by a lattice structure but he failed to express it mathematically, for the fact that a lattice form collapse under the action of continuous the Lorentz group.Until 1947, the physicist H. Snyder [8] during his work on problem of the interaction matter and QFT where he proposed an suggestion that the usual four dimensional space-time may not be continuous but discrete or quantized adjacent points (lattice) but at the same time maintaining Lorentz invariant. Here space-time became Lorentz convarianty non-commutative and this modification effected Heisenberg uncertainty to so-called Generalized Uncertainty Princeple (GUP) and this agree totally with the assumption of existence a smallest unit of length [9]. This new model of quantum theory of space-time removed the problem of UV divergence caused by the infinite density near the horizon [10]. Later on, the work of Kempf, Mangano, and Mann in 1995 [11] by developing the mathematical basis for the Snyder model (the GUP approch) in QM and QFT made it the strongest candidate that could be virtually a combination of GR and QM, it address the Planck scale phenomena.

The study of the implications of these modifications took a great importance, it can be useful to describe non-point-like particles, composite particles such as hadrons in nuclear physics. In this context, many papers were published studies to embrace the minimum length to relativistic quantum mechanics through the study of different quantum system in space with the new Heisenberg principle, we mention : Klein-Gordon equation with Coulomb potential in the presence of a minimal length [12], the minimal length case of the Klein-Gordon equation with hyperbolic cotangent potential using Nikivorof-Uvarof method [13], three-dimensional Dirac oscillator with minimal length : novel phenomena for quantized energy [14], Harmonic oscillator in relativistic minimal length quantum mechanics [15].

In this presented work, the main purpose is to treat in the framework of relativistic quantum mechanics via the GUP formalism, the massless relativistic particles in this deformed algebra formalism which are represented by the Klein-Gordon equation with Kempf non-cummutative algebra, and we are going to study how such a deformation can affect the main properties : energy spectrum of a simple physical system, using analytical methods with ”special functions” such as Gegenbauer polynomials and the Hermite polynomials to find the wave function and the energy in this space.

Our thesis is essentially composed of three chapters : the first chapter is dedicated as general introduction that gives us a brief history on the motivations of this minimal length. In the second chapter, we have made a recall to the ordinary and regular spaces, starting with brief view on classical mechanics and some properties, then we introduced the mathematical framework of quantum mechanics. After that, we presented minimal length quantum mechanics formalism in (1+1) and (1+3) kempf algebra form in momentum space, we defined the so-called maximum localized states notion for the Position representation and wave function on this states. In the third chapter we have established the equation of Klein Gordon oscillator with the scalar potential of a charged particle in a uniform external electric field of specific value ε in the GUP formalism.later on we derived the thermic properties of the deformed Klein Gordon oscillator using the Euler-MacLaurin method.

Chapitre 2

The mathematical framework of minimal length quantum mechanics

2.1 Regular spaces and Heisenberg incertainty principal

2.1.1 The mathematical framework of quantum mechanics

A brief look at classical mechanics (CM) :

In classical mechanics (CM), any physical system described by a state at time t is given by a point $\omega(t)$ and it time evolution equation in the phase space Ω, which is mathematically a symplectic manifold $W = (T^{*}M, \omega)$, where $T^{*}M$ is the cotangent bundle over configuration space M (i.e. $\mathbb{R}^{n}$) and ω is a symplectic form $d\omega = \sum dp_i \wedge dq_i$, in the local Darboux coordinates for standard symplectic forms $\{q_i, p_i\}$: $q_i = x_i$ generalized coordinates for the position and $p_i = m\frac{dq_i}{dt}$ their canonically conjugate generalized momenta for $(q, p) \in \Omega$ together defined as pure states of the system. A general dynamical variable $f(q, p)$ represents the physical properties of system (classical observable :"C-observable") in time t is a scalar (real-value) smooth (C^{∞}function) on Ω modelled by $2-n$ first-order ordinary differential equations (for the Hamiltonian system) [16] :

$$\dot{q_i} = \frac{dH}{dp_i} = \{q_i, H\} \quad ,\dot{p_i} = -\frac{dH}{dq_i} = \{p_i, H\}, \tag{2.1}$$

where $\{f, g\}$ (f,g smooth functions) is the familiar Poisson braket's of f with g :

$$\{f, g\} = \sum_k \{\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}\}, \tag{2.2}$$

we identified (2.1) as Hamilton's Equations.

In the study in CM of N-particles system (thermostatistics) we use description on the phase space ($\Omega = \mathbb{R}^{6n}$) as a probability distribution of the behavior of the system at certain time t can be obtain as $\rho(q, p)$ (or $p(d\omega, t)$) in the following we shall reserve the term state to the probability density $\rho(q, p)$ to describe or to define a general or mixed state to explain that system lies in small $\triangle\omega$ region of Ω and it can be also expressed as a convex sum of mixed or pure states $\omega_0(q_0, p_0) \in \Omega$ interpreted in terms of probability density by using the Dirac delta function

$$\rho_{\omega_0}(q, p) = \delta^n(q - q_0)\delta^n(p - p_0), \tag{2.3}$$

$$\rho(q, p) = \sum_i P_i\rho_i(q, p) \quad P_i > 0, \quad \sum_i P_i = 1, \tag{2.4}$$

where P_i is the probabilities of the classical system at any state$\rho_i(q, p)$.

A C-observable $f(q, p)$ has an expectation value $\langle f\rangle$ and dispersion $\langle(\triangle f)^2\rangle$ given by the following equations

$$\langle f\rangle = \int\int d^nqd^np\rho(q, p)f(q, p), \tag{2.5}$$

$$\begin{aligned}\langle(\triangle f)^2\rangle &= \langle f(q, p) - \langle f\rangle\rangle^2; \\ &= \langle f(q, p)^2\rangle - \langle f\rangle^2 \geq 0.\end{aligned} \tag{2.6}$$

The equations (2.1) for a general state representation of Hamiltonian system is obtained by the Liouville equation

$$\frac{\partial}{\partial t}\rho(q,p;t) + \{\rho(q,p;t), H(q,p;t)\} = 0. \tag{2.7}$$

Here $\{\rho(q,p;t), H(q,p;t)\}$ is computed by (2.2) Liouville equation is more general then equations (2.1), it describes non-Hamiltonian system also.

b-Ordinary quantum mechanics

Quantum mechanics (QM) is fundamentally build on the postulate that the phase space is a Hilbert space H, the Hilbert space of Lebesgus square integrable functions $L^2(M)$ on a manifold M (we often mean by Hilbert space specifically for "wave mechanics " : the extended Hilbert space that called the rigged Hilbert space RHS [17, 18]). This implies a different evolution equation, the classical image of observables, states and laws of motion are replaced by more sophisticated structures, the deterministic theory turned to probabilistic one. Here the physical systems are described by the self-adjoint part of a C*-algebra A over H, namely algebras of operators (observables) on Hilbert space.

b$_1$)-Hilbert space H

We call a complex Hilbert space H, the space that satisfies the following :

(a)-H is a linear space $(H, +, \cdot)$

A linear space (also called vector space) H over the complex numbers $\mathbb{C}$ is a set of elements $\varphi, \psi, \chi \ldots$ with the addition rule $(+)$ of any two elements (vectors) and the multiplication rule $(\cdot)$ of vectors by scalars (in this case we mean complex numbers $\lambda \in \mathbb{C}$), for thus two rules we have the following properties :

The addition rule properties (structure of an abelian group) :

(1)-$\varphi + \psi = \psi + \varphi$, $\forall \varphi, \psi \in H$,

(2)- $(\varphi + \psi) + \phi = \varphi + (\psi + \phi)$, $\forall \varphi, \psi, \phi \in H$,

(3)- There exists a $0 \in H$ such that $0 + \psi = \psi$, $\forall \psi \in H$,

(4)-$\forall \varphi \in H$ there exists $\psi \in H$ such that $\varphi + \psi = 0$ (we write $\psi = -\varphi$).

The multiplication of vectors by scalars properties :

(5)- $(\lambda\mu)\,\psi = \lambda\,(\mu\psi)$, $\forall \lambda, \mu \in \mathbb{C}$,$\forall \psi \in H$,

(6)-$(\lambda + \mu)\,\psi = \lambda\psi + \mu\psi$, $\forall \lambda, \mu \in \mathbb{C}$, $\forall \psi \in H$,

(7)- $\lambda\,(\varphi + \psi) = \lambda\varphi + \lambda\psi$, $\forall \lambda \in \mathbb{C}$, $\forall \varphi, \psi \in H$,

(8)- $1\psi = \psi$, $\forall \psi \in H$.

(b)- H is a scalar product space $(H,+,\cdot,(.,.))$

A linear space is called a Euclidean space (or scalar product space or pre-Hilbert space) if for each pair of vectors $\varphi,\psi\in H$ we can define a complex number (φ,ψ) satisfying the following properties :

(1)- $(\varphi,\psi)=\overline{(\psi,\varphi)},\ \ \forall\varphi,\psi\in H,$

(2)- $(\varphi,\alpha\psi_1+\beta\psi_2)=\alpha\left(\varphi,\psi_1\right)+\beta\left(\varphi,\psi_2\right)$, $\forall\varphi,\psi_1,\psi_2\in H$, $\forall\alpha,\beta\in\mathbb{C}$,

(3)-$(\psi,\psi)\geq 0$, and $(\psi,\psi)=0$ if and only if (iff) $\psi=0$.

The properties (1) and (3) holds definition of a norm of a vector $\parallel\psi\parallel=\sqrt{(\psi,\psi)}$

We defined the Cauchy–Schwarz inequality which is always satisfied in these spaces

$$\mid(\varphi,\psi)\mid^2\leq(\psi,\psi)\left(\varphi,\varphi\right). \tag{2.8}$$

The Parallelogram identity and Polarization identity are also satisfied naturally in scalar product space :

Parallelogram identity

$$\parallel\psi+\varphi\parallel^2+\parallel\psi-\varphi\parallel^2:=2\left(\parallel\psi\parallel^2+\parallel\varphi\parallel^2\right). \tag{2.9}$$

Polarization identity

$$(\varphi,\psi)=\frac{\parallel\varphi+\psi\parallel^2-\parallel\varphi-\psi\parallel^2}{4}+i\left(\frac{\parallel\varphi+i\psi\parallel^2-\parallel\varphi-i\psi\parallel^2}{4}\right). \tag{2.10}$$

Hermitian form : is a complex-valued function $h\left(\varphi,\psi\right)$ of two vector arguments :

(1)-$h\left(\varphi,\psi\right)=\overline{h\left(\psi,\varphi\right)}$,

(2)-$h\left(\varphi,\lambda\psi\right)=\lambda h\left(\varphi,\psi\right)$,

(3)-$h\left(\varphi_1+\varphi_2,\psi\right)=h\left(\varphi_1,\psi\right)+h\left(\varphi_2,\psi\right)$.

h is a positive definite Hermitian form, if h satisfies :

(a)-$h\left(\psi,\psi\right)\geq 0$,$\forall\psi\in H$,

(b)-$h\left(\psi,\psi\right)=0$ for $\psi=0$.

A positive definite Hermitien form is a scalar product .

For positive Hermitien form the Cauchy–Schwarz inequality is also satisfied $\mid h(\varphi,\psi)\mid^2 \leq h(\psi,\psi)\, h(\varphi,\varphi)$

(c)- H is separable space

There exists a Cauchy sequence $\psi_n \in H$ such that for every ψ of H $(n = 1, 2, 3...)$ and $\varepsilon > 0$, there exists at least one ψ_n of the sequence for which

$$\|\psi - \psi_n\| \leq \varepsilon. \tag{2.11}$$

(d)-H is complete space

Every Cauchy sequence $\psi_n \in H$ converges to an element of H . That is, for any ψ_n, the relation

$$\lim_{n,m\longrightarrow\infty} \|\psi_n - \psi_m\| = 0, \tag{2.12}$$

defines a unique limit ψ of H such that

$$\lim_{n\longrightarrow\infty} \|\psi - \psi_n\| = 0. \tag{2.13}$$

b$_2$)- Dirac notation (bra ,ket)

Let us consider a complex Hilbert space H (Ref [19]).

$$H = \{|\phi\rangle, |\varphi\rangle, |\psi\rangle, \ldots\}$$

The elements of H are called a ket vectors or kets .

A linear function $\varphi : H \to \mathbb{C}$ defined by

$$\varphi(\alpha\,|\psi_1\rangle + \beta\,|\psi_2\rangle) = \alpha\varphi(|\psi_1\rangle) + \beta\varphi(|\psi_2\rangle) \qquad \forall \alpha, \beta \in \mathbb{C} \ \ , \ \ |\psi_i\rangle \ \in H. \tag{2.14}$$

We write the linear function as $\langle\varphi\,|$and the action as $\langle\varphi \mid \psi\rangle \in \mathbb{C}$. The set of linear functions is itself a vector space called the dual vector space of H, denoted H^*. An element of H^* is called a bra vector or bra.

Let $\{|e_1\rangle, |e_2\rangle, \ldots\}$ be a basis of H (the expansion of Euclidean space vectors in terms of the basis vectors "Himal basis"). A ny vector $|\psi\rangle \in H$ is then expended as $|\psi\rangle = \sum_n \psi_n\,|e_n\rangle$ where $\psi_n \in \mathbb{C}$ is called the nth component of $|\psi\rangle$. Now let us introduce a basis $\{\langle\varepsilon_1|, \langle\varepsilon_2|, \ldots\}$ in H^*. We require that this basis be a dual basis of $\{|e_n\rangle\}$ that is

$$\langle\varepsilon_i \mid e_j\rangle = \delta_{ij} \tag{2.15}$$

Then an arbitrary linear function $\langle\varphi|$ is expanded as $\langle\varphi| = \sum_n \varphi_n \langle\varepsilon_n|$, where $\varphi_n \in \mathbb{C}$ is the nth component of $\langle\varphi|$. The action of $\langle\varphi| \in H^*$ on $|\psi\rangle \in H$ is now expressed in terms of their components as

$$\langle\varphi \mid \psi\rangle = \sum_{ij} \varphi_i \psi_j \langle\varepsilon_i \mid e_j\rangle = \sum_{ij} \varphi_i \psi_j \delta_{ij} = \sum_i \varphi_i \psi_i \tag{2.16}$$

One may consider $|\psi\rangle$ as column vector and $\langle\varphi|$as row vector so that $\langle\varphi \mid \psi\rangle$ is regarded as matrix multiplication of a row vector and a column vector yielding a scalar.

It is possible to introduce a one-to-one correspondence between elements $\langle\varphi| = \sum_n \psi^* \langle\varepsilon_n| \in H^*$ the reason for the complex conjugation of ψ_n becomes clear shortly. Then it is possible to introduce an inner product between two elements of H in this notation. Let $|\varphi\rangle$, $|\psi\rangle \in H$. Their inner product is defined by

$$(|\varphi\rangle, |\psi\rangle) \equiv \langle\varphi \mid \psi\rangle = \sum_n \varphi_n^* \psi_n \tag{2.17}$$

We customarily use the same letter to denote corresponding bras and kets. The norm in this notation of inner product is expressed by $\||\psi\rangle\| = \sqrt{\langle\psi \mid \psi\rangle}$ and we call a normalized ket vector, the vector that verifies $\||\psi\rangle\|^2 = \langle\psi \mid \psi\rangle = 1$.

b_3) - Linear operators

In the following we are going to focus on H as Euclidean space vectors $(H, +, \cdot, \langle ., . \rangle)$

A linear operators (transformation) in a vector space $(H, +, \cdot)$, that maps each vector $|\psi\rangle$ in a vector space H into a vector $|\varphi\rangle \in H$ (or another vector space), $A|\psi\rangle = |\varphi\rangle$, is called a linear operator. If for every $|\varphi\rangle$, $|\psi\rangle \in H$ and $\lambda \in \mathbb{C}$ it fulfills the conditions :

(1)-$A(|\varphi\rangle + |\psi\rangle) = A|\varphi\rangle + A|\psi\rangle$;

(2)-$A(\lambda|\psi\rangle) = \lambda(A|\psi\rangle)$.

Let's consider two operators A and B, we can easily defined in the following operation :

$$(A+B)|\psi\rangle := A|\psi\rangle + B|\psi\rangle \ , \ (\lambda A)|\psi\rangle := \lambda(A|\psi\rangle) \ , \ (AB)|\psi\rangle := A(B|\psi\rangle) . \tag{2.18}$$

The definition of the operation : $A + B$ and AB is more complicated and involves questions on the domains and on the ranges of the operators.

Adjoint operator : For every linear operator A in vector space H ,there exists a unique linear operator $A^{\dagger}$ on the elements $|\varphi\rangle$ in for which

$$(|\psi\rangle, A|\varphi\rangle) = \left(A^{\dagger}|\psi\rangle, |\varphi\rangle\right), \tag{2.19}$$

for all $|\psi\rangle, |\varphi\rangle \in H$.

The operator $A^{\dagger}$ is called the adjoint operator of A. An operator for which $A^{\dagger} = A$ is called self-adjoint or Hermitian. On the other hand, An operator is anti-Hermitian or skew hermitian if $A^{\dagger} = -A$

Identity operator : We call Identity operator $\mathbb{1}$ on H , the linear operator such that $\mathbb{1}(|\psi\rangle) = |\psi\rangle$ for all $|\psi\rangle \in H$.

The zero operator, denoted 0 in H , $0|\psi\rangle = 0$ for all $|\psi\rangle \in H$.

The inverse operator : An operator B is called the inverse of an operator B if $BA = AB = \mathbb{1}$. The operator B is denoted by A^{-1}.

Projection operator : An operator is said to be a projection operator if it is Hermitien and satisfies

$$A^{\dagger} = A \quad , \quad A^2 = A, \tag{2.20}$$

the identity operator is projection operator $\mathbb{1}^{\dagger} = \mathbb{1}$, $\mathbb{1}^2 = \mathbb{1}$

Unitary operator : An operator is call unitary operator if $A = A^{-1}$

$$AA^{\dagger} = A^{\dagger}A = \mathbb{1}. \tag{2.21}$$

Normal operator : An operator A is said to be normal if

$$A^{\dagger}A = AA^{\dagger}. \tag{2.22}$$

Spectral theorem : An operator A in H is diagonalizable iff A is normal.

Eigenvector and eigenvalue : A nonzero vector $|\psi\rangle \in H$ is called an eigenvector of the linear operator A if

$$A|\psi\rangle = \lambda|\psi\rangle \quad \text{with} \quad \lambda \in \mathbb{C}; \tag{2.23}$$

λ is called the eigenvalue of A corresponding to the eigenvector $|\psi\rangle$.

For a given operator A, there may be many (perhaps infinitely many) different eigenvectors with different eigenvalues. There may also be n (finite or infinite) many different eigenvectors with the same eigenvalue λ. In this case, λ is called n-fold degenerate.

If A is a Hermitian operator defined on a Hilber space or pre-Hilbert space H, then eigenvectors and eigenvalues have the following properties :

(1)-All eigenvalues are real.

(2)- If $|\psi_1\rangle$ and $|\psi_2\rangle$ are eigenvectors of A with eigenvalues λ_1 and λ_2, respectively, and if $\lambda_1 \neq \lambda_2$, then $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal to each other $\langle\psi_1 \mid \psi_2\rangle = 0$.

Let A and B be two operators in H, the commutator of A and B is defined by

$$[A, B] \equiv AB - BA \quad \text{or} \quad [A, B]\,|\psi\rangle = AB\,|\psi\rangle - BA\,|\psi\rangle \qquad ,\forall\,|\psi\rangle \in H, \tag{2.24}$$

A and B are said to commute if $[A, B] \equiv AB - BA = 0$.

-We assume A is a set of all operators $(A, B, C...)$ define all over the scalar product space H.

C*-algebra A over H :

Let's consider a set A is an (associative) algebra with unit element iff

(1)- A is a vector space,

(2)- For every pair $A, B \in A$, a product $AB \in A$ is defined such that :

(a)-$(AB)\,C = A\,(BC)$;

(b)-$A(B + C) = AB + AC$;

(c)-$(A + B)C = AC + BC$;

(d)-$(\lambda A)B = A(\lambda B) = \lambda AB$,

(3) There exists an element $\mathbb{1} \in A$ such that

(a)-$\mathbb{1}A = A\mathbb{1} = A \;\;,\;\; \forall A \in A$,

(4)- An algebra A is called a $*-$algebra if we have on the algebra a $\dagger-$operation (involution), $A \longrightarrow A^\dagger$, that has the following defining properties :

(a)-$(\lambda A + \mu B)^\dagger = \lambda^* A^\dagger + \mu^* B^\dagger$;

(b)-$(AB)^\dagger = B^\dagger A^\dagger$;

(c)-$\left(A^\dagger\right)^\dagger = A$;

(d)-$\mathbb{1}^\dagger = \mathbb{1}$.

Thus the set of linear operators defined on the whole vector space H forms a $*-$algebra. A subalgebra of this algebra is called an "operator $*-$algebra". It can be shown that in a certain sense every $*-$algebra can be realized as an operator $*-$algebra in a scalar-product space.

-We call a subset A_1 of an algebra A a subalgebra of A if A_1 is an algebra with the same definitions of the operations $A, B \in A_1$ and $\lambda \in \mathbb{C}$, it follows that $A + B \in A_1$, $\lambda A \in A_1$, and $AB \in A_1$(for $A, B \in A$).

b$_4$) - Axioms of quantization

A simple isolated classical system a be quantized through the following axioms :

A$_1-$The quantum system and the state of system is described by vectors $|\psi\rangle \in H$, in this sense $|\psi\rangle$ is also called the state or a state vector, the state $|\psi\rangle$ and $c\,|\psi\rangle$ $(c \in \mathbb{C}, c \neq 0)$ describe the same state.

A$_2-$ A physical quantity (observable) is represented by a Hermitien operator $\hat{A}$ acting on H and the value that can take is one of its eigenvalues.

A$_3-$The Poisson bracket in CM (1.2) is replaced by the commutation (commutator algebra)

$$\left[\hat{A}, \hat{B}\right] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}. \tag{2.25}$$

multiplied by $\frac{i}{\hbar}$. The fundamental commutation relation are :

$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0 \qquad , \qquad [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}. \tag{2.26}$$

The Hamilton equation of motion (1.1) became

$$\dot{\hat{q}}_i = \frac{i}{\hbar}\left[\hat{q}_i, \hat{H}\right] \quad , \quad \dot{\hat{p}}_i = \frac{i}{\hbar}\left[\hat{p}_i, \hat{H}\right], \tag{2.27}$$

which is for $\hat{A}(t)$ given by Heisenberg equation of motion (will be clearified later in the next subsections), hence the time independent C-observable A (the classical quantity) satisfies the same equation as equation (1.7) (as we defiened in the previous subsection)

$$\frac{d\hat{A}\,(t)}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{A}\,(t)\right]. \tag{2.28}$$

A$_4-$Let $|\psi\rangle$ be an arbitrary state. The (real) number a state assigns to an Hermitien element is interpreted as the expectation value of the corresponding observable $\hat{A}(t)$

$$\langle A\rangle_t = \frac{\langle\psi \mid \hat{A}(t) \mid \psi\rangle}{\langle\psi \mid \psi\rangle}. \tag{2.29}$$

A$_5-$ For any physical state $|\psi\rangle \in H$, there exists an operator for which $|\psi\rangle$ is one of the eigenstates.

b_5) -Discrete bases representation : Operators

In the discrete representation every state vector $|\psi\rangle$ of the Hilbert space H is expressed by it components, using term of complete set of base kets with the orthonormality condition.

Since H is separable space and there are at most a countably infinite number of vectors in the basis. It is possible to construct an onthonormal basis $\{|\phi_n\rangle\}$ (Souder basis) such that :

$$(|\phi_m\rangle, |\phi_n\rangle) = \langle\phi_m \mid \phi_n\rangle = \delta_{nm} \tag{2.30}$$

where δ_{nm} is the Kronecker delta symbol defined by

$$\delta_{nm} = \begin{cases} 1 \text{ for } & m = n \\ 0 \text{ for } & m \neq n \end{cases}, \tag{2.31}$$

by using the scalar product between two ket vectors. Suppose $|\psi\rangle = \sum_{n=1}^{\infty} \psi_n |\phi_n\rangle$ by multiplying $\langle\phi_n|$ for the left one obtains $\langle\phi_n \mid \psi\rangle = \psi_n$ then $|\psi\rangle$ is expressed as

$$|\psi\rangle = \sum_{n=1}^{\infty} \langle\phi_n \mid \psi\rangle |\phi_n\rangle = \sum_{n=1}^{\infty} \langle\phi_n \mid \psi\rangle |\phi_n\rangle = \sum_{n=1}^{\infty} |\phi_n\rangle \langle\phi_n \mid \psi\rangle. \tag{2.32}$$

Since this is true for any $|\psi\rangle$, we have obtained the completeness relation

$$\mathbb{1} = \sum_{n=1}^{\infty} |\phi_n\rangle \langle\phi_n| . \tag{2.33}$$

Where $\mathbb{1}$ is the identity operator of H.
Or simply (2.29) with the use of identity operator one can obtain

$$\begin{aligned} |\psi\rangle &= \mathbb{1} |\psi\rangle ; \\ &= \left(\sum_{n=1}^{\infty} |\phi_n\rangle \langle\phi_n| \right) |\psi\rangle ; \\ &= \sum_{n=1}^{\infty} \psi_n |\phi_n\rangle . \end{aligned} \tag{2.34}$$

The bra-ket is given by

$$\langle \psi \mid \varphi \rangle = \sum_{n=1}^{\infty} \psi_n^* \varphi_n. \tag{2.35}$$

For each linear operator A, we can easily write

$$\begin{aligned} \hat{A} = \mathbb{1}\hat{A}\mathbb{1} &= \left(\sum_{n=1}^{\infty} |\phi_n\rangle \langle \phi_n| \right) \hat{A} \left(\sum_{n=1}^{\infty} |\phi_n\rangle \langle \phi_n| \right); \\ &= \sum_{nm} A_{nm} |\phi_n\rangle \langle \phi_n| , \end{aligned} \tag{2.36}$$

where A_{nm} is the nm matrix element of the operator $\hat{A}$

$$A_{nm} = \langle \phi_n| \hat{A} |\phi_m\rangle .$$

The matrix representation of $|\phi\rangle = \hat{A}|\psi\rangle$ can be obtien in the discrete representation by inserting a completeness relation and simplifying as follows

$$\begin{aligned} |\phi\rangle &= \hat{A}|\psi\rangle ; \\ \left(\sum_{n=1}^{\infty} |\phi_n\rangle \langle \phi_n| \right) |\phi_n\rangle &= \left(\sum_{n=1}^{\infty} |\phi_n\rangle \langle \phi_n| \right) \hat{A} \left(\sum_{n=1}^{\infty} |\phi_n\rangle \langle \phi_n| \right) |\psi\rangle ; \\ \sum_{n=1}^{\infty} b_n |\phi_n\rangle &= \sum_{n=1}^{\infty} \psi_n |\phi_n\rangle \langle \phi_n| A |\phi_m\rangle = \sum_{n=1}^{\infty} \psi_n A_{nm} |\phi_n\rangle ; \end{aligned} \tag{2.37}$$

where $b_n = \langle \phi_n \mid \phi\rangle$.
Similarly we calculate $\langle\varphi| \hat{A} |\psi\rangle$ through applying

$$\langle\varphi| \hat{A} |\psi\rangle = \langle\varphi| \mathbb{1}\hat{A}\mathbb{1} |\psi\rangle ;$$

$$\begin{aligned} \langle\varphi| \hat{A} |\psi\rangle &= \sum_{m,n} \langle\varphi \mid \phi_n\rangle \langle \phi_n| \hat{A} |\phi_m\rangle \langle \phi_m \mid \psi\rangle; \\ &= \sum_{n,m} b_n^* A_{nm} a_m. \end{aligned} \tag{2.38}$$

Now, we introduce another set $\left\{ \left| \phi_n^{'} \right\rangle \right\}$ and it has to be different from $\{|\phi_n\rangle\}$. The change of basis is represented by expressing kets $|\phi_n\rangle$ of the old basis by the new one

$$
\begin{aligned}
|\phi_n\rangle &= \left(\sum_{n=1}^{\infty} \left|\phi_m^{'}\right\rangle \left\langle \phi_m^{'}\right|\right) |\phi_n\rangle\,; \\
&= \sum_{n=1}^{\infty} U_{mn} \left|\phi_m^{'}\right\rangle, \qquad (2.39)
\end{aligned}
$$

where $U_{mn} = \langle \phi_m^{'} \mid \phi_n\rangle$ is matrix element.

The eigenvalues and eigenvectors of operator $\hat{A}$ are $\hat{A}\,|\psi\rangle = \lambda\,|\psi\rangle$, in this representation are expressed with use of identity operator by

$$
\sum_n A_{mn}\langle \phi_n \mid \psi\rangle = \lambda \sum_n \langle \phi_n \mid \psi\rangle \delta_{nm}, \qquad (2.40)
$$

it can be written

$$
\sum_n \left[A_{mn} - \lambda\delta_{nm}\right] \langle \phi_n \mid \psi\rangle = 0, \qquad (2.41)
$$

with $A_{mn} = \langle \phi_m|\, \hat{A}\, |\phi_n\rangle$.

The change of basis form are given by

$$
\begin{aligned}
A_{mn}^{'} &= \left\langle \phi_m^{'}\right| \left(\sum_{j=1}^{\infty} |\phi_j\rangle \langle \phi_j|\right) \hat{A} \left(\sum_{n=1}^{\infty} |\phi_l\rangle \langle \phi_l|\right) \left|\phi_n^{'}\right\rangle; \\
&= \sum_{jl} U_{mj} A_{jl} U_{nl}^{*}. \qquad (2.42)
\end{aligned}
$$

b_6) - Continues bases representation : Wave function The elements $|\psi\rangle$of H are interpreted as representing physical states and each state vector corresponds to one wave function :

For a moving particle on real line $\mathbb{R}$, we define on a pair of continuous bases : the continuous basis in the coordinate and momentum representation denoted as ($\{|x\rangle\}$ and $\{|p\rangle\}$) respectively, with $x, p \in \mathbb{R}$. We consider :

X as the position operator with the eigenvalue x and the corresponding eigenvector $|x\rangle$; $\hat{X}\,|x\rangle = x\,|x\rangle$. The eigenvectors are normalized as

$$
\langle x \mid x^{'}\rangle = \delta(x - x^{'}). \qquad (2.43)
$$

Similarly let p be the eigenvalue of $\hat{P}$ with the eigenvector $|p\rangle$; $\hat{P}\,|p\rangle = p\,|p\rangle$ with the obtained normalization

$$\langle p \mid p^{'}\rangle = \delta(p - p^{'}). \tag{2.44}$$

The inner product $\psi(x) \equiv \langle x \mid \psi\rangle$ is the component of $|\psi\rangle$ in the basis $|x\rangle$

$$\begin{aligned} |\psi\rangle &= \int |x\rangle\,\langle x \mid \psi\rangle dx, \\ &= \int \psi(x)\,|x\rangle\,dx, \end{aligned} \tag{2.45}$$

we intoduce the following identity

$$\mathbb{1} := \int_{-\infty}^{+\infty} |x\rangle\,\langle x|\,dx. \tag{2.46}$$

We define the coefficient $\langle x \mid \psi\rangle \in \mathbb{C}$ by the wave function, it is the probability amplitude of finding the particle at x in the state $|x\rangle$ namely $\mid \langle x \mid \psi\rangle \mid^2 dx$ the probability of finding the particle in the interval $[x, x + dx]$, the normalization condition in this representation is expressed

$$\int_{-\infty}^{+\infty} dx \mid \langle x \mid \psi\rangle \mid^2 dx = \langle\psi \mid x\rangle\langle x \mid \psi\rangle = 1. \tag{2.47}$$

Since the probability of finding the particle anywhere on the real line is always units.

Similarly $\psi(p) \equiv \langle p \mid \psi\rangle$ is the probability amplitude of finding the momentum of the particle in the interval $[p, p + dp]$ is $\mid \psi(p) \mid^2 dp$.

The scalar product of two arbitrary states $|\psi\rangle$, $|\varphi\rangle$ of H in terms of the wave-function is

$$\begin{aligned} \langle\psi \mid \varphi\rangle &= \int_{-\infty}^{+\infty} \psi^*\,(x)\,\varphi\,(x)\,dx; \\ &= \int_{-\infty}^{+\infty} \langle\psi \mid x\rangle\langle x \mid \varphi\rangle dx; \\ \langle\psi \mid \varphi\rangle &= \int_{-\infty}^{+\infty} \langle\psi \mid p\rangle\langle p \mid \varphi\rangle dp = \int_{-\infty}^{+\infty} \psi^*(p)\varphi(p)dp, \end{aligned} \tag{2.48}$$

with the following identity

$$\mathbb{1} := \int_{-\infty}^{+\infty} |p\rangle \langle p| \, dp. \tag{2.49}$$

Now, we write down the operators in the basis $|x\rangle$. From the defining equation $\hat{X}|x\rangle = x|x\rangle$, one obtains $\langle x|\hat{X} = \langle x| x$. Which yields after multiplication by $|\psi\rangle$ from the right

$$\begin{aligned} \langle x| \hat{X} |\psi\rangle &= x\langle x \mid \psi\rangle; \\ &= x\psi(x), \\ \hat{X}(\psi(x)) &= x\psi(x), \end{aligned} \tag{2.50}$$

the momentum operator for any state $|\psi\rangle$ of H, one obtains

$$\begin{aligned} \langle x| \hat{P} |\psi\rangle &= -i\frac{d}{dx}\langle x \mid \psi\rangle; \\ &= -i\frac{d}{dx}\psi(x). \end{aligned} \tag{2.51}$$

This is also written as

$$\hat{P}(\psi(x)) = -i\frac{d\psi(x)}{dx}. \tag{2.52}$$

Similarly if one uses a basis $|p\rangle$, one will have the momentum representation of the operators as

$$\hat{X}|p\rangle = -i\frac{d}{dp}|p\rangle \quad , \quad \hat{P}|p\rangle = p|p\rangle, \tag{2.53}$$

$$\langle p| \hat{X} |\psi\rangle = i\frac{d}{dp}\psi(p) \quad , \quad \langle p| \hat{P} |\psi\rangle = p\psi(x), \tag{2.54}$$

$$\langle x \mid p\rangle = \frac{1}{\sqrt{2\pi}}\exp(ipx) \quad , \quad \langle p \mid x\rangle = \frac{1}{\sqrt{2\pi}}\exp(-ipx), \tag{2.55}$$

$\psi(p)$ could be defined as Fourier transformation of $\psi(x)$:

$$\psi(p) = \langle p \mid \phi \rangle = \int dx \; \langle p \mid x \rangle \langle x \mid p \rangle;$$
$$= \int \frac{dx}{\sqrt{2\pi}} \exp(-ipx)\psi(x). \tag{2.56}$$

we can define the same in 3D

Let's consider a pair of continuous bases : the continuous basis in the coordinate and momentum representation denoted as $(\{|\overrightarrow{r}\rangle\}$ and $\{|\overrightarrow{p}\rangle\})$ which are respectively eigenkets of the position $\overrightarrow{\hat{R}}$ and momentum operator $\overrightarrow{\hat{P}}$

$$\overrightarrow{\hat{R}}\,|\overrightarrow{r}\rangle = \overrightarrow{r}\,|\overrightarrow{r}\rangle \quad , \quad \overrightarrow{\hat{P}}\,|\overrightarrow{p}\rangle = \overrightarrow{p}\,|\overrightarrow{p}\rangle\,, \tag{2.57}$$

with the following normalization for position and momentum

$$\langle \overrightarrow{r} \mid \overrightarrow{r}' \rangle = \delta(\overrightarrow{r} - \overrightarrow{r}');$$
$$\langle \overrightarrow{r} \mid \overrightarrow{r}' \rangle = \delta(x - x^{'})\delta(y - y^{'})\delta(z - z^{'}), \tag{2.58}$$

$$\langle \overrightarrow{p} \mid \overrightarrow{p}' \rangle = \delta(\overrightarrow{p} - \overrightarrow{p}');$$
$$= \delta(p_x - p_x^{'})\delta(p_y - p_y^{'})\delta(p_z - p_z^{'}), \tag{2.59}$$

the component of $|\psi\rangle$ in the basis of $|\overrightarrow{r}\rangle$ are written as follows

$$|\psi\rangle \equiv \int \psi(\overrightarrow{r})\,|\overrightarrow{r}\rangle\, dr^3. \tag{2.60}$$

The 3D position representation inner product is given by

$$\langle \psi \mid \varphi \rangle = \int_{-\infty}^{+\infty} \psi^*(\overrightarrow{r})\,\varphi(\overrightarrow{r})\, dr^3, \tag{2.61}$$

with the following completeness relation

$$\mathbb{1} := \int_{-\infty}^{+\infty} |\overrightarrow{r}\rangle\,\langle \overrightarrow{r}|\, dr^3. \tag{2.62}$$

We also can define this relation

$$\langle \overrightarrow{r} \mid \overrightarrow{p} \rangle = \frac{1}{\sqrt[3]{2\pi\hbar}} \exp(i\overrightarrow{p}\cdot\overrightarrow{r}) \qquad , \quad \langle \overrightarrow{p} \mid \overrightarrow{r} \rangle = \frac{1}{\sqrt[3]{2\pi\hbar}} \exp(-i\overrightarrow{p}\cdot\overrightarrow{r}). \tag{2.63}$$

The action of momentum operator on the wave function in position representation expressed by

$$\begin{aligned}
\langle \overrightarrow{r} | \overrightarrow{\hat{P}} |\psi\rangle &= \int \langle \overrightarrow{r} | \overrightarrow{\hat{P}} |\overrightarrow{p}\rangle \langle \overrightarrow{p} \mid \psi\rangle dp^3; \\
&= \frac{1}{\sqrt[3]{2\pi\hbar}} \int \overrightarrow{p} \exp(i\overrightarrow{p}\cdot\overrightarrow{r})\psi(p)\, dp^3; \\
&= -i\hbar\nabla \int \frac{1}{\sqrt[3]{2\pi\hbar}} \exp(i\overrightarrow{p}\cdot\overrightarrow{r})\psi(p)\, dp^3; \\
\langle \overrightarrow{r} | \overrightarrow{\hat{P}} |\psi\rangle &= -i\hbar\nabla\langle \overrightarrow{r} \mid \psi\rangle,
\end{aligned} \tag{2.64}$$

where $\hat{P} = -i\hbar\overrightarrow{\nabla}$.

We derive the Schrödinger differential equation, which $\psi(\overrightarrow{r})$ is a function of continuos position satisfies by applying $\langle \overrightarrow{r}|$ on from the left, we obtain

$$\langle \overrightarrow{r} | i\frac{d}{dt} |\psi(t)\rangle = \langle \overrightarrow{r} | \hat{H} |\psi(t)\rangle \tag{2.65}$$

Now, we obtain the time-dependent Schrödinger equation for the Hamiltonian of the type $\hat{H} = \frac{\hat{p}^2}{2m} + v(\overrightarrow{r})$ by

$$\begin{aligned}
i\frac{d}{dt}\psi(\overrightarrow{r},t) &= \langle \overrightarrow{r} | \frac{\overrightarrow{\hat{P}}^2}{2m} + v(\overrightarrow{r}) |\psi(t)\rangle; \\
&= -\frac{1}{2m}\nabla^2\psi(\overrightarrow{r},t) + v(\overrightarrow{r})\,\psi(\overrightarrow{r},t),
\end{aligned} \tag{2.66}$$

where $\psi(\overrightarrow{r},t) \equiv \langle \overrightarrow{r} \mid \psi(t)\rangle$.

b_7) -Pictures and evolution equations In quantum mechanics we identify two type of representations (pictures) with two different evolution equations :

The first evolution equation is given by the Heisenberg equation (2.28), where the formal solution of this equation is easily obtained as

$$A(t) = \exp(i\hat{H}t)\hat{A}(0)\exp(-i\hat{H}t), \tag{2.67}$$

where the operators $\hat{A}\,(t)$ and $\hat{A}\,(0)$ in (2.67) are related by the unitary operator

$$U\,(t) = e^{-i\hat{H}t}. \tag{2.68}$$

-This representation called the Heisenberg picture, where we consider that the state of the system at a given moment is determined by the values of all the observables on this state at this moment. It is the algebra of observables that becomes the central element (the formulation of matrix mechanics). The evolution equation is done in this algebra, and an observable depends on time through the Heisenberg equation.

Now to the second evolution equation, let us write down the expectation value of $\hat{A}$ with respect to the state $|\psi\rangle$ as

$$\langle A\rangle_t = \langle\psi|\exp(i\hat{H}t)\hat{A}\exp(-i\hat{H}t)\,|\psi\rangle\,. \tag{2.69}$$

If we write $|\psi(t)\rangle \equiv e^{-i\hat{H}t}\,|\psi\rangle$ we find that the expectation value at t is also expressed as

$$\langle A\rangle_t = \langle\psi\,(t)|\,\hat{A}\,(0)\,|\psi\,(t)\rangle\,. \tag{2.70}$$

-This is the so-called the Schrödinger picture, where it is interpreted as state function evolution $i\frac{d}{dt}\,|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle$ where the evolution equation is a differential of normalized time-dependent state associated to a self-adjoint operator $\hat{H}$ on the Hilbert space which is the Hamiltonian. The expectation value of an observable on a normalized state $|\psi\rangle$ in this representation is expressed at time t by (2.70) this expectation value is real. Here in this representation wave mechanics becomes the main element.

Generalize uncertainty relation Let's take the expectation values of two observables $\hat{A}$ and $\hat{B}$ on a normalize state vector $|\psi\rangle$: $\langle\hat{A}\rangle = \langle\psi|\,\hat{A}\,|\psi\rangle$, $\langle\hat{B}\rangle = \langle\psi|\,\hat{B}\,|\psi\rangle\,.$

We introduce the operators $\triangle\hat{A}$ and $\triangle\hat{B}$:

$$\triangle\hat{A} = \hat{A} - \langle\hat{A}\rangle \quad , \triangle\hat{B} = \hat{B} - \langle\hat{B}\rangle. \tag{2.71}$$

Where

$$\left(\triangle\hat{A}\right)^2 = \hat{A}^2 - 2\hat{A}\langle\hat{A}\rangle + \langle\hat{A}\rangle^2,$$
$$\left(\triangle\hat{B}\right)^2 = \hat{B}^2 - 2\hat{B}\langle\hat{B}\rangle + \langle\hat{B}\rangle^2. \tag{2.72}$$

We have $\langle\hat{A}^2\rangle = \langle\psi|\,\hat{A}^2\,|\psi\rangle$, $\langle\hat{B}^2\rangle = \langle\psi|\,\hat{B}^2\,|\psi\rangle$, the uncertainties $\triangle A$ and $\triangle B$ is defined by

$$\triangle A = \sqrt{\langle\left(\triangle\hat{A}\right)^2\rangle} = \sqrt{\langle\hat{A}^2\rangle + \langle\hat{A}\rangle^2},$$
$$\triangle B = \sqrt{\langle\left(\triangle\hat{B}\right)^2\rangle} = \sqrt{\langle\hat{B}^2\rangle + \langle\hat{B}\rangle^2}. \tag{2.73}$$

The action of thus two observables on any state $|\psi\rangle$ is given as follows

$$|\varphi\rangle = \triangle\hat{A}\,|\psi\rangle = \left(\hat{A} - \langle\hat{A}\rangle\right)|\psi\rangle \ \ , \ |\phi\rangle = \triangle\overset{\frown}{B}\,|\psi\rangle = \left(\hat{B} - \langle\hat{B}\rangle\ \right)|\psi\rangle\,,$$
$$\langle\varphi| = \langle\psi|\left(\hat{A} - \langle\hat{A}\rangle\ \right) \qquad , \qquad \langle\phi| = \langle\psi|\left(\hat{B} - \langle\hat{B}\rangle\ \right). \tag{2.74}$$

We can construct the famous Heisenberg uncertainty relation for arbitrary obrervables $\hat{A}$ and $\hat{B}$ by defining

$$\triangle\hat{A}^2\triangle\hat{B}^2 = \langle\varphi \mid \varphi\rangle\langle\phi \mid \phi\rangle. \tag{2.75}$$

Now we can apply the Cauchy-Schwarz inequality (1.8), which is true for any inner product with the following relation

$$\operatorname{Im}(\langle\varphi \mid \phi\rangle) = \left[\frac{1}{2i}\left(\langle\varphi \mid \phi\rangle - \langle\phi \mid \varphi\rangle\right)\right], \tag{2.76}$$

which is also valid for any complex number $\langle\varphi \mid \phi\rangle \in \mathbb{C}$.

We insert to (2.75) the inner product formula introduced above and with relations (1.8) and (1.76), we obtain

$$\triangle\hat{A}^2\triangle\hat{B}^2 \geq \left[\frac{1}{2i}\left(\langle\varphi \mid \phi\rangle - \langle\phi \mid \varphi\rangle\right)\right]^2;$$
$$\triangle\hat{A}^2\triangle\hat{B}^2 \geq \left[\frac{1}{2i}\left(\langle\psi|\left(\hat{A} - \langle\hat{A}\rangle\ \right)\left(\hat{B} - \langle\hat{B}\rangle\ \right)|\psi\rangle - \langle\psi|\left(\hat{B} - \langle\hat{B}\rangle\ \right)\left(\hat{A} - \langle\hat{A}\rangle\ \right)|\psi\rangle\right)\right]^2; \tag{2.77}$$

it's easy to simplify (1.77) as

$$\triangle\hat{A}^2\triangle\hat{B}^2 \geq \Big[\frac{1}{2i}((\langle\psi|\,\hat{A}\hat{B} - \langle\hat{A}\rangle\hat{B} - \langle\hat{B}\rangle\ \hat{A} + \langle\hat{A}\rangle\langle\hat{B}\rangle\,|\psi\rangle$$
$$- \langle\psi|\,\hat{B}\hat{A} - \langle\hat{B}\rangle\ \hat{A} - \langle\hat{A}\rangle\hat{B} + \langle\hat{B}\rangle\langle\hat{A}\rangle\,|\psi\rangle)\Big]^2;$$

$$\triangle\hat{A}^2\triangle\hat{B}^2 \geq \Big[[\frac{1}{2i}((\langle\psi|\,\hat{A}\hat{B}\,|\psi\rangle - \langle\hat{A}\rangle\,\langle\psi|\,\hat{B}\,|\psi\rangle - \langle\hat{B}\rangle\,\langle\psi|\,\hat{A}\,|\psi\rangle + \langle\hat{A}\rangle\langle\hat{B}\rangle\ \langle\psi\mid\psi\rangle$$
$$- \langle\psi|\,\hat{B}\hat{A}\,|\psi\rangle - \langle\hat{B}\rangle\ \ \langle\psi|\,\hat{A}\,|\psi\rangle - \langle\hat{A}\rangle\,\langle\psi|\,\hat{B}\,|\psi\rangle + \langle\hat{B}\rangle\langle\hat{A}\rangle\langle\ \psi\mid\psi\rangle)\Big]^2;$$

$$\triangle\hat{A}^2\triangle\hat{B}^2 \geq \Big[\frac{1}{2i}((\langle\psi|\,\hat{A}\hat{B}\,|\psi\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle - \langle\hat{B}\rangle\langle\hat{A}\rangle\ + \langle\hat{A}\rangle\langle\hat{B}\rangle$$
$$- \langle\psi|\,\hat{B}\hat{A}\,|\psi\rangle + \langle\hat{B}\rangle\ \langle\hat{A}\rangle + \langle\hat{A}\rangle\langle\hat{B}\rangle - \langle\hat{B}\rangle\langle\hat{A}\rangle)\Big]^2;$$

$$\triangle\hat{A}^2\triangle\hat{B}^2 \geq \Big[\frac{1}{2i}\Big(\langle\psi|\,\hat{A}\hat{B}\,|\psi\rangle - \langle\psi|\,\hat{B}\hat{A}\,|\psi\rangle\Big)\Big]^2;$$
$$\triangle\hat{A}^2\triangle\hat{B}^2 \geq \Big[\frac{1}{2i}\Big(\langle\psi|\,\hat{A}\hat{B}\ \ - \hat{B}\hat{A}\,|\psi\rangle\Big)\Big]^2;$$
$$\triangle\hat{A}^2\triangle\hat{B}^2 \geq \Big[\frac{1}{2i}\Big(\langle\big[\hat{A},\hat{B}\big]\rangle\Big)\Big]^2;$$
$$\triangle\hat{A}^2\triangle\hat{B}^2 \geq \mid \frac{1}{2i}\langle\big[\hat{A},\hat{B}\big]\rangle\mid^2;$$

$$\triangle\hat{A}\triangle\hat{B} \geq \frac{1}{2}\mid\langle\big[\hat{A},\hat{B}\big]\rangle\mid \tag{2.78}$$

The measurement of $\hat{B}$ made us lose all the information we had obtained on $\hat{A}$ during the first measurement : the observable $\hat{A}$ and $\hat{B}$ do not commute (incompatible), i.e. we cannot measure it simultaneously (problem of simultaneous diagonalization of operators).

2.2 General Heisenberg uncertainty principal and deformed algebra

2.2.1 Momentum space and general Hisenberg uncertainty principal

In the introduction, we have mentioned that quantum gravity is characterized by the Planck length [20], at the Planck length gravitational effect can not be ignored, the extreme energy concentration in a small space will create a black hole with an event horizon. To this fact, several studies where minimal length played an essential role such as non-commutative geometries [21] , the Cosmological Constant Problem [22] , and string theory [23] have proposed small corrections to the Heisenberg uncertainty principle of the form :$(\triangle X)(\triangle P) \geq \frac{\hbar}{2}\left(1+\beta(\triangle p)^2+...\right)$, this form known as generalized uncertainty principle (GUP). This correction has as a consequences, the minimum non-zero uncertainty $(\triangle x)_{\min}$ which can be related to the size of the particles and the modification of the canonical commutation relation between the position operator and the momentum operator which become : $\left[\hat{X},\hat{P}\right] = i\hbar\left(1+\beta p^2+...\right)$, where β is a small positive parameter called the deformation parameter.

Minimal length deformed QM (ml-QM)

The (1+1) dimensional structure of ml-QM In the present work of Kempf [11], the one dimonsionl commutation relation between position and momentum is written

$$\left[\hat{X},\hat{P}\right] = i\hbar(1+\beta p^2), \tag{2.79}$$

the relation (2.79) implies the appearance of a non-zero minimal uncertainty $\triangle X_0$ in position.

The respective uncertainty relation (GUP) to (2.79) is given below

$$\triangle X \triangle P \geq \frac{\hbar}{2}(1+\beta(\triangle p)^2+\gamma), \tag{2.80}$$

where β and γ are two positive parameters independent of X and P but which depend on the mean value of the operators X and P ($\gamma = \beta\langle p\rangle^2$), we can note that in the case where these quantities are zero we find ourselves back to the Heisenberg uncertainty relation of ordinary quantum mechanics

$$\triangle X \triangle P \geq \frac{\hbar}{2}. \tag{2.81}$$

This fact explains the random characteristic of the variation of $\triangle X$ and $\triangle P$ (the increase of $\triangle X$ implies the decrease of $\triangle P$). On the other hand, the dependence of the relation (2.80) in $\beta(\triangle p)^2$ shows that $\triangle X$ cannot take any small value in an arbitrary way. This gives rise to the minimal uncertainty in the position measurement $\triangle X_0$

$$\triangle X_0 \geq \frac{\hbar\sqrt{2\beta}}{4}(3+\gamma). \tag{2.82}$$

The expression (2.80) is satisfied in the interval : $[-\triangle p, +\triangle p]$ for fixed $\triangle X$

$$\triangle P_{\pm} = \frac{\triangle X}{\hbar\beta} \pm \sqrt{\frac{\triangle X}{\hbar\beta} - \frac{1}{\beta} - \langle p\rangle^2}. \tag{2.83}$$

The minimal value $(\triangle X)_{\min}$ is obtained by

$$\begin{aligned}(\triangle X)_{\min}(\langle p\rangle) &= \hbar\sqrt{\beta}\sqrt{1+\beta\langle p\rangle^2};\\ &= \hbar\sqrt{\beta}, \quad \text{with } \langle p\rangle = 0,\end{aligned} \tag{2.84}$$

where the smallest value of $(\triangle X)_{\min}$ denoted by $\triangle X_0$ is non-zero

$$\triangle X_0 = \hbar\sqrt{\beta}. \tag{2.85}$$

In a similar way to ordinary QM Heisenberg canonical commutation relation , we can find a representation of $\hat{X}$ and $\hat{P}$ which verifies the modified commutation relation above (2.79) described as

$$\hat{X} = i\hbar(1+\beta p^2)\partial_p \ , \ \hat{P} = \hat{p} \ . \tag{2.86}$$

In the momentum space, we can denote the position $\hat{X}$ and momentum $\hat{P}$ operators and $\psi(p)$ is the wave function, which is defined on the momentum space parameterized by p as

$$\hat{P}\psi(p) = p\psi(p) \ , \ \hat{X}\psi(p) = i\hbar(1+\beta p^2)\frac{\partial}{\partial p}\psi(p). \tag{2.87}$$

As we notice, $\hat{P}$ is still obviously symmetric

$$\left(\langle\psi|\,\hat{P}\right)\varphi\rangle = \langle\psi\left(\hat{P}\,|\varphi\rangle\right), \tag{2.88}$$

but we can not say the same thing about the symmetry of $\hat{X}$, it's can be seen and satisfied only through new scalar product formula in this deformed algebra

$$\langle\psi \mid \varphi\rangle = \int_{-\infty}^{+\infty} \frac{dp}{(1+\beta p^2)}\psi^*(p)\varphi(p). \tag{2.89}$$

Now, lets show the symmetry of $\hat{X}$ with the use of deformed algebra scalar product (2.89)

$$\begin{aligned}\langle\psi|\left(\hat{X}\,|\varphi\rangle\right) &= \int_{-\infty}^{+\infty} \frac{dp}{1+\beta p^2}\psi^*(p)\left[i\hbar\left(1+\beta p^2\right)\partial_p\right]\varphi(p);\\ &= i\hbar\int_{-\infty}^{+\infty} dp\psi^*(p)\partial_p\varphi(p);\\ &= \left[i\hbar\psi^*(p)\varphi(p)\right]_{-\infty}^{+\infty} - i\hbar\int_{-\infty}^{+\infty} dp\left(\partial_p\psi^*(p)\right)\psi(p) = \left(\langle\psi|\,\hat{X}\right)\varphi\rangle.\end{aligned} \tag{2.90}$$

On the other hand with the same way (integrating by parts) we obtain

$$\begin{aligned}\left(\langle\psi \mid \hat{X}\right) \mid \varphi\rangle &= \int_{-\infty}^{+\infty} \frac{dp}{1+\beta p^2}\left[i\hbar\left(1+\beta p^2\right)\partial_p\psi\left(p\right)\right]^*\varphi\left(p\right);\\ &= -i\hbar\int_{-\infty}^{+\infty} dp\left(\partial_p\psi^*\left(p\right)\right)\varphi\left(p\right).\end{aligned} \tag{2.91}$$

The modification of this product implies a new completeness relation, which is written by

$$\int_{-\infty}^{+\infty} \frac{dp}{1+\beta p^2}\,|p\rangle\,\langle p| = \mathbb{1}. \tag{2.92}$$

The scalar product of momentum eignstates is given by

$$\langle p \mid p^{'}\rangle = (1+\beta p^2)\delta(p-p^{'}), \tag{2.93}$$

and it's equal to

$$\langle p \mid p^{'} \rangle = \delta(\frac{1}{\sqrt{\beta}}\arctan(\sqrt{\beta}p) - \frac{1}{\sqrt{\beta}}\arctan(\sqrt{\beta}p^{'})). \tag{2.94}$$

A brief review on the ml-QM structure in the form of (3+1) dimension Kempf algebra The development of the field of research related to the minimum uncertainty problem for relativistic and non-relativistic quantum systems has been mainly realized by [11] , which have established generalized commutation relations according to the dimensions of the momentum representation space. We shall see that the 3D generalization of the QM introduced in previous subsection with respect to the commutation relation of the three dimensional deformed Kempf algebra takes the following tensor form [15] :

$$[X_i, P_j] = i\hbar[\delta_{ij}(1 + \beta P^2) + \beta^{'} p_i p_j], \tag{2.95}$$

$$[X_i, X_j] = -i\hbar[2\beta - \beta^{'} + (2\beta + \beta^{'})\beta P^2]\epsilon_{ijk} L_k, \tag{2.96}$$

$$[P_i, P_j] = 0, \tag{2.97}$$

where β , $\beta^{'}$and γ are very small non negative parameters.

From the generalized commutation relations above the position and momentum operators in the momentum space are defined the following form

$$X_i = i\hbar[(1 + \beta P^2)\frac{\partial}{\partial p_i} + \beta^{'} p_i p_j \frac{\partial}{\partial p_j} + \gamma p_i] \qquad , \quad \widehat{P}_i = p_i. \tag{2.98}$$

The angular momentum operator is given in this representation by

$$L_i = (1 + \beta P^2)^{-1}\epsilon_{ijk} X_j P_k \qquad\qquad i = 1, 2, 3. \tag{2.99}$$

The (2.97) satisfy the well-known commutation relation as follows

$$[L_i, X_j] = ih\epsilon_{ijk} X_k \qquad , \qquad [L_i, P_j] = ih\epsilon_{ijk} P_j. \tag{2.100}$$

We can express modified form of GUP (2.84) as

$$\triangle X_i \triangle P_i \geq \frac{\hbar}{2}[1 + 3\beta(\triangle P_i)^2 + \beta^{'}(\triangle P_i)^2]. \tag{2.101}$$

As for the generalized minimal uncertainty relation in 3D (2.79), the minimal uncertainty in position is isotropic

$$(\triangle X)_{\min}(\langle p\rangle) = \hbar\sqrt{3\beta+\beta'}. \tag{2.102}$$

The completeness relation (2.91) in momentum space becomes

$$\mathbb{1} = \int_{-\infty}^{+\infty} \frac{d^3p}{\left(\left[1-(\beta+\beta')p^2\right]^{1-\frac{\gamma-\beta'}{\beta+\beta'}}\right)} \mid p\rangle\langle p \mid . \tag{2.103}$$

Finally, the scalar product in (2.89) turns to

$$\langle\psi \mid \varphi\rangle = \int_{-\infty}^{+\infty} \frac{d^3p}{\left[1-(\beta+\beta')p^2\right]^{1-\frac{\gamma-\beta'}{\beta+\beta'}}} \psi^*(p)\varphi(p). \tag{2.104}$$

2.2.2 Configuration space representation in presence of minimal length

In the presence of a minimal measurable length, as we have seen in previous section and also in the introduction, all literature ([13, 15]) represented the calculation in the momentum spaces, that goes back to the fact that we can introduce our QM systems on momentum space without any difficulty meanwhile, in the position space representation of the Hilbert space of standard QM collapses and breaks down. More exactly (Ref [11]) :

In ordinary quantum mechanics continuous representation, we have defined momentum space element $\psi(p) \equiv \langle p \mid \psi\rangle$ and position matrix element $\psi(x) \equiv \langle x \mid \psi\rangle$ (as we explaind prevouislly), where $|x\rangle$ is position eigenstate and $|p\rangle$ is momentum eigenstate. These eigenstates can be approximated to arbitrary precision by sequences $|\psi_n\rangle$ of physical states of increasing localization in position or momentum space

$$\begin{aligned} \lim_{n\longrightarrow 0} \triangle X_{|\psi_n\rangle} &= 0, \\ \lim_{n\longrightarrow 0} \triangle P_{|\psi_n\rangle} &= 0. \end{aligned} \tag{2.105}$$

But with the presence of a minimal measurable length, the situation gets different because of the uncertainties $\triangle X_0 \geq 0$ and $\triangle P \geq 0$. A non-zero minimal uncertainty in position implies that there cannot be any physical state which is a position eigenstate since an eigenstate would have zero uncertainty in position, we have $\forall \left|\psi\right\rangle$

$$\begin{aligned}\left\langle\psi\right|(\triangle X)^2\left|\psi\right\rangle &= (\triangle X)^2\,\psi;\\ &= \left\langle\psi\right|\left(\hat{X}-\left\langle\psi\right|\hat{X}\left|\psi\right\rangle\right)^2\left|\psi\right\rangle \geq \triangle X_0\ ,\end{aligned} \tag{2.106}$$

The position operator $\hat{X}$ here is no longer self-adjoint but only symmetric (as we shown above) which means then there are no more position eigenstates $\left|x\right\rangle$ in the representation of the Heisenberg algebra. The position space representation in the presence of minimal length can be represented only through the maximally localized states or quasi-space representation. This representation has a direct interpretation in terms of position measurements, although it does not mean that $\hat{X}$ is diagonalised.

The Brau reduction

The commutation relation presented by Kempf are not unique [25], the same goes for the Kempf representation that we have introduced (in previous section). Now we introduce a new representation known as the Brau representation this representation satisfies the condition (2.79) and it's given as follows

$$\hat{X}=\hat{x} \quad , \quad \hat{P}=f(\hat{p}), \tag{2.107}$$

from (2.106) we notice that unlike the Kempf representation, the Brau representation preserves the ordinary nature of the position operator and defines a symmetric position operator $\hat{X}$ and moment operator $\hat{P}$ taken by form expansion of an injective function

$$f(\hat{p})=\hat{p}(1+\frac{\beta}{3}\hat{p}^2+\ldots), \tag{2.108}$$

the operator $\hat{P}$ can be defined by obtaining $f(\hat{p})$ in the first order in expansion form

$$\hat{X}=\hat{x} \qquad , \qquad \hat{P}=\hat{p}(1+\frac{\beta}{3}\hat{p}^2). \tag{2.109}$$

The (2.108) is introduced in general form [25] by Stetsko and Tkachuk ($\beta=2\beta'$) as

$$\hat{X}_i = \hat{x}_i + \frac{2\beta - \beta^{'}}{4}\left(\hat{p}^2\hat{x}_i + \hat{x}_i\hat{p}^2\right),$$
$$\hat{P}_i = \hat{p}_i(1 + \beta^{'}\hat{p}^2). \tag{2.110}$$

Quasi-position space representation : the maximal localized states

The states with maximum localized around the position x are, by definition, states $\left|\psi_x^{ml}\right\rangle$ satisfying the two following conditions [11]

$$\left\langle\psi_x^{ml}\right|\hat{X}\left|\psi_x^{ml}\right\rangle = \widehat{x} \quad , \quad (\triangle X)_{\left|\psi_x^{ml}\right\rangle} = (\triangle X)_{\min}, \tag{2.111}$$

given that the smallest value of the minimum uncertainty $(\triangle X)_{\min}$ in formula (2.84) corresponds to $\langle p\rangle = 0$, starting from this expression $\|\ (\hat{X} - \langle\hat{X}\rangle)^2 + \left(\frac{|\langle[\hat{X},\hat{P}]\rangle|}{2(\triangle P)^2}\right)^2\left(\hat{P} - \langle\hat{P}\rangle^2\right)|\psi\rangle\ \| \geq 0$ we can establish the uncertainty relation $\left[\hat{X},\hat{P}\right]$ being imaginary, then

$$\langle\psi|\,(\hat{X} - \langle\hat{X}\rangle)^2 + \left(\frac{|\ \langle\left[\hat{X},\hat{P}\right]\rangle\ |}{2\left(\triangle P\right)^2}\right)^2\left(\hat{P} - \langle\hat{P}\rangle^2\right)|\psi\rangle \geq 0, \tag{2.112}$$

which immediately implies the uncertainty relation

$$\triangle X \triangle P \geq \frac{1}{2}\ |\ \langle\left[\hat{X},\hat{P}\right]\rangle\ |, \tag{2.113}$$

so the state $|\psi\rangle$ can be said to verify $\triangle X \triangle P \geq \frac{1}{2}\ |\ \langle\left[\hat{X},\hat{P}\right]\rangle\ |$ if and only if it satisfies

$$\left(\hat{X} - \langle\hat{X}\rangle + \frac{\langle\left[\hat{X},\hat{P}\right]\rangle}{2\left(\triangle P\right)^2}\left(\hat{P} - \langle\hat{P}\rangle\right)\right)|\psi\rangle = 0. \tag{2.114}$$

This expression is injected into the phase space and we obtain

$$\left[i\hbar\left(1 + \beta p^2\right)\partial_p - \langle\hat{X}\rangle + \frac{i\hbar\left(1 + \beta\left(\triangle p\right)^2 + \beta\langle\hat{P}\rangle\right)}{2\left(\triangle P\right)^2}\left(\hat{P} - \langle\hat{P}\rangle\right)\right]\psi\left(p\right) = 0, \tag{2.115}$$

whose solution is expressed by

$$\psi\left(p\right)=N\left[\frac{\left(\frac{\langle\hat{X}\rangle}{i\hbar\sqrt{\beta}}-\frac{1+\beta(\triangle p)^{2}+\beta\langle\hat{P}\rangle^{2}\langle\hat{P}\rangle}{2(\triangle P)^{2}\sqrt{\beta}}\right)}{\left(1+\beta p^{2}\right)^{\frac{1+\beta(\triangle p)^{2}+\beta\langle\hat{P}\rangle^{2}}{2\beta\langle\triangle P\rangle^{2}}}}\right]. \tag{2.116}$$

The maximum localized states correspond to the case $\langle\hat{P}\rangle=0$ where $(\triangle X)_{\min}=\hbar\sqrt{\beta}$. The relation (2.79) implies $\triangle P=\frac{1}{\beta}$ that we can obtain the maximal localization states maximal

$$\psi_{x}^{ml}\left(p\right)=N\left(1+\beta p^{2}\right)^{\frac{-1}{2}}\exp\left(-i\frac{x}{\hbar\beta}\arctan\left(\sqrt{\beta}p\right)\right)\qquad;\quad N=\sqrt{\frac{2\sqrt{\beta}}{\pi}}. \tag{2.117}$$

The states (2.116) is a generalization of the maximum localized states in ordinary quantum mechanics. Now, the states are physical states ; and we can see that the divergence of the mean value of energy is absorbed. In effect

$$\int_{-\infty}^{+\infty}\frac{dp}{1+\beta p^{2}}\frac{P^{2}}{2m}=\frac{1}{2m\beta}. \tag{2.118}$$

The scalar product of states maximum localization $\langle\psi_{x}^{ml}\mid\psi_{x}^{ml}\rangle$ as a function of $\left(x-x^{'}\right)$ defined

$$\begin{aligned}\langle\psi_{x}^{ml}\mid\psi_{x}^{ml}\rangle&=\frac{2\sqrt{\beta}}{\pi}\int_{-\infty}^{+\infty}\frac{dp}{\left(1+\beta p^{2}\right)^{2}}\exp\left(-i\frac{\left(x-x^{'}\right)}{\hbar\beta}\arctan\left(\sqrt{\beta}p\right)\right);\\&=\frac{1}{\pi}\left[\frac{\left(x-x^{'}\right)}{2\pi\sqrt{\beta}}-\left(\frac{x-x^{'}}{2\hbar\sqrt{\beta}}\right)^{3}\right]^{-1}\sin\left(\frac{x-x^{'}}{2\hbar\sqrt{\beta}}\right).\end{aligned} \tag{2.119}$$

The maximum localized states are not generally orthogonal and this is of origin of the new structure of our modified space.

The Quasi-position representation in position space : Position wave function

The introduction of a minimal uncertainty on the position caused in direct way to the non-existence of a complete basis of the eigenstates $\{|x\rangle\}$

of the position operator $\widehat{X}$. However the maximum localized states $\left|\psi_x^{ml}\right\rangle$ can be used to project arbitrary states$|\varphi\rangle$ .The projections $\varphi(x) = \langle\psi_x^{ml} \mid \varphi\rangle$ will be considered as wave functions in a representation to what's called "quasi-configuration representation", where $|\ (\varphi(x))\ |^2$ will be interpreted as the probability amplitude for the particle to be localized with uncertainty $(\triangle X)_{\min}$ around the position x

$$\varphi(x) = \langle\psi_x^{ml} \mid \varphi\rangle = \sqrt{\frac{2\sqrt{\beta}}{\pi}} \int_{-\infty}^{+\infty} \frac{dp\varphi(p)}{(1+\beta p^2)^{\frac{3}{2}}} \exp\left(\frac{ix \arctan\left(\sqrt{\beta}p\right)}{\hbar\sqrt{\beta}}\right). \tag{2.120}$$

This relation represents the generalized Fourier transform, is the same as relation (2.56) in standard QM allow the passage from the representation of the momentum to the quasi-representation of the position.

The modified dispersion relation corresponding to this "generalized plane wave" is

$$k = \frac{2\pi}{\lambda} = \frac{\arctan\left(\sqrt{\beta}p\right)}{\hbar\sqrt{\beta}}, \tag{2.121}$$

$$\lambda(E) = \frac{2\pi\hbar\sqrt{\beta}}{\arctan(\sqrt{2m\beta E})}. \tag{2.122}$$

Similarly, the Fourier transformation of a quasi-position wavefunction into a momentum space wave function is given by

$$\varphi(p) = \frac{1}{\sqrt{8\pi\sqrt{\beta}\hbar}} \int_{-\infty}^{+\infty} dx \left(1+\beta p^2\right)^{\frac{1}{2}} \exp\left(\frac{-ix \arctan\left(\sqrt{\beta}p\right)}{\hbar\sqrt{\beta}}\right) \varphi(x). \tag{2.123}$$

Using (2.119) and (2.89) we introduce the scalar product of states in terms of the quasi-position wave function that have being represented on the space of quasi-position wave function

$$
\begin{aligned}
\langle \psi \mid \phi \rangle &= \int_{-\infty}^{+\infty} \frac{dp}{1+\beta p^2} \psi^* (p) \, \phi (p) \, ; \\
&= \frac{1}{\sqrt{8\pi\sqrt{\beta}\hbar}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dp dx dx^{'} \exp \left(\left(x - x^{'} \right) \frac{\arctan \left(\sqrt{\beta} p \right)}{\hbar\sqrt{\beta}} \right) \psi^* (x) \, \phi \left(x^{'} \right) .
\end{aligned} \tag{2.124}
$$

The completeness relation on this space can be written as

$$
\int_{-\infty}^{+\infty} dx \left| \psi_x^{ml} \right\rangle \left\langle \psi_x^{ml} \right| = 4\hbar\sqrt{\beta} \left(1 + \beta p^2 \right)^{-1} . \tag{2.125}
$$

We immediately deduce the representation of the operator $\hat{P}$ in this configuration space from (2.119) as

$$
\hat{P}\psi (x) = \frac{\tan \left(-i\hbar\sqrt{\beta}\partial_x \right)}{\sqrt{\beta}} \psi (x) . \tag{2.126}
$$

Now, for the position operator acting on the functions, it can be deduced by using equation (2.119)

$$
\hat{X}\psi (x) = \left(x + \beta \frac{\tan \left(-i\hbar\sqrt{\beta}\partial_x \right)}{\sqrt{\beta}} \right) \psi (x) . \tag{2.127}
$$

2.2.3 A brief review on ml-quantum system in path integral formalism

The path integral approch in momentum space for any ml-quantum system can be obtained through constructing the transition amplitude with the following steps :

We have

$$
\begin{aligned}
(p_a t_a \mid p_b t_b) &= \langle p_b \mid U (t_b, t_a) \, | p_a \rangle \, ; \\
&= \lim_{N\to\infty} \langle p_b | \, U (t_b, t_a) \, | p_a \rangle \, ,
\end{aligned} \tag{2.128}
$$

with the infinitisimal evelution operator

$$U\left(t_n, t_{n-1}\right) = \exp\left(-\frac{i\varepsilon}{\hbar}\hat{H}\left(t_n\right)\right), \tag{2.129}$$

where $\epsilon = t_j - t_{j-1} = \frac{t_b - t_a}{N+1}$. Now we insert the completness relation (2.91) between each pair of infinitisimal evolution operators we obtain

$$\left(p_a t_a \mid p_b t_b\right) = \lim_{N\to\infty}\prod_{j=1}^{N}\int\frac{dp_n}{1+\beta p_n^2}\prod_{j=1}^{N+1}\left(p_j t_j \mid p_{j-1}t_{j-1}\right), \tag{2.130}$$

where the infinitisimal amplitude is defined by

$$\left(p_j t_j \mid p_{j-1}t_{j-1}\right) = \langle p_j|\exp -\frac{i\varepsilon}{\hbar}\hat{H}\left(t_j\right)|p_{j-1}\rangle, \tag{2.131}$$

using the completness relation for the formal eigenvectors and we obtain the following phase space path integral

$$\left(p_a t_a \mid p_b t_b\right) = \int\frac{dx_n}{2\pi\hbar}\exp\left(-\frac{i\varepsilon}{\hbar}\hat{H}\left(t_n\right)\right)\exp\left\{\frac{ix_n}{\hbar\sqrt{\beta}}\left(\arctan\left(\sqrt{\beta}p_n\right)-\arctan\left(\sqrt{\beta}p_{n-1}\right)\right)\right\}. \tag{2.132}$$

Substituting in (2.129) we get the final expression for the path integral representation of the transition amplitude for a nonrelativistic particle with nonzero minimum position uncertainty submitted to the potential $V\left(x\right)$

$$\begin{aligned}\left(p_a t_a \mid p_b t_b\right) = \lim_{N\to\infty}\prod_{j=1}^{N}\int\frac{dp_n}{1+\beta p_n^2}\prod_{j=1}^{N+1}\int\frac{dx_n}{2\pi\hbar}\exp\Bigg\{-\frac{i\varepsilon}{\hbar}-\frac{i\varepsilon}{\hbar}\Bigg[\frac{ix_n}{\hbar\sqrt{\beta}}\left(\arctan\left(\sqrt{\beta}p_n\right)\right.\\ \left.-\arctan\left(\sqrt{\beta}p_{n-1}\right)\right)\Bigg]-\frac{p_n^2}{2m}-V\left(x_n\right)\Bigg\}.\end{aligned} \tag{2.133}$$

2.3 The free particle of Klein Gordon equation

Relativistic quantum mechanics was extracted from non-relativistic quantum mechanics and not from classical theory, using the most natural approach to describe the state of a relativistic particle by application of the Heisenberg principle and the energy-momentum dispersion relation, which allows to replace classical observable by quantum mechanical differential operators acting on the wave functions. We first introduce the differential operators $\hat{X}^{\mu}$ and $\hat{P}^{\nu}$ in such way that their commutators obey the rules (Heisenberg algebra)

$$\left[\hat{X}^{\mu}, \hat{X}^{\nu}\right] = 0 \quad , \quad \left[\hat{X}^{\mu}, \hat{P}^{\nu}\right] = -i\hbar g^{\mu\nu} \quad , \quad \left[\hat{P}^{\mu}, \hat{P}^{\nu}\right] = 0 \tag{2.134}$$

and $g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is the Minkowski space mertic

it can be done if we take $\hat{X}^{\mu} = x^{\mu}$ and $\hat{P}^{\nu} = i\hbar\partial^{\nu}$. Expressively

$$\hat{X}^0 = x^0 = ict \ , \hat{X}^1 = x^1 = x \ , \hat{X}^2 = x^2 = y \ \ , \hat{X}^3 = x^3 = z, \tag{2.135}$$

$$\hat{P}^0 = i\hbar\frac{\partial}{\partial x^0} = i\hbar\frac{\partial}{\partial ct}, \ \hat{P}^1 = i\hbar\frac{\partial}{\partial x^1} = -i\hbar\frac{\partial}{\partial x} \ , \ \hat{P}^2 = i\hbar\frac{\partial}{\partial x^2} = -i\hbar\frac{\partial}{\partial y} \ ,$$

$$\hat{P}^3 = i\hbar\frac{\partial}{\partial x^3} = -i\hbar\frac{\partial}{\partial z}. \tag{2.136}$$

The direct approach of having a linear equation from the dispersion relation energy-momentum relation of the restricted relativity $E - \sqrt{\hat{p}^2c^2 + m_0^2c^4} = 0$ where m_0 is the rest of mass of the particle and c the velocity of light in vacumm .with the squring both of parts of the dispersion relation to write it in the form $E^2 = \hat{p}^2c^2 + m_0^2c^4$. Now, with the correspondence principle (2.133) we obtain the Klein-Gordon equation of a free particle by :

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{m_0^2c^2}{\hbar^2}\right)\psi\left(x^{\mu}\right) = 0, \tag{2.137}$$

where the covariant form of this equation giving by

$$\left(\partial^{\mu}\partial_{\mu} - \left(\frac{m_0 c}{\hbar}\right)^2\right)\psi\left(x^{\mu}\right) = 0, \tag{2.138}$$

with $\partial_{\mu} = \left(\frac{\partial}{\partial t}, \overrightarrow{\nabla}\right)$ and $\partial^{\mu} = \left(\frac{\partial}{\partial t}, \overrightarrow{-\nabla}\right)$.

Replacing the plane wave of the type $\psi\left(\overrightarrow{x},t\right) = \exp\left(\frac{i}{\hbar}\left(p\cdot\overrightarrow{x}-Et\right)\right)$ into the Klein Gordon equation with the following de Broglie and Einstein relations $P=\hbar k$ and $E=\hbar\omega$,we obtain :

$$-\mid k\mid^2+\frac{\omega^2}{c^2}=\frac{m^2c^2}{\hbar^2}, \tag{2.139}$$

$$\langle p\rangle=\langle\psi\mid -i\hbar\nabla_x\mid\psi\rangle=\hbar k \quad , \quad \langle E\rangle=\langle\psi\mid i\hbar\partial_t\mid\psi\rangle=\hbar\omega, \tag{2.140}$$

we insert (2.139) in (2.138) we get the classical relativistic equation

$$E^2=m^2c^2+p^2c^4 \qquad \Longrightarrow \quad E=\pm\sqrt{m^2c^2+p^2c^4} \tag{2.141}$$

We finally get a plane wave solution to the Klein-Gordon equation that corresponds to energy in the equation

$$\psi\left(x,t\right)=\exp(-i(\pm\mid E\mid t-p\cdot x)/\hbar) \tag{2.142}$$

The negative energy solutions represnt the unoccupied states of negative energy describe ”antiparticles”.

Chapitre 3

The Klein-Gordon Oscillator slutions in different cases

So far, we have exposed : quantum mechanics with regular Heisenberg algebra which implies the standard uncertainty relation, then quantum mechanics in non commutative Heisenberg algebra which implies directly the Generalized uncertainty principle. In this chapter, we shall apply the formalism introduced in previous chapter. We are going to study the exact solutions of simple examples of relativistic scalar particles in thus two algebras, we recall the following relations :

Standard Heisenberg algebra

$$\left[\hat{X},\hat{P}\right]=i\hbar \tag{3.1}$$

$$\hat{X}=x \quad ,\hat{P}=-i\hbar\partial_x \tag{3.2}$$

Non standard Heisenberg algebra

$$\left[\hat{X}_i,\hat{P}_i\right]=i\hbar\left(\delta_{ij}+\beta\hat{P}^2\delta_{ij}+\beta^{'}\hat{P}_i\hat{P}_j\right) \tag{3.3}$$

$$\hat{X}_i=i\hbar\left[\left(1+\beta P^2\right)\frac{\partial}{\partial p_i}+\beta^{'}P_iP_j\frac{\partial}{\partial p_i}+\gamma P_i\right]\ ,\ \hat{P}_i=p_i \tag{3.4}$$

In this work, we are going to set $\beta^{'}=\gamma=0$, which is directly equivalent to one dimension case.

3.1 The (1+1) Klein Gordon oscillator solutions in regular space

In this section, we are going to solve a one-dimensional Klein Gordon oscillator in regular space with standard algebra as an example, the equation of the oscillator is given for

$$\hat{P} \longrightarrow \hat{P} - im\omega x, \tag{3.5}$$

as follows (for $\hbar = c = 1$)

$$\left[E^2 + \left(\hat{P} + im\omega x\right)\left(\hat{P} - im\omega x\right) - m^2\right]\psi\left(x,t\right) = 0, \tag{3.6}$$

where m is the rest mass of the particle, ω is the classical frequency of the oscillator.

We introduce the following wave function

$$\psi\left(x,t\right) = \phi\left(x\right)\exp\left(-iEt\right), \tag{3.7}$$

with regular space algebra

$$\begin{cases} \hat{x} = x \\ \hat{P} = -i\partial_x \end{cases}. \tag{3.8}$$

The (time-independent) Klein-Gordon equation is obtain by inserting (3.7) in (3.6)

$$[(\hat{p} + im\omega x)(\hat{p} - im\omega x) + m^2 - E^2]\phi(x) = 0; \tag{3.9}$$

$$\frac{d^2\phi(x)}{dx^2} + (E^2 + m\omega - m^2 - m^2\omega^2x^2)\phi(x) = 0. \tag{3.10}$$

We set $\varepsilon = E^2 + m\omega - m^2$ and transform our equation to the standard harmonic oscillator equation by multiplying by the factor $\frac{1}{2m}$ as in the Ref [26]

$$\frac{1}{2m}\frac{d^2\phi(x)}{dx^2} + \left(-\frac{1}{2}m\omega^2x^2 + \zeta\right)\phi(x) = 0, \tag{3.11}$$

where $\zeta = \frac{\varepsilon}{2m}$.

By rescaling the variables as $z = \sqrt{m\omega}x$, $\lambda = \frac{\zeta}{\omega}$, one arrives at

$$\frac{d^2\Phi(x)}{dx^2} + (\lambda - z^2)\phi(x) = 0. \tag{3.12}$$

The solutions of this type of equations (3.12) are given by

$$\frac{d^2h_\nu(\tau)}{d\tau^2} - (\nu + \frac{1}{2} - \tau^2)h_\nu(\tau) = 0, \tag{3.13}$$

where the function $h_\nu(\tau)$ for ν is a non-negative integer, one finds Hermite polynomial

$$h_n\left(z\right) = \exp\left(\frac{-z^2}{2}\right) H_n\left(z\right). \tag{3.14}$$

The energy spectrum also can be calculated through

$$\lambda = n + \frac{1}{2}, \tag{3.15}$$

where we replace $\lambda = \frac{\varepsilon}{2m\omega}$ and obtain the energy of this oscillator

$$E = \pm m\left[1 + 2\frac{\omega}{m}n\right]^{\frac{1}{2}}. \tag{3.16}$$

The wave function of the equation (3.12) is expressed as follows

$$\phi\left(x\right) \sim N\exp(-\frac{m\omega x^2}{2})H_n\left(\sqrt{m\omega}x\right), \tag{3.17}$$

where N is the normalization constant, we have to use orthogonality condition to calculate this constant

$$\int_{-\infty}^{\infty} \psi^*\left(x\right)\phi\left(x\right)dx = 1, \tag{3.18}$$

$$\int_{-\infty}^{\infty} \exp\left(-x^2\right) H_m\left(x\right) H_n\left(x\right) dx = \begin{cases} 0 \text{ for } & m \neq n \\ 2^n n! \sqrt{\pi} & \text{for } m = n \end{cases}, \tag{3.19}$$

we obtain the following normalization constant

$$N = \sqrt{\sqrt{\frac{m\omega}{\pi}}\frac{1}{2^n n!}}. \tag{3.20}$$

The wave function of this system is taken by

$$\phi(x)=\sqrt{\frac{m\omega}{2^n n!\sqrt{\pi}}}\left[\exp(-\frac{m\omega}{2}x^2)\right]H_n\left(\sqrt{m\omega}x\right). \tag{3.21}$$

3.2 The (1+1) Klein-Gordon oscillator solutions in a uniform electric field of specific strength ε with presence of minimal length

The stationary equation describing the one dimensional Klein-Gordon oscillator in a uniform electric field of specific strength ε in momentum representation is given by

$$[(\hat{p}+im\omega\hat{x})(\hat{p}-im\omega\hat{x})+m^2-(i\frac{d}{dt}-q\varepsilon\hat{x})^2]\psi(p)=0, \tag{3.22}$$

where $V(x)=q\varepsilon\hat{x}$ is the vectorial potential, ε is the external electric field and q is the charge of the particle.

The stationary case can be defined by $\psi(p)=\phi(p)\exp\{-iEt\}$, with the following algebra

$$\begin{cases}\hat{x}=i\hbar(1+\beta p^2)\frac{d}{dp}\\ \hat{P}=\hat{p}\end{cases}. \tag{3.23}$$

Now, we introduce the algebra $\left(\hat{x},\hat{P}\right)$ and the stationary case into Eq.(3.22) and also we set $(\hbar=c=1)$ to obtain

$$[m^2\omega^2\hat{x}^2+(1-m\omega\beta)\hat{P}^2+m^2-E^2-m\omega-q^2\varepsilon^2\hat{x}^2+2Eq\varepsilon\hat{x}]\phi(p)=0, \tag{3.24}$$

otherwise

$$[(m^2\omega^2-q^2\varepsilon^2)\hat{x}^2+2Eq\varepsilon\hat{x}+(1-m\omega\beta)\hat{p}^2+m^2-E^2-m\omega]\phi(p)=0, \tag{3.25}$$

as will, we set $\Omega^2=(m^2\omega^2-q^2\varepsilon^2)$, with $m^2\omega^2>q^2\varepsilon^2$

$$\left[\Omega^2\hat{x}^2+2Eq\varepsilon\hat{x}+(1-m\omega\beta)\,\hat{p}^2+m^2-E^2-m\omega\right]\phi\,(p)=0. \tag{3.26}$$

Substituting the momentum space realization of the operators $\hat{x}^2$ and $\hat{p}$, we get the following equation

$$\Big[-\Omega^2\left((1+\beta p^2)^2\frac{\partial^2}{\partial p^2}+2\beta(1+\beta p^2)p\frac{\partial}{\partial p}\right)+2iEq\varepsilon(1+\beta p^2)\frac{\partial}{\partial p}+(1-m\omega\beta)p^2$$

$$+m^2-E^2-m\omega\Big]\,\phi\,(p)=0 \tag{3.27}$$

With the aid of the variable change (Ref [15] for $\beta^{'}=0$)

$$u=\frac{1}{\sqrt{\beta}}\arctan(\sqrt{\beta}p) \tag{3.28}$$

where $p\in[-\infty,\infty]$, $u\in\left[\frac{-\pi}{2\sqrt{\beta}},\frac{\pi}{2\sqrt{\beta}}\right]$, with this variable change the equation (3.27) becomes :

$$[-\Omega^2\frac{\partial^2}{\partial u^2}+2iEq\varepsilon\frac{\partial}{\partial u}+(\frac{1-m\omega\beta}{\beta})\tan^2(\sqrt{\beta}u)+m^2-E^2-m\omega]\phi(u)=0 \tag{3.29}$$

We can eliminate the imaginary term from (3.29), by introducing a 2^{nd} variable change to the equation as follows

$$\phi(u)=\left[\exp\left\{i\frac{Eq\varepsilon}{(m^2\omega^2-q^2\varepsilon^2)}u\right\}\right]f\;(u), \tag{3.30}$$

$$[-\Omega^2\frac{\partial^2}{\partial u^2}+\tan^2(\sqrt{\beta}u)(\frac{1-m\omega\beta}{\beta})-\frac{q^2E^2\varepsilon^2}{\Omega^2}+m^2-E^2-m\omega]f\;(u)=0. \tag{3.31}$$

At this stage, we introduce another change of variable defined by

$$z=\sin(\sqrt{\beta}u), \tag{3.32}$$

$$\left[((1-z^2)\frac{\partial^2}{\partial z^2}-z\frac{\partial}{\partial z})-\frac{z^2}{1-z^2}\frac{\left(\frac{1}{\beta}-m\omega\right)}{\beta\Omega^2}+\frac{1}{\beta\Omega^2}\left(m^2-E^2\right)+\frac{q^2\varepsilon^2E^2}{\Omega^4\beta}\right]f\ (z)=0. \tag{3.33}$$

We set a change of function defined by

$$f\ (z)=(1-z^2)^{\frac{\lambda}{2}}g(z), \tag{3.34}$$

the equation (3.33) becomes

$$\left[(1-z^2)\frac{\partial^2}{\partial z^2}-z(1+2\lambda)\frac{\partial}{\partial z}+\frac{\lambda\left(\lambda-1\right)-\frac{\left(\frac{1}{\beta}-m\omega\right)}{\beta\Omega^2}}{1-z^2}-\lambda^2+\frac{1}{\beta^2\Omega^2}\right.$$

$$\left.-\frac{1}{\beta\Omega^2}\left(m^2-E^2\right)+\frac{q^2\varepsilon^2E^2}{\Omega^4\beta}\right]g(z)=0, \tag{3.35}$$

we have to eliminate the coefficient that followed by $\frac{1}{1-z^2}$ through calculating delta solution of the following second degree equation

$$\lambda\left(\lambda-1\right)-\frac{\left(\frac{1}{\beta}-m\omega\right)}{\beta\Omega^2}=0.$$

Now, the differential equation (3.35) reduces to the type of Gegenbauer differential equation

$$\left[(1-z^2)\frac{\partial^2}{\partial z^2}-(2\lambda+1)z\frac{\partial}{\partial z}+n(n+2\lambda)\right]g(z)=0. \tag{3.36}$$

Our physical solution can be satisfied only by taking λ_+ solution

$$\lambda=\lambda_+=\frac{1}{2}+\frac{1}{2}\sqrt{1-4\frac{[\beta m\omega-1]}{\beta^2\Omega^2}}. \tag{3.37}$$

The energy spectrum of this system is given by the following relation

$$\frac{-1}{\beta\Omega^2}\left(m^2-E^2\right)+\frac{q^2\varepsilon^2E^2}{\Omega^4\beta}-\lambda^2+\frac{1}{\beta^2\Omega^2}=n(n+2\lambda).$$

We simplify, to get

$$E_n^2 = \frac{\Omega^4}{m^2\omega^2}\left[\beta\left(n^2+2\lambda n+\lambda^2\right)+\frac{m^2}{\Omega^2}-\frac{1}{\beta\Omega^2}\right]. \tag{3.38}$$

Then

$$E_n = \pm\left(1-\frac{q^2\varepsilon^2}{m^2\omega^2}\right)^{\frac{1}{2}}\left[m^2+\beta(m^2\omega^2-q^2\varepsilon^2)\left(n^2+2\lambda n+\lambda^2\right)-\frac{1}{\beta}\right]^{\frac{1}{2}}. \tag{3.39}$$

To deducing the particular cases, we calculate the following limits :

3.2.1 Case one : absent of deformation $(\beta = 0)$

$$\lim_{\beta\longrightarrow 0}(\beta\lambda) = \sqrt{\frac{1}{m^2\omega^2-q^2\varepsilon^2}}; \tag{3.40}$$

$$\lim_{\beta\longrightarrow 0}\left(\beta\Omega^2\lambda\right) = \sqrt{(m^2\omega^2-q^2\varepsilon^2)}; \tag{3.41}$$

$$\lim_{\beta\longrightarrow 0}\left(\left(\beta\Omega^2\lambda^2\right)-\frac{1}{\beta}\right) = \sqrt{(m^2\omega^2-q^2\varepsilon^2)}-m\omega; \tag{3.42}$$

Replacing (3.42) and (3.43) in (3.40), we get the energy spectrum in the absence of minimal length

$$\lim_{\beta\longrightarrow 0}E_n = \pm\left(1-\frac{q^2\varepsilon^2}{m^2\omega^2}\right)^{\frac{1}{2}}\left[m^2-m\omega+\sqrt{(m^2\omega^2-q^2\varepsilon^2)}(2n+1)\right]^{\frac{1}{2}}. \tag{3.43}$$

3.2.2 Case two : Absent of the electric field $(\varepsilon = 0)$

For $\varepsilon = 0$, the energy spectrum is given by

$$\lim_{\varepsilon\longrightarrow 0}E_n = E_n = \pm\left[m^2+\beta m^2\omega^2\left(n^2+2\lambda n+\lambda^2\right)-\frac{1}{\beta}\right]^{\frac{1}{2}}. \tag{3.44}$$

3.2.3 Case three : The pure Klein-Gordon Oscillator ($\beta = 0$ and $\varepsilon = 0$)

To confirm our result obtained in ordinary space, we calculate the limit of the energy spectrum E_n of equation

$$\lim_{(\beta,\varepsilon)\longrightarrow 0} E_n = \pm m \left[1 + 2\frac{\omega}{m} n\right]^{\frac{1}{2}}, \tag{3.45}$$

which agrees with the result obtained above and the result in Ref [27] .

We obtain the wave function as

$$g(z) = N C_n^{\lambda}(z). \tag{3.46}$$

Then, for our system the wave function is defined by the following relationship by

$$\phi_n(p) = N_n^{\lambda} \left[\cos\left(\arctan\left(\sqrt{\beta} p\right)\right)\right]^{\left(\frac{1}{4}+\frac{1}{4}\sqrt{1-4\frac{m\omega\beta-1}{\beta^2\Omega^2}}\right)}$$

$$\times\left(\exp\left\{\frac{iEq\varepsilon}{\sqrt{\beta}(m^2\omega^2 - q^2\varepsilon^2)} \arctan\left(\sqrt{\beta} p\right)\right\}\right) C_n^{\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-4\frac{m\omega\beta-1}{\beta^2(m^2\omega^2-q^2\varepsilon^2)}}\right)} \sin\left(\arctan\left(\sqrt{\beta} p\right)\right). \tag{3.47}$$

where N_n is the normalization constant, given by (Ref [28])

$$N_n^{\lambda} = (\Gamma(\lambda))^2 \left(\frac{2^{2\lambda-1} n! \,(n+1)\sqrt{\beta}}{\pi\Gamma(n+2\lambda)}\right). \tag{3.48}$$

For the study of the thermal properties of the deformed Klein Gordon oscillator, which is equivalent to the zero ε case "absence of the electric field", we use a numerical method based on the Euler-Maclaurin formula to compute the partition function and as consequence we obtain the thermodynamic properties of the system. In the canonical ensemble, the partition function of this oscillator in deformd space is given by the energy spectrum formula

$$Z = \sum_{n=1}^{\infty} \exp\left\{-\left(\frac{E_n - E_0}{k_B T}\right)\right\}, \tag{3.49}$$

where k_B is the Boltzmann constant. The sum (3.49) can be evaluate with the help of the Euler-Maclaurin summation formula

$$\sum_{n=0} F(n) = \frac{1}{2} F(0) + \int_0^\infty F(x)\,dx - \sum_{p=1} \frac{B_{2p}}{(2p)!} F^{(2p-1)}(0), \tag{3.50}$$

where $\frac{B_{2p}}{(2p)!}$ are the Bernoulli numbers multiplied by the $(2p-1)$ order derivations of $F(0)$. We obtain the partition function

$$Z = \frac{1}{2}\left[1 - \sqrt{\frac{\pi}{\beta\lambda^4\omega^2}}\left(\left(\frac{T}{T_0}\right)^{\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{\delta^n}{n}\left(\frac{T_0}{T}\right)^{\frac{2n-1}{2}}\right)\right], \tag{3.51}$$

where the parameters λ and δ in the case are defined by

$$\lim_{\varepsilon \longrightarrow 0} \lambda = 1 - \frac{1}{\beta m\omega}, \tag{3.52}$$

$$\delta = \frac{\beta\omega^2\lambda^2}{2} + \frac{1}{2\beta m^2}. \tag{3.53}$$

Now, we can calculate all thermodynamics quantities for our system such as free energy, entropy, total energy and specific heat from the partition function as the follows

$$F = -k_\beta T \ln\left(\frac{1}{2}\left[1 - \sqrt{\frac{\pi}{\beta\lambda^4\omega^2}}\left(\left(\frac{T}{T_0}\right)^{\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{\delta^n}{n}\left(\frac{T_0}{T}\right)^{\frac{2n-1}{2}}\right)\right]\right),$$

$$S = \frac{k_\beta T\sqrt{\frac{\pi}{\beta\lambda^4\omega^2}}\left[\frac{1}{2\left(\frac{T}{T_0}\right)T_0} - \frac{T_0\delta}{2T^2\sqrt{\frac{T_0}{T}}} - \sum_{n=1}^{\infty}\frac{(2n+1)\delta^{n+1}}{2nT^2}\left(\frac{T_0}{T}\right)^{\frac{2n-1}{2}}\right]}{2Z} + k_\beta \ln(Z),$$

$$U = \frac{k_\beta T^2\sqrt{\frac{\pi}{\beta\lambda^4\omega^2}}\left[\frac{1}{2\left(\frac{T}{T_0}\right)T_0} - \frac{T_0\delta}{2T^2\sqrt{\frac{T_0}{T}}} - \sum_{n=1}^{\infty}\frac{(2n+1)\delta^{n+1}}{2nT^2}\left(\frac{T_0}{T}\right)^{\frac{2n-1}{2}}\right]}{2Z},$$

$$C = \frac{\pi\left[\frac{1}{2\left(\frac{T}{T_0}\right)T_0} - \frac{T_0\delta}{2T^2\sqrt{\frac{T_0}{T}}} - \sum_{n=1}^{\infty}\frac{(2n+1)\delta^{n+1}}{2nT^2}\left(\frac{T_0}{T}\right)^{\frac{2n-1}{2}}\right]}{4\beta^2\lambda^4\sqrt{\frac{\pi}{\beta\lambda^4\omega^2}}\,\omega^2 Z}. \tag{3.54}$$

3.3 Deformed Klein Gordon oscillator in path integral formalism

In this section, we propose to use the path integral formalism in relativistic quantum mechanics to calculate the Green's function of the same system studied " the zero ε case" or what we can call "deformed Klein-Gordon oscillator" in the energy-momentum space.

Let's consider the Green function operator of our system

$$\left[\hat{P}_0^2+\left(\hat{P}_i+im\omega\hat{x}_i\right)\left(\hat{P}^i-im\omega\hat{x}^i\right)-m^2\right]\hat{G}^{(\beta)}=I, \tag{3.55}$$

where

$$\hat{G}^{(\beta)}=\frac{i}{\left[\hat{P}_0^2+\left(\hat{P}_i+im\omega\hat{x}_i\right)\left(\hat{P}^i-im\omega\hat{x}^i\right)-m^2\right]+i\epsilon}. \tag{3.56}$$

So the propagator is

$$G^{(\beta)}\left(p_b;p_a\right)=\left\langle p_a,p_{0a}\right|\hat{G}^{(\beta)}\left|p_b,p_{0b}\right\rangle. \tag{3.57}$$

The Hamiltonian operator of this ml-quantum system in the momentum space is given by

$$\tau\hat{H}^{(\beta)}=-\tau\left[\hat{P}_0^2-\left(1-m\omega\beta\right)\hat{P}^2+m\omega-m^2-m^2\omega^2\hat{X}^2\right]. \tag{3.58}$$

Now, we follow the standard discretization method for the kernel (3.58) where the time interval τ get devided to $N+1$ infinitesimal equal parts to $\varepsilon=\frac{\tau}{N+1}$ and we apply the formula

$$\langle p_f,p_{0f}\mid\exp\left(-i\tau\left(\hat{H}^{(\beta)}\right)\right)\mid p_i,p_{0i}\rangle=\lim_{N\longrightarrow\infty}\langle p_f,p_{0f}\mid\left[\exp\left(-i\varepsilon\hat{H}^{(\beta)}\right)\right]^{N+1}\mid p_i,p_{0i}\rangle, \tag{3.59}$$

$$G=i\int d\tau\lim_{N\longrightarrow\infty}\prod_{n=1}^{N}\int\frac{dp_n}{\left(1+\beta p_n^2\right)^{1-\alpha}}dp_{0n}\prod_{n=1}^{N+1}\left\langle p_n,p_{0n}\right|\exp\left\{i\varepsilon\left(\hat{P}_0^2-\left(1-m\omega\beta\right)\hat{P}\right)\right.$$

$$\left.+m\omega-m^2-\left(m\omega\right)^2\hat{X}^2\right)\Big\}\left|p_{n-1},p_{0n-1}\right\rangle. \tag{3.60}$$

-We introduce the algebra

$$\begin{cases} \hat{x} = i\hbar(1+\beta p^2)\frac{d}{dp} + \gamma p \\ \hat{P} = \hat{p} \end{cases}, \tag{3.61}$$

to the equation (3.61) with following momentum space representation :

The scalar product of P and P_0 in the relativistic case, where we assume that the deformation does not affect on the time component P_0. Which are given by of momentum eigenstates

$$\begin{aligned} \langle p_j \mid p_{j-1}\rangle &= \left(1+\beta p_j^2\right)^{-\frac{\alpha}{2}} \left(1+\beta p_{j-1}^2\right)^{-\frac{\alpha}{2}} \delta\left(\frac{1}{\sqrt{\beta}}\arctan\left(\sqrt{\beta}p_j\right) - \frac{1}{\sqrt{\beta}}\arctan\left(\sqrt{\beta}p_{j-1}\right)\right) \\ &= \left(1+\beta p_j^2\right)^{\frac{1-\alpha}{2}} \left(1+\beta p_{j-1}^2\right)^{\frac{1-\alpha}{2}} \delta\left(p_j - p_{j-1}\right), \end{aligned} \tag{3.62}$$

$$\langle p_{0j} \mid p_{0j-1}\rangle = \delta\left(p_{0j} - p_{0j-1}\right), \tag{3.63}$$

with the following completeness relation

$$\int \frac{dp}{\left(1+\beta p_j^2\right)^{1-\alpha}} \left|p\right\rangle \left\langle p\right| = \mathbb{1}, \tag{3.64}$$

$$\int dp_0 \left|p_0\right\rangle \left\langle p_0\right| = \mathbb{1}, \tag{3.65}$$

where $\alpha = \frac{\gamma}{\beta}$.

We insert the scalar product relation (3.63) and completness relation (3.65) N times, then we get the Lagrangian path integral representation for the Green function

$$\begin{aligned} G = i\lim_{N\longrightarrow\infty} \int d\tau \prod_{n=1}^{N} \int \frac{dp_n}{\left(1+\beta p_n^2\right)^{1-\alpha}} dp_{0n} \prod_{n=1}^{N+1} \exp\left\{i\varepsilon\left(\hat{p}_{0n}^2 - \left(1-m\omega\beta\right)\hat{p}^2 + m\omega - m^2\right)\right\} \\ \left\langle p_n, p_{0n}\right| \exp\left(-i\varepsilon\left(m\omega\right)^2 \hat{q}_n^2\right) \left|p_{n-1}, p_{0n-1}\right\rangle, \end{aligned} \tag{3.66}$$

The integral representation of $\langle p_n, p_{0n} \mid p_{n-1}, p_{0n-1}\rangle$ is defined by

$$\langle p_n, p_{0n} \mid p_{n-1}, p_{0n-1} \rangle = \int\int \frac{dt_n}{2\pi}\frac{dq_n}{2\pi}\frac{\exp\left(it_n\left(p_{0n}-p_{0n-1}\right)\right)}{\left(1+\beta p_n^2\right)^{\frac{\alpha}{2}}\left(1+\beta p_{n-1}^2\right)^{\frac{\alpha}{2}}}$$

$$\times \exp\left\{iq_n\left(\frac{1}{\sqrt{\beta}}\arctan\left(\sqrt{\beta}p_n\right)-\frac{1}{\sqrt{\beta}}\arctan\left(\sqrt{\beta}p_{n-1}\right)\right)\right\}. \qquad (3.67)$$

Now by inserting (3.68) in the propagator formula, we get

$$G = -i\int d\tau \lim_{N\longrightarrow\infty}\prod_{n=1}^{N}\int\frac{dp_n dp_{0n}}{\left(1+\beta p_n^2\right)^{1-\alpha}}\prod_{n=1}^{N+1}\int\int\frac{dt_n}{2\pi}\frac{dq_n}{2\pi}\frac{\exp\left\{it_n\left(p_{0n}-p_{0n-1}\right)\right\}}{\left(1+\beta p_n^2\right)^{\frac{\alpha}{2}}\left(1+\beta p_{n-1}^2\right)^{\frac{\alpha}{2}}}$$
$$\exp\left\{i\left[\frac{1}{\sqrt{\beta}}\left(\arctan\left(\sqrt{\beta}p_n\right)-\arctan\left(\sqrt{\beta}p_{n-1}\right)\right)q_n-\varepsilon\left(m\omega\right)^2 q_n^2\right]\right\}$$

$$\times \exp\left\{-i\varepsilon\left(p_{0n}^2-\left(1-m\beta\omega\right)p_n^2-m^2+m\omega\right)\right\}, \qquad (3.68)$$

with the use of the Gaussian integrations, the propagator become

$$\int dq_n \exp\left[\frac{1}{\sqrt{\beta}}\left(\arctan\left(\sqrt{\beta}p_n\right)-\arctan\left(\sqrt{\beta}p_{n-1}\right)\right)q_n-\varepsilon\left(m\omega\right)^2 q_n^2\right]$$
$$=\left(\frac{1}{\sqrt{4i\varepsilon\left(m\omega\right)^2\pi}}\right)\exp\left\{i\left(\frac{\triangle\arctan\left(\sqrt{\beta}p_n\right)}{\beta 4\varepsilon\left(m\omega\right)^2}\right)^2\right\}, \qquad (3.69)$$

$$G = i\left(1+\beta p_f^2\right)^{\frac{\alpha}{2}}\left(1+\beta p_i^2\right)^{\frac{\alpha}{2}}\int d\tau \lim_{N\longrightarrow\infty}\prod_{n=1}^{N}\frac{dp_n}{\left(1+\beta p_n^2\right)}p_{0n}\prod_{n=1}^{N+1}\frac{\delta\left(p_{0n}-p_{0n-1}\right)}{\sqrt{4i\pi\varepsilon\left(m\omega\right)^2}}$$
$$\times\exp\left\{-i\varepsilon\left(p_{0n}^2-\left(1-m\omega\beta\right)p_n^2-m^2+m\omega\right)\right\}$$

$$\times\exp\left\{i\frac{\left(\arctan\left(\sqrt{\beta}p_n\right)-\arctan\left(\sqrt{\beta}p_{n-1}\right)\right)^2}{4\beta\varepsilon\left(m\omega\right)^2}\right\}. \qquad (3.70)$$

Let's introduce now the variable change

$$\theta_n = \frac{1}{\sqrt{\beta}} \arctan\left(\sqrt{\beta} p_n\right),$$
$$\theta_{0n} = p_{0n}, \tag{3.71}$$

with this variable change, we get the new propagator formula

$$G^{(\beta)}\left(\theta_n, \theta_{n-1}, \tau\right) = i\left(1+\tan^2\left(\sqrt{\beta}\theta_f\right)\right)^{\frac{\alpha}{2}}\left(1+\tan^2\left(\sqrt{\beta}\theta_i\right)^{\frac{\alpha}{2}}\right)$$
$$\int d\tau \lim_{N\longrightarrow\infty} \prod_{n=1}^{N} \int d\theta_{0n} \delta\left(\theta_{0n}-\theta_{0n-1}\right) \exp\left\{-i\varepsilon\left(\theta_{0n}^2 - m + m\omega\right)\right\}$$

$$\prod_{n=1}^{N} \int d\theta_n \prod_{n=1}^{N+1} \frac{1}{\sqrt{4i\pi\varepsilon\left(m\omega\right)^2}} \exp\left\{i\left(\frac{\left(\theta_n-\theta_{n-1}\right)^2}{4\varepsilon\left(m\omega\right)^2} - \varepsilon\left(\frac{1}{\beta} - m\omega\right)\tan^2\left(\sqrt{\beta}\theta_n\right)\right)\right\}, \tag{3.72}$$

the term $\theta_n - \theta_{n-1}$ can be written as $\triangle\theta_n$

$$\triangle\theta_n = \theta_n - \theta_{n-1}, \tag{3.73}$$

$$G^{(\beta)}\left(\theta_n, \theta_{n-1}, \tau\right) = i\left(1+\tan^2\left(\sqrt{\beta}\theta_f\right)\right)^{\frac{\alpha}{2}}\left(1+\tan^2\left(\sqrt{\beta}\theta_i\right)^{\frac{\alpha}{2}}\right)$$
$$\int d\tau \lim_{N\longrightarrow\infty} \prod_{n=1}^{N} \int d\theta_{0n} \delta\left(\theta_{0n}-\theta_{0n-1}\right) \exp\left\{-i\varepsilon\left(\theta_{0n}^2 - m + m\omega\right)\right\}$$

$$\prod_{n=1}^{N} \int d\theta_n \prod_{n=1}^{N+1} \frac{1}{\sqrt{4i\pi\varepsilon\left(m\omega\right)^2}} \exp\left\{i\left(\frac{\left(\triangle\theta_n\right)^2}{4\varepsilon\left(m\omega\right)^2} - \varepsilon\left(\frac{1}{\beta} - m\omega\right)\tan^2\left(\sqrt{\beta}\theta_n\right)\right)\right\}. \tag{3.74}$$

The solution of the is type of equation is given in Ref [28,29],

$$G = i\left(1+\tan^2\left(\sqrt{\beta}\theta_i\right)\right)^{\frac{\alpha}{2}}\left(1+\tan^2\left(\sqrt{\beta}\theta_f\right)\right)^{\frac{\alpha}{2}} \sum_{n=0}^{\infty} N_n^{\lambda} \int d\tau \delta\left(\theta_{0b}-\theta_{0a}\right)$$
$$\exp\left\{-i\tau\left(\theta_{0n}^2 - m^2 + m\omega\right)\right\} \exp\left\{i\tau\beta\left(m\omega\right)^2\left(n^2 + (2n+1)\lambda\right)\right\}$$

$$\left(\cos\left(\sqrt{\beta}\theta_a\right)\cos\left(\sqrt{\beta}\theta_b\right)\right)^{\lambda} C_n^{\lambda}\left(\sin\left(\sqrt{\beta}\theta_a\right)\right) C_n^{\lambda}\left(\sin\left(\sqrt{\beta}\theta_b\right)\right), \quad (3.75)$$

where the normalization constant is given by

$$N_n^{\lambda} = \Gamma(\lambda)^2 \left(\frac{2^{2\lambda-1} n! \,(n+1)\sqrt{\beta}}{\pi \Gamma(n+2\lambda)}\right). \quad (3.76)$$

The physical solution are obtain always as we mentioned before for positive λ^+

$$\lambda = \lambda^+ = \frac{1}{2}\left(1 + \sqrt{1 + 4\frac{1-\beta(m\omega)}{\beta^2 (m\omega)^2}}\right). \quad (3.77)$$

The propagator became

$$G = i\left(1+\tan^2\left(\sqrt{\beta}\theta_f\right)\right)^{\frac{\alpha}{2}}\left(1+\tan^2\left(\sqrt{\beta}\theta_i\right)^{\frac{\alpha}{2}}\right)\sum_{n=0}^{\infty} N_n^{\lambda}\int d\tau \delta\left(\theta_{0b}-\theta_{0a}\right)$$
$$\exp\left\{-i\tau\left(\theta_{0n}^2 - m^2 + m\omega\right)\right\}\exp\left\{i\tau\beta (m\varepsilon)^2\left(n^2 + (2n+1)\lambda\right)\right\}$$

$$\times\left(\cos\left(\sqrt{\beta}\theta_a\right)\cos\left(\sqrt{\beta}\theta_b\right)\right)^{\lambda} C_n^{\lambda}\left(\sin\left(\sqrt{\beta}\theta_a\right)\right) C_n^{\lambda}\left(\sin\left(\sqrt{\beta}\theta_b\right)\right). \quad (3.78)$$

To evaluate the wave functions and energy spectrum, let us integrate over the τ variable

$$G = i\left(1+\tan^2\left(\sqrt{\beta}\theta_f\right)\right)^{\frac{\alpha}{2}}\left(1+\tan^2\left(\sqrt{\beta}\theta_i\right)^{\frac{\alpha}{2}}\right)$$
$$\sum_{n=0}^{\infty} N_n^{\lambda}\frac{\delta\left(\theta_{0b}-\theta_{0a}\right)\left(\cos\left(\sqrt{\beta}\theta_a\right)\cos\left(\sqrt{\beta}\theta_b\right)\right)^{\lambda} C_n^{\lambda}\left(\sin\left(\sqrt{\beta}\theta_a\right)\right) C_n^{\lambda}\left(\sin\left(\sqrt{\beta}\theta_b\right)\right)}{\left(\theta_{0n}^2 - m^2 + m\omega\right) - \beta (m\omega)^2\left(n^2 + (2n+1)\lambda\right).} \quad (3.79)$$

Finally evaluating exactly the propagator expression, it is convenient to write the Fourier transformation (3.80) for the variables $\{\theta_{0b}\}$ and $\{\theta_{0a}\}$. The first integral on the delta is immediate, we get

$$G=i\left(1+\tan^2\left(\sqrt{\beta}\theta_f\right)\right)^{\frac{\alpha}{2}}\left(1+\tan^2\left(\sqrt{\beta}\theta_i\right)^{\frac{\alpha}{2}}\right)\sum_{n=0}^{\infty}N_n^{\lambda}\int\frac{dE}{2\pi}e^{-iE\left(t_f-t_i\right)}$$
$$\frac{\left(\cos\left(\sqrt{\beta}\theta_f\right)\cos\left(\sqrt{\beta}\theta_i\right)\right)^{\lambda}C_n^{\lambda}\left(\sin\left(\sqrt{\beta}\theta_i\right)\right)C_n^{\lambda}\left(\sin\left(\sqrt{\beta}\theta_f\right)\right)}{\left(E^2-m^2+m\omega\right)-\beta\left(m\omega\right)^2\left(n^2+\left(2n+1\right)\lambda\right)}, \tag{3.80}$$

wich can be written

$$G=i\left(1+\tan^2\left(\sqrt{\beta}\theta_f\right)\right)^{\frac{\alpha}{2}}\left(1+\tan^2\left(\sqrt{\beta}\theta_i\right)^{\frac{\alpha}{2}}\right)\sum_{n=0}^{\infty}N_n^{\lambda}\int\frac{dE}{2\pi}e^{-iE\left(t_f-t_i\right)}$$
$$\frac{\left(\cos\left(\sqrt{\beta}\theta_f\right)\cos\left(\sqrt{\beta}\theta_i\right)\right)^{\lambda}C_n^{\lambda}\left(\sin\left(\sqrt{\beta}\theta_i\right)\right)C_n^{\lambda}\left(\sin\left(\sqrt{\beta}\theta_f\right)\right)}{E^2-\left(m^2+\beta\left(m\omega\right)^2n^2+2n\left(m\omega\right)\right)}. \tag{3.81}$$

This can be converted to a complex integration along the special contour C, and then using the residue theorem, we get

$$\oint\frac{dE}{2\pi}\frac{e^{-iE(t_f-t_i)}}{E^2-E_n^2}=\frac{-i}{2E_n}\left[\Theta\left(t_f-t_i\right)e^{-iE_n^{(\beta)}\left(t_f-t_i\right)}-\Theta\left(t_i-t_f\right)e^{iE_n^{(\beta)}\left(t_f-t_i\right)}\right], \tag{3.82}$$

where the energy eigenvalues are given by

$$E_{n,\pm}^{(\beta)}=\pm\sqrt{m^2+\beta\left(m\omega\right)^2n^2+2n\left(m\omega\right)}, \tag{3.83}$$

the propagator now might be written as

$$G=\left(1+\tan^2\left(\sqrt{\beta}\theta_f\right)\right)^{\frac{\alpha}{2}}\left(1+\tan^2\left(\sqrt{\beta}\theta_i\right)^{\frac{\alpha}{2}}\right)\sum_{n=0}^{\infty}\frac{N_n^{\lambda}}{2E_n}\left(\cos\left(\sqrt{\beta}\theta_f\right)\cos\left(\sqrt{\beta}\theta_i\right)\right)^{\lambda}$$

$$\times C_n^{\lambda}\left(\sin\left(\sqrt{\beta}\theta_i\right)\right)C_n^{\lambda}\left(\sin\left(\sqrt{\beta}\theta_f\right)\right)\left[\Theta\left(t_f-t_i\right)e^{-iE_n^{(\beta)}\left(t_f-t_i\right)}-\Theta\left(t_i-t_f\right)e^{iE_n^{(\beta)}\left(t_f-t_i\right)}\right], \tag{3.84}$$

we use $P=\frac{1}{\sqrt{\beta}}\tan\left(\sqrt{\beta}\theta\right)$ as a variable change to the propagator (3.85)

$$\cos\left(\sqrt{\beta}\theta\right) = \frac{1}{\sqrt{1+\beta P^2}},$$
$$\sin\left(\sqrt{\beta}\theta\right) = \frac{\sqrt{\beta}P}{\sqrt{1+\beta P^2}},$$

we get

$$G\left(p_i, p_f\right) = \sum_{n=0}^{\infty} \frac{N_n^\lambda}{2E_n} \left(\left(1+\beta P_f\right)^{\frac{\alpha-\lambda}{2}} \left(1+\beta P_i\right)^{\frac{\alpha-\lambda}{2}} C_n^\lambda \left(\frac{\sqrt{\beta}P_i}{\sqrt{1+\beta P_i}} \right) \right.$$
$$\left. C_n^\lambda \left(\frac{\sqrt{\beta}P_f}{\sqrt{1+\beta P_f}} \right) \right) \left[\Theta\left(t_f - t_i\right) e^{-iE_n^{(\beta)}\left(t_f - t_i\right)} - \Theta\left(t_i - t_f\right) e^{iE_n^{(\beta)}\left(t_f - t_i\right)} \right]. \tag{3.85}$$

Conclusion

In the present thesis, we have presented the necessary tools and techniques for the relativistic quantum theory in presence of minimal length which includes a generalized uncertainly principal (GUP). In the framework of minimum length we applied this method on the equation of the Klein-Gordon oscillator with a uniform electric field ε of specific strength ε, where we have calculated the energy spectrum E_n and the correlated wave function $\phi_n(p)$. The energy spectrum $E_n$depends on the deformation parameter β as well as the power of n which explains the minimal length effect, as well the wave function is obtaind by the Gegenbaouer polynomials. The limits cases in the regular space are deduced, which are compatible with the other results in the references and we also confirmed our calculated results.

In the last of this work, for a better understanding to the behavior of particles in presence of minimal length according to the analytically calculated formulas and their graphic representation, we have calculated the thermal properties : such as free energy F, entropy S and specific heat C from the partition function Z in the high tempture in the absence of the electric field ε. After that as final stage to this work, we presented what may look the Green's function of this system (in the absence of the electric field ε).

Bibliography

[1] B. A. Robson : Introductory Chapter : Standard Model of Cosmology. Redefining Standard Model Cosmology. (2019).

[2] N. Zettili : Quantum Mechanics Concepts and Applications, Second Edition.

[3] A.Zee :Quantum Field theory in a Nutshell. second Edition.

[4] V. Nikolaev Todorinov : Relativistic generalized uncertainty principle and its implication. University of Lethbridge Canada (2020).

[5] J.Madore : Introduction to Non-Commutative Geometry. Proceedings of Corfu Summer Institute on Elementary Particle Physics. (1999).

[6] S. Hossenfelder : Minimal Length Scale Scenarios for Quantum Gravity. Living Reviews in Relativity.16 (2013).

[7] H. Kragh : Arthur March, Werner Heisenberg, and the search for a smallest length. Revue d'histoire des sciences. 48, 401-434 (1995).

[8] H. S. Snyder : Quantized Space-Time. Physical Review. 71, 38-41(1947).

[9] M.Roushan, K.Nozari : Particle processes in a discrete spacetime and GW170814 event. The European Physical Journal C. 79-3 (2019).

[10] Meng-Sen Ma, Hui-Hua Zhao : Quantized space-time and its influence on two physical problems. Chinese Physics C. 38, 045102 (2019).

[11] A. Kempf,G. Mangano, B.Mann : Hilbert Space Representation of the Minimal length uncertainty relation. Physical Review D. 58, 1108-1118 (1995).

[12] D. Bouaziz : Klein-Gordon equation with Coulomb potential in the presence of a minimal length. arXiv preprint arXiv :1311.7405 (2013).

[13] D. Bouaziz : The minimal length case of the Klein Gordon equation with hyperbolic cotangent potential using Nikivorof Uvarof method. Journal of Physics : Theories and Applications. Vol 4, No 1 (2020)

[14] Pouria Pedram : Nonperturbative effects of the minimal length uncertainty on the relativistic quantum mechanics. Physics Letters B. 710, 478-485

(2012).

[15] M. Betrouche, M. Maamache, Jeong Ryeol Choi : Three-Dimensional Dirac Oscillator with Minimal Length : Novel Phenomena for Quantized Energy.Advances in High Energy Physics.2013 1-10(2013).

[16] K .S Mallesh, S. Chaturvedi, R.Simon and N.Mukunda : States of Physical Systems in Classical and Quantum Mechanics . Resonance. 17, 53-75 (2012).

[17] Jean-Pierre Antoine, R.C. Bishop, A. Bohm, S. Wickramasekara : Rigged Hilbert Spaces in Quantum Physics. Compendium of Quantum Physics. 640-650 (2009).

[18] E. Celeghini, M. Gadella , M. A. Del Olmo : Groups, Special Functions and Rigged Hilbert Spaces. Axioms. 8, 89 (2019).

[19] Mikio Nakahara : Geometry, Topology and Physics. second Edition. CRC Press (2017).

[20] L.J. Garay : Quantum Gravity And Minimum Length. International Journal of Modern Physics A. 10, 145-165 (1995).

[21] D.J.Gross,P. F.Mende : String theory beyond the Planck scale. Nuclear Physics B. 303, 407-454 (1988).

[22] A.Kempf : On quantum field theory with nonzero minimal uncertainties in positions and momenta. Journal of Mathematical Physics. 38, 1347-1372 (1997).

[23] Lay Nam Chang, Djordje Minic, Naotoshi Okamura, TatsuTakeuchi : The Effect of the Minimal Length Uncertainty Relation on the Density of States and the Cosmological Constant Problem.Physical Review D. 65-12 (2002).

[24] M. M. Stetsko and V. M. Tkachuk : Perturbation hydrogen-atom spectrum in deformed space with minimal length, Physical Review A. 74, 012101 (2006).

[25] A. Kempf : Non-pointlike particles in harmonic oscillators. Journal of Physics A : Mathematical and General. 30, 2093 (1997).

[26] A. Boumali, A. Hafdallah and A.Toumi : Comment on ‘Energy profile of theone-dimensional Klein–Gordon oscillator’.Phys. Scr. 84, 037001 (2011).

[27] S.Bruce, P. Minning. : The Klein-Gordon Oscillator. ILNuovo Cimento. 106, 712 (1993).

[28] H. Benzair, T. Boudjedaa and M. Merad : Path Integral for Dirac oscillator with generalized uncertainty principle. Journal of Mathematical Physics 53, 123516 (2012).

[29] D.C. Khandekar and S.V. Lawande : Path integrals : some exact results and application, Physics.Rep137, 115 (1986).

Appendix A

Hermite polynomials

The Hermite polynomials $H_n(x)$ are set of orthogonal polynomials over the domain $[-\infty, +\infty]$ with weighting function e^{-x^2}, known by :

$$\begin{aligned} H_n(x) &= (2x)^n\, {}_2F_0\left(-\frac{1}{2}n, -\frac{1}{2}(n-1),\ , x^{-2}\right) \\ &= 2^n x^n \left(x^2\right)^{\frac{-n}{2}} U\left(\frac{-1}{2}n, \frac{1}{2}, x^2\right) \end{aligned} \tag{3.86}$$

where $U(a, b, z)$ is a confluent hypergeometric function of the second kind. The Hermite polynomials satisfy the symmetry condition

$$H_n(-x) = (-1)^n H_n(x) \tag{3.87}$$

The Rodrigues formula of the polynomials is defiend by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{3.88}$$

The recurrence relations of this polynomials are given by

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{3.89}$$

The first few Hermite polynomials are
$H_0(x) = 1$
$H_1(x) = 2x$
$H_2(x) = 4x^2 - 2$
$H_3(x) = 8x^3 - 12x$
$H_4(x) = 16x^4 - 48x^2 + 12$

They are orthogonal in the range $[-\infty, +\infty]$ with respect to the weighting function e^{-x^2}

$$\int_{-\infty}^{+\infty} H_m(x)\, H_n(x)\, e^{-x^2} = \delta_{mn} 2^n n! \sqrt{\pi} \tag{3.90}$$

and the polynomials also satisfy the second-order differential equation

$$y^{"}(x) - 2xy^{'}(x) + 2ny(x) = 0 \tag{3.91}$$

The functions $h_n(x) = e^{-\frac{x^2}{2}} H_n(x)$ satisfy the differential equation

$$h_n^{"}(x) + \left((2n+1) - x^2\right) h_n(x) = 0 \tag{3.92}$$

Hermite polynomials are relevant for the analysis of the quantum harmonic oscillator.

Appendix B

Gegenbauer polynomials

Gegenbauer polynomials $C_n^{\lambda}(x)$ are n degree polynomial with real coefficients a class of orthogonal polynomials on the interval $[-1,1]$, well known in it hypergeometric function term :

$$\begin{aligned} C_n^{(\lambda)}(x) &= \binom{n+2\lambda+1}{n} {}_2F_1\left(-n, n+2\lambda, \lambda+\frac{1}{2}, \frac{1}{2}(1-x)\right) \\ &= \binom{n+2\lambda+1}{n}\left(\frac{x+1}{2}\right)^n {}_2F_1\left(-n, -n-\lambda+\frac{1}{2}, \lambda+\frac{1}{2}, \frac{(1-x)}{x+1}\right) \end{aligned} \tag{3.93}$$

where

$$C_n^{(\lambda)}(-x) = (-1)^n C_n^{(\lambda)}(x) \tag{3.94}$$

for $\lambda > -\frac{1}{2}$
The first few Gegenbauer polynomials are
$C_0^{(\lambda)}(x) = 1$
$C_1^{(\lambda)}(x) = 2\lambda x$
$C_2^{(\lambda)}(x) = -\lambda + 2\lambda(1+\lambda)x^2$
$C_3^{(\lambda)}(x) = -2\lambda(1+\lambda)x + \frac{4}{3}\lambda(1-\lambda)(2+\lambda)x^3$
The polynomials satisfy the recurrence relation

$$C_n^{(\lambda)}(x) = \frac{1}{n}\left[2x(n+\lambda-1)C_{n-1}^{(\lambda)}(x) - (n+2\lambda-2)C_{n-2}^{(\lambda)}(x)\right] \tag{3.95}$$

$C_{(n)}^{(\lambda)}(x)$ is a solution of the following Gegenbauer differential equation :

$$\left(1-x^2\right)y^{"}-(2\lambda+1)xy^{'}+n\left(n+2\lambda\right)y=0 \tag{3.96}$$

When $\lambda=\frac{1}{2}$, the equation reduces to the Legendre equation, and the Gegenbauer polynomials reduce to the Legendre polynomials.

When $\lambda=1$, the equation reduces to the Chebyshev differential equation, and the Gegenbauer polynomials reduce to the Chebyshev polynomials of the second kind.

Gegenbauer polynomials $C_{(n)}^{(\lambda)}\left(x\right)$ are normalized by

$$\int_{-1}^{1}\left(1-x^2\right)^{\lambda-\frac{1}{2}}\left[C_n^{(\lambda)}\left(x\right)\right]^2dx=2^{1-2\lambda}\pi\frac{\Gamma\left(n+2\lambda\right)}{\left(n+\lambda\right)\Gamma^2\left(\lambda\right)\Gamma\left(n+1\right)} \tag{3.97}$$

Résumé

Dans ce mémoire, nous avons traité le problème de l'oscillateur Klein-Gordon (KGO) avec le principe d'incertitude généralisé (GUP) dans l'espace déformé. Dans le premier cas, nous avons traité le problème de la particule scalaire dans le cas de l'oscillateur Klein-Gordon libre ($\varepsilon = 0$), le spectre d'énergie E_n est déduit en fonction de n et la fonction d'onde $\phi_n(x)$ est déterminée en fonction de polynome Hermite $H_n(x)$.

Dans le 2ème cas, nous avons résolué l'équation de l'oscillateur Klein-Gordon en la présence du champs électrique externe ε dans l'espace déformé, le spectre d'énergie E_n est donné en fonction de puissance de n qui expliqué par la lenguer minimal et le fonction d'onde $\phi_n(p)$ est déterminée en fonction de polynome Gegenbaouer $C_n^{\mu}(p)$. Les cas limites sont déduits et confirm les résultats obtenus, et on a calculé les probabilites termiques Z, U, F, C, S. En conclusion de ce travail, nous avons introduit le traitement par les intégrals de chemin de l'oscillateur de Klein-Gordon en l'absence du champ électrique externe ε dans l'espace déformé.

Mots-clés : mécanique quantique relativiste, équation de Klein Gordon, espaces régulier, espaces déformés, longueur minimale.

الملخص

في هذه الأطروحة عالجنا مشكلة مذبذب كلاين - جوردون بمدأ عدم اليقين المعمم في الفضاء المشوه .

في الحالة الاولى درسنا مشكلة الجسيم السلمي في حالة مذبذب كلاين - جوردون الحر $(\varepsilon = 0)$ ، طيف الطاقة E_n معطى بدلالة n و الدالة الموجبة $\phi_n(x)$ ممثلة بكثير حدود هرميت $H_n(x)$.

في الحالة الثانية حللنا معادلة مذبذب كلاين - جوردون في وجود المجال الكهربائي الخارجي ε في الفضاء المشوه. حيث طيف الطاقة E_n في هذه الحالة معطى كدالة بدلالة قوى n مما يوضح تأثير الحد الأدنى للطول و الدالة الموجبة $\phi_n(p)$ المعرفة في هذه الحالة بكثير حدود غيغنبور $C_n^{\lambda}(p)$. تم استنتاج الحالات الحدودية و تأكيدها بالنتائج المتحصل عليها ثم حسبنا الخصائص الترموديناميكية Z, U, F, C, S .

في ختام هذا العمل عالجنا معادلة مذبذب كلاين - جوردون في غياب المجال الكهربائي ε باستعمال طريقة تكامل المسار في هذا الفضاء المشوه .