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Robustness and Stability Analysis of the Hovering Motion of a Helicopter

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Dedication

I dedicate this work to
My mother Z.Mezrig and my father B.Habbar, source of my happiness and success in life. May Allah bless them .

To my brothers, and my sisters especially Nour Houda for their unconditional support and encouragement to purse my interests.

To my extended family and the many friends,particularly Amina, Aldjia, Khadidja, Akila who have been so supportive and encouraged the fulfillment of this work.

To my best thesis partner Fatima Zohra Saadaoui.

To all my friends and colleagues of the class: Avionics

To anyone who encouraged or helped me during my studies.

To all those who believed in me and pried for my success.

Nassima Habbar

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I dedicate this work to
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and so much could be done with little.

To my father
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In the Name of Allah, the Most Merciful, the Most Compassionate all praise be to Allah, the Lord of the worlds; and prayers and peace be upon Mohammed His servant and messenger.

Abstract

Control engineering plays a fundamental role in modern technological systems. The benefits of improved control in industry can be immense. However, control engineering is not limited to engineering discipline but it is applicable to aeronautical, mechanical, chemical, and electrical engineering. It is based on the foundations of feedback theory and linear system analysis that have arisen during the past century. Multi-input, multi-output systems are aimed to modify the behavior of an existing system to perform a best characteristics in terms of the time response performances, the robustness, the stability. All of this in a desired way using the feedback control theory. In this case, the control design is allowed to handle the desired objectives by using two different methodologies i.e(control strategies) named the block control forms and the general forms. Our application consists of simulating the dynamics of a helicopter at lateral motion using Matlab software. All these results are then compared to select the best state-feedback gain matrix satisfying the required criteria mentioned above. The comparison showed that the system performance is considerably affected by both the choice of the input sequence and the arrangement of the eigenvalues in the desired matrix.

Keywords: MIMO system, State feedback controller, Block pole placement technique, General forms, Stability, Robustness.

ملخص

تلعب هندسة التحكم دورًا أساسيا في النظم التكنولوجية الحديثة. فوائد تحسين السيطرة يمكن أن تكون ضخمة. ومع ذلك ، فإنه ينطبق على هندسة الطيران والميكانيكية والكيميائية والكهربائية. وهي تستند إلى أسس نظرية التغذية الراجعة وتحليل النظام الخطي التي نشأت خلال القرن الماضي. تهدف الأنظمة متعددة المدخلات والمخرجات إلى تعديل سلوك النظام الحالي لأداء استجابة الوقت والمتانة والاستقرار. كل هذا بطريقة مرغوبة باستخدام نظرية التحكم في الملاحظات. في هذه الحالة ، يسمح لتصميم التحكم بمعالجة الأهداف المرغوبة باستخدام منهجين مختلفين ، أي (استراتيجيات التحكم). يتكون تطبيقنا من محاكاة ديناميات طائرة هليكوبتر في الحركة الجانبية باستخدام برنامج Matlab . ثم تتم مقارنة كل هذه النتائج بأفضل مصفوفة لتعليقات الحالة. أظهرت المقارنة أن أداء النظام يتأثر إلى حد كبير باختيار تسلسل الإدخال وترتيب القيم الذاتية في المصفوفة المطلوبة.

الكلمات المفتاحية: نظام MIMO ، وحدة تحكم ردود الفعل ، تقنية وضع القيم الذاتية للقطب ، والأشكال العامة ، الاستقرار ، المتانة.

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Notations and Acronyms

A	State matrix.
B	Input matrix.
C	The output matrix.
D	The feedforward matrix.
n	Number of states.
m	Number of inputs.
p	Number of outputs.
$\dot{x}(t)$	Derivative of the state with respect to time.
$x(t)$	State of the system.
$u(t)$	The control input.
$y(t)$	The output of the system.
$(.)^{-1}$	Inverse .
$(.)^{T}$	Transpose.
$(.)^{+}$	Pseudo-inverse.
$(.)^{\star}$	Conjugate.
$(.)^{(i)}$	i^{th} derivative.
I_{ij}	Identity matrix of dimension $i \times j$.
$A(.)/A_R/A_L$	Matrix polynomial/Right/Left.
λ_i i^{th}	eigenvalue/latent value.
Λ	Spectral matrix of eigenvalues/latent values.
V_R/V_L	Right/ Left block Vandermonde matrix.
R_i/L_i	Right/ Left Solvent (block Root).
σ	Spectrum of a rational or polynomial matrix.
δ_{ij}	Dirac delta function.

δ_d	desired matrix polynomial.
$\omega_c, \mathcal{P}$	Controllability matrix.
$\omega_o, \mathcal{Q}$	Observability matrix.
k_i	Controllability index.
v_i	Observability index.
T_c/T_{c1}	Controller similarity transformation matrices.
$\{A_c, B_c, C_c\}$	Block Controller state-space system.
T_o/T_{o1}	Observer similarity transformation matrices.
$\{A_o, B_o, C_o\}$	Block Observer state-space system.
$\{A_d, B_d, C_d\}$	Block Diagonal state-space system.
$D(\lambda)/D_R/D_L$	Denominator (Right / Left) of MFD system.
$N(\lambda)/N_R/N_L$	Numerator (Right /Left) of MFD system.
$T(\lambda)$	Matrix transfer function.
K	Gain feedback matrix.
L	Gain matrix for observer design.
$H(\lambda)$	Open loop system transfer function.
$C(\lambda)$	Compensator transfer function.
$D_c(\lambda)/N_c(\lambda)$	Denominator and numerator of $C(\lambda)$.
$C_1(\lambda)/C_0(\lambda)$	Compensators of input-output feedback configuration.
$H_p(\lambda)$	Pre-compensator transfer function.
$H_{cl}(\lambda)$	Closed loop system transfer function.
$D_l(\lambda)$	Desired denominator.
$L_c(\lambda)/M_c(\lambda)$	Compensator numerators.
V_i	Right eigenvectors.
W_i	Left eigenvectors.
g	Gravity,$[m/s^2]$.
$Ixx; Izz; Ixz; Ixz$	Moment of inertias according to axes.
m	the mass of the helicopter, [Kg]

U	Forward velocity in the body axis set.
V	Side velocity in the body axis set.
W	Vertical velocity in the body axis set.
p	Roll rate.
q	Pitch rate.
r	Yaw rate.
ϕ	angle of roll,[rad]
ψ	angle of yaw, [rad]
θ	angle of Pitch,[rad]
L	Rolling moment,[N:m]
N	Yawing moment, [N:m]
M	Pitching moment, [N:m]
$I_v, I_p, ...$	Non-dimensional normalised rolling moment derivatives.
$m_u, m_w, ..$	Non-dimensional normalised pitching moment derivatives.
$n_v, n_w, ...$	Non-dimensional normalised yawing moment derivatives.
$x_u, x_w, ...$	Non-dimensional OX force derivatives.
$y_v, y_p, ..$	Non-dimensional OY force derivatives.
$z_u, z_w, ...$	Non-dimensional OZ force derivativesl.

GENERAL INTRODUCTION

General Introduction

Unlike other engineering specialties whose subject of study is a specific engineering system such as an engine system or an airborne system, control systems theory studies only a general mathematical model of engineering systems. The reason that systems control theory has concentrated mainly on linear time-invariant systems is that only the mathematical models of this kind of systems can have general and explicit solutions. Furthermore, only the general and explicit understanding of the system can be used to guide generally, systematically, and effectively the complicated control system design [9]. The purpose and requirement of control systems is generally the control of plant system output (or response) Y(s) so that it can quickly reach and stabilize to its desired state, State space control theory provides distinctly general, accurate, and clear analysis on linear time-invariant systems, especially their performance and sensitivity properties. Only this kind of analysis and understanding can be used to guide generally and effectively the design of complex control systems.[9],[23]

A large-scale MIMO system, described by state equations, is often decomposed into small subsystems, from which the analysis and design of the MIMO system can be easily performed. Similarity block transformations are developed to transform a class of linear time-invariant MIMO state equations, for which the systems described by these equations have the number of inputs dividing exactly the order of the state, into block companion forms so that the classical lines of thought for SISO systems can be extended to MIMO systems [24]. Such systems can be studied via the eigenstructure, eigenvalues and eigenvectors, of the state matrix A. The eigenvalues and eigenvectors can determine system performance and robustness far more directly and explicitly than other indicators. Hence their assignment should improve feedback system performance and robustness distinctly and effectively. [9]

The critical importance of system poles (eigenvalues of system dynamic matrix) on system performance are determined and examined by the location of those roots, however in a complementary the sensitivity of eigenvalues is determined by their corresponding eigenvectors which is a basic result of numerical linear algebra. Unfortunately numerical linear algebra, has not been commonly used in the existing textbooks on control systems, is a branch of study which concentrates on the sensitivity of linear algebraic computation with respect to the initial data variation and computational round-off errors [Fox, 1964]. Because linear algebra is the basic mathematical tool in linear control systems theory, the results of numerical linear algebra can be used directly in analyzing linear system sensitivities. When the controlled system is multi-input multi-output then an infinite number of gain matrices K may be found which will provide the required stability characteristics. Consequently, an alternative and very powerful method for designing feedback gains for auto-stabilization systems is the right and/or left block pole placement method. The method is based on the manipulation of the equations of motion in block state space form and makes full use of the appropriate computational tools in the analytical process. The design of state feedback control in MIMO systems leads to the so-called matrix polynomials assignment. [25]

The use of block poles constructed from a desired set of closed-loop poles offers the advantage of assigning a characteristic matrix polynomial rather than a scalar one [26]. The desired characteristic matrix polynomial is first constructed from a set of block poles selected among a class of similar matrices, and then the state feedback is synthesized by solving matrix equations. The forms of the block poles used in our work are the diagonal, the controller and the observer forms. Robustness is assessed, in each case, using the infinity norm, the singular value of the closed loop transfer matrix and the condition number of the closed-loop transfer matrix. Time response is assessed by plotting the step response and comparing the time response characteristics. A comparison study is conducted to determine, in light of the above criteria, the best choice of the form of the block poles.[9]

Our thesis is split into four chapters which are defined in the following:

- Chapter 1 introduces the basic concepts and the theory of two state space design methods in multivariable control systems.

- Chapter 2 involves the criteria design used to compare between the block pole and the general form.

- Chapter 3 describes the model under study which is the helicopter model and its dynamics and modelization.

- Chapter 4 presents the results of the performed simulations, and the comparative study between these control methods.

We finish this work with a general conclusion along with some potential perspectives.

Chapter 1

Canonical Forms In MIMO Systems

1.1 Introduction

One of the most popular and well-known techniques in Multi-Input Multi-Outputs (MIMO) system used to assign the eigenvalues of the closed-loop system at any desired location is state feedback method. In the case of multivariable systems, the feedback gain matrix permitting the assignment of the desired set of poles (block pole assignment), to the fact that different block poles can be constructed from the same set of eigenvalues to satisfy some desired closed-loop performances (the system response characteristics, robustness, tracking, decoupling, regulation, etc.). In this case the system can be transformed into block and general forms. Similarity block transformations are developed to transform a class of linear time-invariant MIMO state equations.[1]

The main goal of the state feedback design is to force the system to behave in a desired way, since the eigenvalues describe the behavior of the system, we will assign new desired closed loop eigenvalues using two method which are the block pole placement using the state feedback and general form in multivariable system.[2]
First of all we given some definitions related to state feedback control:

1.2 Basic Concepts

Definition 1.1: The state of a dynamic system is the minimum set of variables describing the internal dynamics of system, and can be used to determine the future response, given the excitation inputs and the equations describing the dynamics. The states are not necessary measurable.[3]

Definition 1.2: A system is said to be completely reachable if, for any t_0, any initial state $X(t_0) = X_0$ and any given final state X_f, there exists a finite time $t_1 > t_0$ and a control $U(t)$, $t_0 \leq t \leq t_1$, such that $X(t_1) = X_f$.[4]

If the system is described by the equations:

$$\dot{x} = Ax + Bu \tag{1.1a}$$

$$y = Cx + Du \tag{1.1b}$$

Where: $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$.

Then a linear time invariant system is said to be completely controllable if and only if the $n \times nm$ controllability matrix M is full rank.[5]

$$M = \begin{bmatrix} B\ AB \dots\ A^{n-1}B \end{bmatrix} \tag{1.2}$$

The solution of the state equation given in (1.1) is given by (1.3), with x_0 being the initial state.[6]

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{1.3}$$

1.3 State Feedback Design Using Block Form Transformation

Before we present this technique, let's give, first some important definitions on material related to control system theory and the elements of matrix polynomials.

1.3.1 The transfer function description

The input-output relationship of MIMO system can be given by:

$$Y(s) = H(s)U(s) \tag{1.4}$$

Where $H(s)$ is the Laplace Transform relation of the output $Y(s)$ and the input $U(s)$ with zero initial conditions, and it is called the transfer function of the system.
From the state space equation, the transfer function $H(s)$ can be obtained as follow:

$$H(s) = [C(sI - A)^{-1}B + D] \tag{1.5}$$

Or the impulse response

$$h(t) = Ce^{At}B + D\delta(t) \tag{1.6}$$

The transfer function describes the effect of each of the inputs on all the outputs. In MIMO systems, $H(s)$ is called a rational transfer function matrix that can be expressed either by:

$$H(s) = N_r(s)D_r^{-1}(s) + D \tag{1.7}$$

Or

$$H(s) = D_l^{-1}(s)N_l(s) + D \tag{1.8}$$

Where $D_l(s)$, $D_r(s)$, $N_r(s)$, $N_l(s)$ are matrix polynomials of an appropriate degrees, the last two equations are the right matrix fraction description (RMFD) and the left matrix fraction description (LMFD), respectively.

1.3.2   Solvents of Matrix Polynomial

Given a set of $m \times m$ complex matrices $\{A_0, A_1, \ldots, A_l\}$ the following matrix valued function of the complex variable λ is called matrix polynomial of degree (index) l and order m: ($A(\lambda)$ is called also λ-matrix.)

$$A(\lambda) = I_m \lambda^l + A_1 \lambda^{l-1} + \cdots + A_{l-1} \lambda + A_l \tag{1.9}$$

where $A_i \in \mathbb{R}^{m \times m}$, $\{i = 1, 2, \cdots, l\}$ and $\lambda \in \mathbb{C}^{m \times m}$

Let X be $m \times m$ complex matrix, the two matrix polynomials, defined by

$$A_R(X) = A_0 X^l + A_1 X^{l-1} + \cdots + A_{l-1} X + A_l \tag{1.10}$$

and

$$A_L(X) = X^l A_0 + X^{l-1} A_1 + \cdots + X A_{l-1} + A_l \tag{1.11}$$

are referred to as the right and the left matrix polynomials associated with the λ-matrix $A(\lambda)$ respectively. Factually, they are the left and right evaluation of $A(\lambda)$ with a complex matrix X.[7]

Definition 1.3: A matrix R (respectively: L) is called a right(respectively: left) solvent of the matrix polynomial if and only if the binomial $(\lambda I - R)$ (respectively: $(\lambda I - L)$) divides exactly $A(\lambda)$ on the right (respectively: left).[1]
A right solvent R of $A(\lambda)$ is defined by

$$A_R(R) = A_0 R^l + A_1 R^{l-1} + \cdots + A_{l-1} R + A_l = 0_m \tag{1.12}$$

and the left solvent L of $A(\lambda)$ id defined by

$$A_L(L) = L^l A_0 + L^{l-1} A_1 + \cdots + L A_{l-1} + A_l = 0_m \tag{1.13}$$

where 0_m is an $m \times m$ null matrix, and R, L are $m \times m$ complex matrices.[1]

1.3.3   Block Vandermonde Matrix

As for an eigenvalue system, a block Vandermonde matrix can be defined for solvents with particular properties. Let a set of l right solvents R_i ($m \times m$ matrices) of a corresponding matrix polynomial $A(\lambda)$. A row block Vandermonde matrix of order l is a $lm \times lm$ matrix defined as:

$$V(R_1, R_2, \ldots, R_l) = \begin{bmatrix} I_m & I_m & \cdot & \cdot & \cdot & I_m \\ R_1 & R_2 & \cdot & \cdot & \cdot & R_l \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ R_1^{l-1} & R_2^{l-1} & \cdot & \cdot & \cdot & R_l^{l-1} \end{bmatrix} \tag{1.14}$$

It is called the right block vandermonde matrix.

The block transpose of left block vandermonde matrix of order l is $lm \times lm$ matrix defined by

$$V^T(L_1, L_2, \ldots, L_l) = \begin{bmatrix} I_m & L_1 & \cdot & \cdot & \cdot & L_1^{l-1} \\ I_m & L_2 & \cdot & \cdot & \cdot & L_2^{l-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ I_m & L_l & \cdot & \cdot & \cdot & L_l^{l-1} \end{bmatrix} \tag{1.15}$$

where $\{L_1, L_2, \ldots, L_l\}$ is a set of $m \times m$ matrices.

For more details, we refer the reader to [8] (Hariche, K.)

1.3.4 The Structure of the Desired Solvents

In MIMO system, the feedback gain is not unique; we can construct different sets of solvents from the same set of desired eigenvalues. There exist different structures of solvent; we will consider the canonical forms: diagonal, controller and observer canonical form.

Diagonal Form

Given a set of the desired real eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, the construction of the solvents in diagonal form is as follow:

$$R(or L) = \begin{bmatrix} \lambda_1 & 0 & \cdot & \cdot & 0 \\ 0 & \lambda_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \lambda_m \end{bmatrix} \qquad 1 < i < m \tag{1.16}$$

In the case when we have a set of eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ where λ_1 and λ_2 are complex conjugate and the others are real, thus $\lambda_1 = \sigma + j\omega$; $\lambda_2 = \sigma - j\omega$ then, the solvent is constructed as follows:

$$R(or L) = \begin{bmatrix} \sigma & \omega & 0 & 0 & 0 \\ -\omega & \sigma & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda_{m-1} & 0 \\ 0 & 0 & 0 & 0 & \lambda_m \end{bmatrix} \qquad 1 < i < m \tag{1.17}$$

Controller Form

Consider the set of desired eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, the characteristic equation constructed from this set of eigenvalues is:

$$\Delta(\lambda) = \prod_{i=1}^{m}(\lambda - \lambda_i) = \lambda^m + a_1\lambda^{m-1} + \ldots + a_{m-1}\lambda + a_m \tag{1.18}$$

the construction of the solvents in controllable canonical form is as follows:

$$R(or L) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_m & -a_{m-1} & -a_{m-2} & \dots & -a_1 \end{bmatrix} \qquad 1 < i < m \qquad (1.19)$$

or

$$R(or L) = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{m-1} & -a_m \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad 1 < i < m \qquad (1.20)$$

Observer Form

the construction of the solvents in observable canonical form is as follows:

$$R(or L) = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_m \\ 1 & 0 & \dots & 0 & -a_{m-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & -a_2 \\ 0 & 0 & \dots & 1 & -a_1 \end{bmatrix} \qquad 1 < i < m \qquad (1.21)$$

or

$$R(or L) = \begin{bmatrix} -a_1 & 1 & \dots & 0 & 0 \\ -a_2 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots \\ -a_{m-1} & 0 & \dots & 0 & 1 \\ -a_m & 0 & \dots & 0 & 0 \end{bmatrix} \qquad 1 < i < m \qquad (1.22)$$

For more details, we see [8].

1.3.5   Construction of the Desired Matrix Polynomial

The placement of block poles in a multivariable system, using state feedback requires the construction of a matrix polynomial, which are basically formed by a given set of right or left solvents. The different right and left solvents are constructed using all forms. From this set of solvents, the matrix coefficients of the desired characteristic matrix polynomial must be found. The desired characteristic matrix polynomial is given by the following equation:

$$D_f(s) = Is^l + D_{f1}s^{l-1} + \dots + D_{fl} \qquad (1.23)$$

Then the complete set of right solvents R_i and the left solvents L_i satisfy, respectively, the following matrix polynomial

$$R_i^l + D_{fl}R_i^{l-1} + \dots + D_{f(l-1)}R_i + D_{fl} = 0_m, \quad i = 1, 2, ..., l \qquad (1.24)$$

and

$$L_i^l + L_i^{l-1}D_{fl} + \cdots + L_i D_{f(l-1)} + D_{fl} = 0_m, \quad i = 1, 2, ..., l$$

Hence, the coefficient matrices of the desired matrix polynomial can be obtained by using either:

$$[D_{fl} \ D_{f(l-1)} \ \cdot \ \cdot \ \cdot \ D_{f1}] = [R_1^l \ R_2^l \ \cdot \ \cdot \ \cdot \ R_l^l] \ V_R^{-1} \tag{1.25}$$

or

$$\begin{bmatrix} D_{fl} \\ D_{f(l-1)} \\ . \\ . \\ . \\ D_{f1} \end{bmatrix} = -V_L^{-B} \begin{bmatrix} L_1^l \\ L_2^l \\ . \\ . \\ . \\ L_l^l \end{bmatrix} \tag{1.26}$$

where V_R and V_L^B are the right Vandermonde and the block transpose of the left Vandermonde matrices, respectively. [6]

1.3.6   Block Form Transformation

Transforming the MIMO systems into useful forms called canonical forms requires the knowledge of two major concepts of modern control system theory : controllability and observability, these concepts will be explained in the next subsections. Consider a linear time-invariant system described by a state equation in general coordinates, described by equation(1.1). Where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$.

the system (1.1) is block controllable of index l if the matrix:

i. $M = [B, AB, A^2B, \ldots, A^{l-1}B]$ has full rank.

ii. $l = n/m$ is an integer.

Theorem 1.1:   *The multivariable control system describes in (1.1) can be transformed into a block controller form if two conditions are satisfied:*

i.   *$l = n/m$ is an integer.*

ii.   *The system is block controllable of index l.*

We refer the reader to see Shieh et al [2].

1.3.6.1   The block controller form:

If the two conditions in theorem 1.1 are satisfied, then the change of coordinates using the similarity transformation T_C

$$x_c(t) = T_c x(t) \tag{1.27}$$

where

$$T_c = \begin{bmatrix} T_{c1}^T & (T_{c1}A)^T & (T_{c1}A^2)^T & \ldots & (T_{c1}A^{l-1})^T \end{bmatrix}^T \tag{1.28}$$

and

$$T_{c1} = [0_m, 0_m, \ldots, I_m][B, AB, \ldots, A^{l-1}B]^{-1} \tag{1.29}$$

transforms the system into the following block controller form

$$\begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c u(t) \\ \quad\quad y(t) = C_c x_c(t) \end{cases} \tag{1.30a}$$

$$A_c = T_c A T_c^{-1} = \begin{bmatrix} 0_m & I_m & 0_m & \dots & 0_m \\ 0_m & 0_m & I_m & \dots & 0_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_m & 0_m & 0_m & \dots & I_m \\ -A_l & -A_{l-1} & -A_{l-2} & \dots & -A_1 \end{bmatrix} \tag{1.30b}$$

$$B_c = T_c B = [0_m, 0_m, \dots, I_m]^T \tag{1.30c}$$

$$C_c = C T_c^{-1} = [C_l, C_{l-1}, \dots, C_1]^{-1} \tag{1.30d}$$

Where A_c, B_c, C_c are the new matrices describing the system in the block controllable companion form, and $x_c \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{m \times m}$, $C \in \mathbb{R}^{p \times m}$, $i = 1, 2, \dots, l$ I_m and O_m are $m \times m$ identity and null matrices respectively, and the superscript T in (1.30.c) denotes the transpose.

1.3.6.2 The construction of the block similarity transformation for controller form

T is a similarity transformation which allow a system to appear in another form, assume that T_c is the controllability passage such that:
$A_c = T_c A T_c^{-1}$ $B_c = T_c B$ and $C_c = C T_c^{-1}$ and $D_c = D$
Let we put :

$$T_c = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_l \end{bmatrix} \quad and \quad A_c = \begin{bmatrix} 0_m & I_m & 0_m & \dots & 0_m \\ 0_m & 0_m & I_m & \dots & 0_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_m & 0_m & \dots & \dots & I_m \\ -A_l & -A_{l-1} & \dots & -A_2 & -A_1 \end{bmatrix} \tag{1.31}$$

$$A_c T_c = T_c A = \begin{bmatrix} T_1 A \\ T_2 A \\ \vdots \\ T_l A \end{bmatrix} \tag{1.32}$$

But from the other hand

$$A_c T_c = \begin{bmatrix} T_2 \\ T_3 \\ \vdots \\ -\Sigma_{i=1}^{l} A_i T_{l-i} \end{bmatrix} \tag{1.33}$$

$$A_c T_c = T_c A \Leftrightarrow \left\{ \begin{array}{c} T_2 = T_1 A \\ T_3 = T_2 A = T_1 A^2 \\ \vdots \\ T_l = T_1 A^{l-1} \end{array} \right\} \Leftrightarrow T_c = \left[\begin{array}{c} T_1 \\ T_1 A^2 \\ \vdots \\ T_1 A^{l-1} \end{array} \right] \tag{1.34}$$

But what is T_1?

$$T_c B = B_c = \left[\begin{array}{c} O_m \\ \vdots \\ O_m \\ I_m \end{array} \right] \Rightarrow \left[\begin{array}{c} T_1 B \\ T_1 AB \\ \vdots \\ T_1 A^{l-1} B \end{array} \right] = \left[\begin{array}{c} O_m \\ \vdots \\ O_m \\ I_m \end{array} \right] \tag{1.35}$$

$$T_c B = B_c \Leftrightarrow T_1 \left[\begin{array}{cccc} B, & AB, & \ldots, & A^{l-1} B \end{array} \right] = \left[\begin{array}{cccc} O_m, & \ldots, & O_m, & I_m \end{array} \right]$$

Finally we find that

$$T_1 M = \left[\begin{array}{ccc} O_m, & \ldots, & O_m, I_m \end{array} \right] \Rightarrow T_1 = \left[\begin{array}{ccc} O_m, & \ldots, & O_m, I_m \end{array} \right] M^{-1}$$

1.3.7   Block-Pole Placement by State Feedback

In control theory, the stability and performance of the system (1.1) depends on the location of the eigenvalues $(\lambda_i, i = 1, \ldots, n)$ of the system matrix in the complex plane. In fact, the system is stable if its eigenvalues are located in the left half plane LHP, meaning that they have negative real parts. The pole placement by state feedback is an effective method for the design of closed loop control systems. For the class of MIMO systems for which the number of inputs divides exactly the order of the state equation,i.e., $n = lm$, given a MIMO system described by the state equation (1.1), and a desired matrix polynomial $D_f(\lambda)$ find an $m \times n$ gain matrix K such that under the state feedback operation

$$u(t) = -Kx(t) \tag{1.36}$$

Where $K \in \mathbb{R}^{m \times n}$, the matrix $(A - BK)$ in the new state equation

$$x(t) = (A - BK)x(t)$$

has the desired characteristic matrix polynomial,

$$D_f(\lambda) = I\lambda^l + D_1 \lambda^{l-1} + \cdots + D_{l-1}\lambda + D_l$$

Under the similarity transformation shown in (1.2), the control signal $u(t)$ becomes:

$$u(t) = -K_c x_c(t) \tag{1.37}$$

where

$$K_c = K T_c^{-1} \tag{1.38}$$

$$K_c = [K_{c1}, K_{c2}, \ldots, K_{cl}] \tag{1.39}$$

with $K_{ci}, i = 1, 2, \ldots, l \in \mathbb{R}^{m \times m}$

Equation (1.1) becomes

$$\dot{x}_c(t) = (A_c - B_c K_c)x_c(t)$$
$$y = C_c x_c(t) \tag{1.40}$$

Where

$$A_c - B_c K_c = \begin{bmatrix} 0_m & I_m & 0_m & \cdots & 0_m \\ 0_m & 0_m & I_m & \cdots & 0_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_m & 0_m & 0_m & \cdots & I_m \\ -A_l - K_{c1} & -A_{l-1} - K_{c2} & -A_{l-2} - K_{c3} & \cdots & -A_1 - K_{cl} \end{bmatrix} \tag{1.41}$$

Since $(A_c - B_c C_c)$ is in block companion form, its characteristic matrix polynomial equation is given by:

$$\Delta(s) = I_m s^l + (A_l + K_{c1})s^{l-1} + \ldots + (A_l + K_{cl}) \tag{1.42}$$

The desired characteristic matrix polynomial, constructed from the desired set of block poles is given by

$$\Delta_d(s) = I_m s^l + D_{dl}s^{l-1} + \ldots + D_{dl} \tag{1.43}$$

where $D_i, i = 1, 2, \ldots, l$ are an $m \times m$ matrices.

By forcing $\Delta(s) = \Delta_d(s)$, then the matrices $K_{c1}, K_{c2}, \ldots, K_{cl}$ are given by

$$K_{ci} = D_{di} - A_i \tag{1.44}$$

for $i = 1, 2, \ldots, l$. Then the original gain matrix is given by

$$K = K_c T_c \tag{1.45}$$
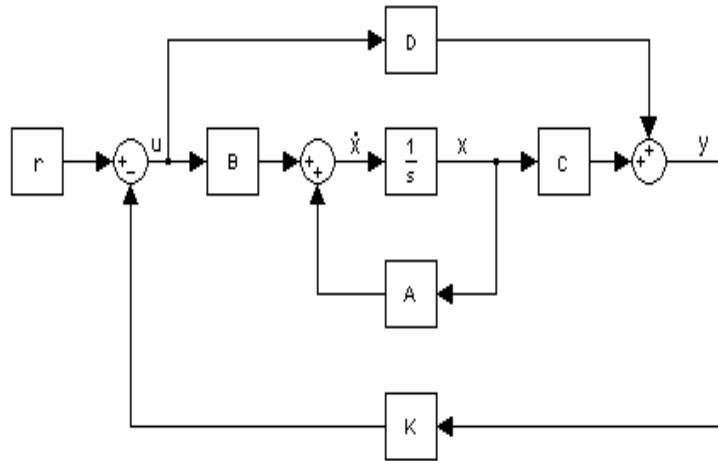
Figure 1.1: State feedback control system

For more details, we see [9].

1.4   State feedback Design Using General Controller Form Transformation

As we have done when we presented the previous technique, we will define some important concepts needed to present the general form transformation.

1.4.1   Controllability Indices

Considering the system described by equations (1.1), its controllability matrix is therefore given by (1.2), and $B = [b_1, b_2, \ldots, b_m]$, where b_i is a column vector of dimension $n \times 1$. Equation (1.2) can be rewrite as:

$$M = (b_1, b_2, \ldots, b_m; Ab_1, Ab_2, \ldots, Ab_m; \ldots; A^{n-1}b_1, A^{n-1}b_2, \ldots, A^{n-1}b_m)$$

We search linearly independent columns of M from left to right. A column is linearly dependent if it can be written as a linear combination of its left hand side columns, otherwise it is linearly independent. At the end of this search process, we will get a set of linearly independent columns after leaving out the linearly dependent ones [1]. Taking this set of vectors and rearranging them, we get the following ordered set:

$$\{b_1, Ab_1, \ldots, A^{k_1-1}b_1; b_2, Ab_2, \ldots, A^{k_2-1}b_2; \ldots; b_m, Ab_m, \ldots, A^{k_m-1}b_m\}$$

Where the integer $k_i(i = 1, 2, \ldots, m)$ represents the number of linearly independent columns associated with b_i in the above vector set. The integer k_i is known as the controllability index, and the set $\{k_1, k_2, \ldots, k_m\}$ is called the set of controllability indices of the system. For the system to be controllable, the following condition has to be satisfied:

$$\sum_{i=1}^{m} k_i = n$$

1.4.2   The General Controller Companion Form

Given the n dimensional system described by equation (1.1), it can be transformed into a general controller companion form by following the steps described below. First, we compute the controllability matrix M defined by (1.2). Then if the system is controllable, we search the linearly independent columns of M in order, from left to right.
After that, we rearrange these columns as

$$E = (b_1, Ab_1, \ldots, A^{k_1-1}b_1; b_2, Ab_2, \ldots, A^{k_2-1}b_2; \ldots; b_m, Ab_m, \ldots, A^{k_m-1}b_m) \qquad (1.46)$$

Where: $k_i(i = 1, 2, \ldots, m)$ are the controllability indices, and m is the number of inputs of the system.

Finally, compute E^{-1} and name its rows as:

$$E^{-1} = \begin{bmatrix} e_{11}^T & e_{12}^T & \ldots & e_{1k_1}^T & \ldots & e_{m1}^T & \ldots & e_{mk_m}^T \end{bmatrix}^T \qquad (1.47)$$

Use the rows $e_{ik_i}(i = 1 \ldots m)$ to from the transformation matrix T_c, and e_i is the σ_i th row of E^{-1} and $\sigma_i = \sum\limits_{j=1}^{i} k_j; i = 1, \ldots, m$.

$$T_c = \left[\; e_1^T \; (e_1 A)^T \; \ldots \; (e_1 A^{k_1-1})^T \; \ldots \; e_m^T \; \ldots \; (e_m A^{k_m-1})^T \; \right]^T \tag{1.48}$$

Once the similarity transformation T_c is known, we can compute the general controller companion form of the matrices A, B and C as as it follows:

$$A_c = T_c A T_c^{-1} = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1m} \\ A_{21} & A_{22} & \ldots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \ldots & A_{mm} \end{bmatrix} \tag{1.49}$$

$$B_c = T_c B = \left[\; B_1^T \; B_1^T \; \ldots \; B_1^T \; \right]^T \tag{1.50}$$

$$C_c = C T_c^{-1} \tag{1.51}$$

The block matrices, A_{ii}, A_{ij} and B_i are respectively of dimensions $k_i \times k_i$, $k_i \times k_j$, and $k_i \times m$ where k_i is a controllability index and $\sum\limits_{i=1}^{m} k_i = n$. The block matrices are of the form given below:

$$A_{ii} = \begin{bmatrix} 0 & 1 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 \\ x & x & x & x & x & x \end{bmatrix} \quad A_{ij} = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ x & x & \cdot & \cdot & \cdot & x \end{bmatrix} \quad B_i = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & x & \cdot & x \end{bmatrix}$$

Where x is a nontrivial element and the first $i \times 1$ columns of the matrix B_i are zero. It is possible to write the controllable matrix $A_c - B_c K_c$ in many different ways depending on the choice of the feedback gain matrix K_c; we may have one or more block diagonal elements, where each block is in companion form. The best case was found to be: we choose K_c so that $(A_c - B_c K_c)$ has the maximum number of block companion forms on the diagonal, with dimensions $(k_1 \times k_1), (k_2 \times k_2), \ldots, (k_m \times k_m)$ respectively. Due to the form of B_c, all rows of A_c, expect the rows denoted by the string element x, are not affected by the state feedback. Thus, the elements of those rows of the matrix $(A_c - B_c K_c)$ can be arbitrary assigned. So we will multiply the matrix B_c by an elementary matrix D_G, in order to zero out the entries marked x in the $\{k_1^{th}, (k_1 + k_2)^{th}, (k_1 + k_2 + k_3)^{th}, \ldots\}$ rows of the matrix B_c.

We will get:

$$B_c D_G^{-1} = E_{bc} = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \tag{1.52}$$

We define also

$$\widetilde{K}_c = D_G K_c \tag{1.53}$$

So that we shall have

$$B_c K_c = B_c D_G^{-1} \widetilde{K}_c = E_{bc} \widetilde{K}_c \tag{1.54}$$

This implies that

$$A_c - B_c K_c = A_c - E_{bc} \widetilde{K}_c = A_d \tag{1.55}$$

Where A_d can have different blocks of companion form on the diagonal. This means that it will have in the rows $[k_1, k_1 + k_2, \ldots, \sum_{i=1}^{m} k_i]$ elements resulting from the choice of the number of blocks and their sizes (the elements of the characteristic equation of each block in each corresponding row) and the other rows just as in A_c.

Note: In our work, we choose the best case, which consists of taking the maximum number of blocks. This number is obtained directly from the controllability indices $k_i (i = 1, 2, \ldots, m)$. We can summarize the state feedback design in four steps:

1. Transform the system into controllable form.

2. Compute $\widetilde{K}_c$ such that $A_c - E_{bc} \widetilde{K}_c$ has a set of desired eigenvalues.

3. Compute $K_c = D_G^{-1} \widetilde{K}_c$ where D_G is obtained from (1.54)

4. Compute K from K_c, such that $K = K_c T_c$.

For more details, we refer the reader to [10].

1.5   Conclusion

In this chapter, we provide that the major advantage of modern control theory is the representation, design and analysis of system models directly in the time domain. This concept is made possible by the structure of the linear state space model which is invariant under similarity transformations: a system represented by a set of first order ordinary differential equations describing its behavior which can be transformed into other more useful forms called canonical forms without changing its structure.

Chapter 2

System Robustness and Sensitivity

2.1 Introduction

It is known from control theory that in the case of single input single output systems, The state feedback gain K is uniquely determined for a particular set of desired eigenvalues. However, for MIMO systems, the state feedback gain matrix, which will allocate the closed-loop poles to some desired positions, is not unique. This non uniqueness of the gain matrix, offers freedom that permits not only to place the closed-loop system eigenvalues, but also to satisfy the closed-loop system robustness to parameter variations [12]. The robustness of the closed-loop system is one of the most important concerns of control system designers. Variations in system parameters due to component aging might result in system performance deterioration, and even in system internal stability concerns. Eigenvalue locations can also be affected by external disturbances and, hence, those disturbances should be considered when designing feedback systems. Height system performance and low sensitivity are two required properties of control systems. Low sensitivity is defined with respect to the system's mathematical model uncertainty and terminal disturbance called robustness.[9]

2.2 Design Criteria

It is well known that high system performance and low sensitivity are the two required properties of control systems, unfortunately, these criteria are usually contradictory to each other i.e; higher performance systems usually have higher sensitivity and worse robustness properties.

The objective of our work is to do a comparative study of the state feedback gains based on four different criteria, and determine the gain matrix, which yields the best response characteristics and system robustness. This chapter will deal with the description of the different criteria used to compare between the block pole and general form are:

- Magnitude of feedback gains.

- System sensitivity and robustness.

- Time domain characteristics.

2.2.1 State-Feedback Gain Magnitude

Small system feedback gains minimize the control energy, and prevent saturation of the controller elements and noise amplification.

2.2.1.1 Norm of a matrix

The concept of norms for vectors can be extended to matrices. The norm of a matrix can provide a scalar measure to the magnitude of the matrix.

Let A be an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \tag{2.1}$$

The norm of A can be defined as:

$$\| A \| = \sup_{x \neq 0} (\frac{\| AX \|}{\| X \|}) = \sup_{\|x\|=1} \| AX \| \tag{2.2}$$

Where "sup" stands for supremum or the least upper bound. This norm is defined through the norm of x. Therefore, it is called an induced norm. For different $\| X \|$, we have different $\| A \|$ According to this definition, we may have many norms.

We refer the reader to see [6].

Case01: If the 1 norm $\| A \|_1$ is used, then:

$$\|A\|_1 = \max_j (\Sigma_{i=1}^n \mid a_{ij} \mid) \tag{2.3}$$

Case02: If the Euclidean norm $\| A \|_2$ is used, then:

$$\| A \|_2 = \max_j (\sqrt{\lambda_j(A^T A)}) \tag{2.4}$$

$\| A \|_2 = $ largest singular value of A
where A^T stands for the transpose of A.

Case03: If the infinite-norm $\| A \|_\infty$ is used, then:

$$\| A \|_\infty = \max_i (\Sigma_{j=1}^n \mid a_{ij} \mid) \tag{2.5}$$

Such that $\| A \|_\infty = $ largest row absolute sum.

Case04: There is another commonly used matrix norm, which is called, the "Frobenius norm" and is defined as follows:

$$\| A \|_F = \sqrt{(\Sigma_{i=j} \mid a_{ij} \mid^2)} = [Trace(A^T)(A)]^{1/2} \tag{2.6}$$

Where the matrix operator "Trace" stands for the sum of all diagonal elements.

We refer the reader to see [9].

2.2.1.2 Condition number

One of the most valuable matrix computations is the condition number as it enables us to estimate the accuracy of the computed results. The condition number can be seen as sensitivity of a function to a perturbed input argument. To illustrate the condition number of a matrix A, we decompose it as:

$$A = X\Lambda X^{-1} \tag{2.7}$$

Can be rewritten as

$$\Lambda = X^{-1}AX \tag{2.8}$$

Now let δA be a small perturbation in the matrix A. Then:

$$A + \delta A = X^{-1}(A + \delta A)X \tag{2.9}$$

Hence:

$$\delta A = X^{-1}\delta A X \tag{2.10}$$

Taking the matrix norms,

$$\parallel \delta\Lambda \parallel \leq \parallel X^{-1} \parallel \parallel X \parallel \parallel \delta A \parallel = \kappa(X) \parallel \delta A \parallel \tag{2.11}$$

Where $\kappa(X)$ is the matrix condition number. For square matrices A we define the condition number to be:

$$\kappa(X) = \parallel A^{-1} \parallel \parallel A \parallel \tag{2.12}$$

We refer the reader to see [11].

2.2.2 System Sensitivity and Robustness

Robustness in control systems is the ability of the system to keep the same or approximately the same behavior under model uncertainties, and since the behavior of the system is described by its eigenvalues, robustness and eigen structure are strongly related. The first robustness result is based upon robust performance; the second is based on the matrix measures approach developed by Chia-Chi Tsui used in robust stability [9]. In the following, we will first analyze the sensitivity of the eigenvalues of system dynamic matrix, and then use this result to analyze the sensitivity of system stability property.

2.2.2.1 The sensitivity of eigenvalues (robust performance)

Robust performance is defined as the low sensitivity of system performance with respect to system model uncertainty and terminal disturbance. It is well known that the eigenvalues of the dynamic matrix determine the performance of the system then from that the sensitivities of these eigenvalues determine the robustness of the system.[9]

A. Overall Eigenvalue Sensitivity:

An overall measure of eigenvalue sensitivity can be derived in terms of the closed-loop right (or left) eigenvectors only. The overall eigenvalue sensitivity of the closed-loop matrix A is defined from the next development. If A becomes $(A + \Delta A)$, then:

$$V^{-1}(A + \Delta A)V = \Lambda + V^{-1}AV = \Lambda + \Delta\Lambda \tag{2.13}$$

$$\parallel \Delta\Lambda \parallel \,\leq\, \parallel V \parallel . \parallel V^{-1} \parallel . \parallel \Delta A \parallel = \kappa(V). \parallel \Delta A \parallel \tag{2.14}$$

Inequality (2.14) indicates that the condition number $\kappa(V)$ of eigenvector matrix V can decide the magnitude of $\Delta\Lambda$. From (2.14), it is reasonable to use the condition number of eigenvector matrix V of the matrix A, $\kappa(V)$,to measure the sensitivity of all eigenvalues (Λ) of matrix A, $s(\Lambda)$. In other words, we define:

$$s(\Lambda) = \kappa(\Lambda) = \parallel V \parallel_2 \times \parallel V^{-1} \parallel_2 = \parallel V \parallel_2 \times \parallel T \parallel_2 \tag{2.15}$$

Where V is the right eigenvector matrix of the closed-loop matrix A, and T is the corresponding left eigenvector matrix.

Even though $s(\Lambda)$ is not an accurate measure of the variation (sensitivity) of each individual eigenvalues. The advantage of this measure is that it is valid for large $\parallel \Delta A \parallel$. In order to obtain a more accurate measure of the sensitivity of individual eigenvalues, first order perturbation analysis is applied and the following result is obtained under the assumption of small $\parallel \Delta A \parallel$.

B. Individual Eigenvalue Sensitivity:

A measure of individual eigenvalue sensitivities which is particularly well known is found by computing a certain function of the closed-loop right and left eigenvectors. The sensitivity of the ith eigenvalue of a closed-loop matrix A to perturbations in some or all of its elements is given by the following theorems:

Theorem 2.1: (Chia-Chi Tsui et.al 2004) *Let λ and λ' be the eigenvalues of the matrices A and $A + \Delta A$ respectively, and let V be the right eigenvectors matrix of A, then Wilkinson has derived the variation in eigenvalues as follows [11]:*

$$\min_i \{\lambda_i - \lambda_i'\} = \min_i \{\Delta(\lambda_i)\} \leq \kappa(V). \parallel \Delta A \parallel \tag{2.16}$$

$\parallel . \parallel$ *Stands for the matrix norm and $\kappa(.)$ Is the condition number.*

Theorem 2.2: (Chia-Chi Tsui et.al 2004) *Let λ_i ,v_i and t_i be the ith eigenvalue, right and left eigenvectors of matrix A, respectively ($i = 1, 2, \cdots, n$).Let $\lambda_i + \Delta\lambda_i$ be the ith eigenvalue of matrix $A + \Delta A$, ($i = 1, 2, \cdots, n$). Then for small enough ΔA,*

$$\Delta\lambda_i \leq \parallel t_i \parallel . \parallel v_i \parallel . \parallel \Delta A \parallel = s(\lambda_i)\Delta A; \qquad (i = 1, 2, \cdots, n). \tag{2.17}$$

This theorem shows that the sensitivity of an eigenvalue is determined by its corresponding left and right eigenvectors and it is valid for small perturbations in the matrix.[9]

C. Relative change:

To study how the eigenvalues are affected by small random perturbations matrix $\parallel \Delta A \parallel$, the relative change r_i is computed as:

$$r_i = \frac{\mid \lambda_i - \lambda_i' \mid}{\mid \lambda_i \mid} = \frac{\mid \Delta \lambda_i \mid}{\mid \lambda_i \mid}; \qquad (i = 1, 2, \cdots, n). \tag{2.18}$$

λ_i are the eigenvalues of the closed loop system $(A - BK)$.

λ_i' are the eigenvalues of the closed loop system $(A + \Delta A - BK)$.

We refer the reader to see [1].

2.2.3   Robust Stability

Stability is the foremost system property. Therefore the sensitivity of this property called robust stability, with respect to system model uncertainty is also critically important. Consequently, a generally accurate numerical measure of this sensitivity is also essential to guide robust stability analysis and design.

The most basic and direct criterion of system stability is that every dynamic matrix eigenvalue has a negative real part. Hence the sensitivity of these eigenvalues with respect to system model uncertainty (or dynamic matrix variation) should be the most direct and critical factor in measuring the sensitivity of system stability (robust stability.[9]

There are three existing robust stability measures using the sensitivity of system poles, they are called M_1, M_2, and M_3.

All three measures are defined such that the more robustly stable the system is, the greater the value of its robust stability measure.

2.2.3.1   The robust stability measure M_1

The measure M_1 is given by:

$$M_1 = \min_{1 \leq \omega \leq \infty} \sigma(Q - j\omega I) \tag{2.19}$$

Where: $Q = A - BK$, Q is an $(n \times n)$ real matrix.

Because σ indicates the smallest possible norm of matrix variation for a matrix to become singular, M_1 equals the smallest possible matrix variation norm for the dynamic matrix A to have an unstable and pure imaginary eigenvalue $j\omega$. Therefore M_1 should be a generally accurate robust stability measure.

For more detail we recommand the reader to see [9].

2.2.3.2   The robust stability measure M_2

The measure M_2 is given by:

$$M_2 = s(\Lambda)^{-1} \mid Re\lambda_n \mid; \qquad (\mid Re(\lambda_n) \mid \leq \cdots \leq \mid Re\lambda_1 \mid) \tag{2.20}$$

In the measure M_2, the term $\mid Re(\lambda_n) \mid$, is the shortest distance between the unstable region and eigenvalues λ_i. Thus, M_2 equals this distance divided by the sensitivity of all the eigenvalue matrix Λ. The lower is the sensitivity $s(\Lambda)$, the greatest is M_2. In other words, M_2 may be considered as the likelihood margin for λ_n to become unstable.

There exist several general and numerical algorithms which can compute state feedback gain matrix K such that the value of $s(\Lambda)^{-1}$ or M_2 is maximized, with arbitrarily assigned eigenvalues in matrix Q.[9]

However, M_2 seems to be less accurate in measuring the likelihood margin for λ_n, to become unstable, because $s(\Lambda)$ is not an accurate measure of the sensitivity of λ_n.

2.2.3.3   The robust stability measure M_3

The measure M_3 is given by:

$$M_3 = \min_{1 \leq i \leq n} s(\lambda_i)^{-1} \mid Re(\lambda_i) \mid \qquad (2.21)$$

In the definition of the measure M_3, the likelihood margins for every eigenvalue to become unstable are considered. The likelihood margin for each λ_i equals $\mid Re(\lambda_i) \mid$ divided by its corresponding sensitivity $s(\lambda_i)$ with $i = 1, 2, \cdots, n$.

From the measures M_1 and M_2 consider only the likelihood margin for λ_n to become unstable,while the instability of any eigenvalue can cause system instability, the $s(\Lambda)$ of M_2 is generally not an accurate measure of individual eigenvalues sensitivity and is not as accurate as the sensitivity $s(\lambda_i)$ of λ_i, itself in measuring the sensitivity of λ_i for $\forall i$ (including $i = n$). Hence, M_3 is more accurate than M_1 and M_2, and reflects the instability likelihood of all eigenvalues.

2.2.4   Time Domain Performance of Control Systems

Primary concerns of control system design are stability and performance. In order to design and analyze control systems, we must first establish adequate performance specifications. Performance specifications can be presented in the time domain or the frequency domain. Time domain specifications generally take the form of setting time, percent overshoot, rise time and steady state error specifications.

The response of a stable system always reaches steady state, a which is called "steady state response" and which is often the desired state of response. The system response before reaching its steady state is called "transient response". This latest is undesired, therefore, the faster and the smoother the transient response, the better the performance of the system.

Time-domain performance specifications are generally given in terms of the transient The response of a system to a given input signal.

the transient portion of the time response is the part which goes to zero (for stable system) as time becomes large. Nevertheless, the transient response of a control system is necessarily important, since both the amplitude and time duration of the transient

response must be kept within prescribed limits.

Performance criteria commonly used for the characterization of linear control systems in the time domain, for a step response, are defined as follows:

- **Maximum overshoot** M_p
  It is the largest deviation of the output over the step input during the transient response. Let $y(t)$ be the unit-step response. Let y_{max} denotes the maximum value of $y(t)$. and y_{ss} be the steady-state of $y(t)$, and $y_{max} > y_{ss}$.

  The maximum overshoot of $y(t)$ is defined as:

  $$Maximum\ overshoot(M_p) = y_{max} - y_{ss} \qquad (2.22)$$

  The maximum overshoot is often represented as a percentage of the final value of the (2.22) step response, that is:

  $$Percent\ maximum\ overshoot = \frac{Maximum\ overshoot(M_p)}{y_{ss}} \times 100\%$$

  A system with large overshoot is usually undesirable. For a step response, an over shoot between 0 % and 10 % is acceptable.

- **Delay time** T_d
  It is defined as the time required for the step response to reach 50% of its final value.

- **Rise time** T_r
  It is defined as the time required for the step response to rise from 10% to 90% of its final value.

- **Settling time** T_s
  It is defined as the time required for the initial state response to decrease and stay within specified tolerance band of the initial state value, normally oscillating bellow $\pm 5\%$.

- **Fall Time** T_f
  It is the shortest time to achieve the final or the steady state value, for the first time. The fall time can also be defined as the time required for the initial state response to decrease from 90% to 10% of the initial state value.
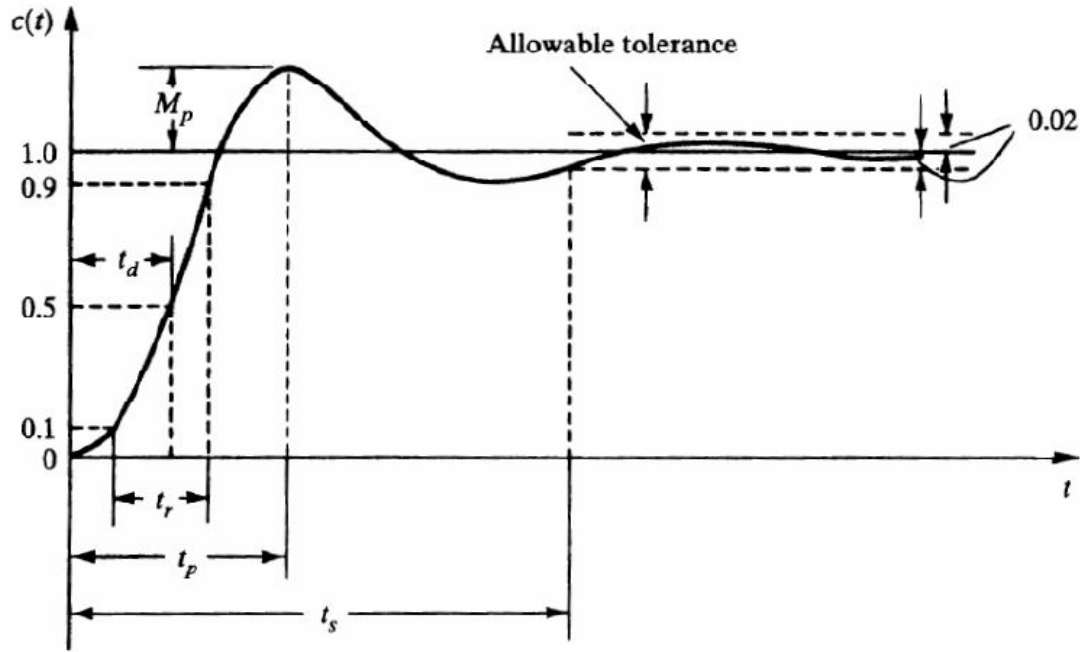
Figure 2.1: Typical unit step response of a control system.

2.3   Time Domain Performance for Initial Condition

However, we are interested in the transient response because we know that the states(with the integral term)will go to zero when t tends to infinity.

$$x_{ss} = \int_0^t \exp A(t - \tau)Bu(\tau)d\tau \longrightarrow 0 \quad , when : t \longrightarrow \infty \qquad (2.23)$$

The elements: maximum overshoot, rise time, fall time and settling time are defined for a step response, in single input single output systems. In these systems, the output should follow a reference input, which is a step. In our case, we are not interested in drawing the outputs,since we are not doing output feed back.But we want to draw the states, because we are feeding them back. This is why we have to define our time response characteristics suited to initial state response. Unlike the step response, the transient response desired value is zero $u(t) = -Kx$ whenever $x = 0$. This means that,under normal conditions, or in the absence of a perturbation, the states are equal to zero. Thus no control is needed. [13]

We have taken the definitions for the step response, given above, and we adapted them to our analysis. Since the desired value, as we have said before, is zero, then taking a percentage of a zero is meaningless. This is why we worked with the initial state value instead. For more details, we refer the reader to see [13].

In many cases, observation of a system is started at a particular point in time. Naturally, the input before that point is unknown or undermined. Evaluating the system response must therefore rely on the state found before the start, and the behaviour of the input signal since then. In the terminology of differential equations and differential calculus this is an initial condition problem.

The response to right-sided signal can be split into an internal and an external term.
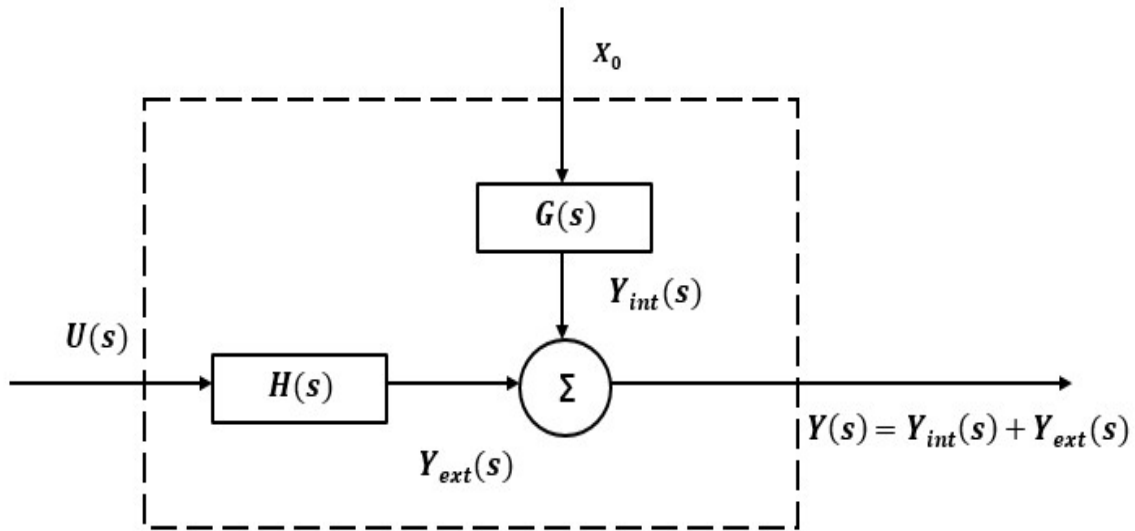
Figure 2.2: The response of the LTI and Initial Condition.

The initial state of a system completely summarises the effects of its past history.

The laplace transform of a state equations with initial conditions is given by:

$$sX(s) - X(0) = AX(s) + BU(s) \tag{2.24}$$

$$Y(s) = CX(s) + DU(s) \tag{2.25}$$

Collecting terms with $X(s)$ together we get:

$$X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}x(0) \tag{2.26}$$

Then the response of the system will be:

$$Y(s) = C(sI - A)^{-1}BU(s) + C(sI - A)^{-1}x(0) + DU(s) \tag{2.27}$$

In the time domain we can say that a system is a contribution of two impulse response to two different inputs.

$$h(t) = Ce^{At}B + D\delta(t) \tag{2.28}$$

With an input $u(t)$.

$$g(t) = Ce^{At} \tag{2.29}$$

With an input of the initial condition $x_0 \ \delta(t)$, "$\delta(t)$" : $dirac\ function$.

2.4   Conclusion

This chapter introduces the different criteria used to compare between the block pole and general form constructed in diagonal, controllable and observable forms. It determine the feedback gain matrix. The norm of gain matrix, the sensitivity of the eigenvalues, the robust stability measures (M_1, M_2 and M_3) and the time domain performance of control systems for initial state response are computed so that the system meet a set of criteria:

1. Smaller feedback gain norm.

2. Good robustness.

3. Better time response characteristics.

Chapter 3

Dynamic Modeling of Helicopter

3.1 Introduction

The helicopter is an aircraft that uses rotating wings to provide lift, propulsion, and control. The rotor blades rotate about a vertical axis, describing a disk in a horizontal or nearly horizontal plane. Aerodynamic forces are created by the motion of a wing surface. The helicopter with its rotary wings can generate these forces even when the initial velocity is zero, unlike fixed-wing aircraft, which require a translational velocity to fly. The helicopter therefore has the capability of vertical flight, including vertical take-off and landing. The efficient accomplishment of vertical flight is the fundamental characteristic of the helicopter rotor.[14]

The rotor must efficiently supply a thrust force to support the helicopter weight. The installed power and fuel consumption of the aircraft are proportional to the power required. Conservation of momentum requires that the rotor lift be obtained by accelerating air downward, because corresponding to the lift is an equal and opposite reaction of the rotating wings against the air. Thus the air left in the wake of the rotor possesses kinetic energy which must be supplied by a power source in the aircraft if level flight is to be sustained. A property of both fixed and rotating wings that constitutes the absolute minimum of power required for equilibrium flight. [14]

Helicopter flight is merely a demonstration of Newton's third law of motion which states that: "For every action there is an equal and opposite reaction". That is, accelerating a large mass of air downwards must produce an equal and opposing force that lifts the helicopter. [15]

3.2 Helicopter Flight Dynamics

A helicopter reference system consists of three mutually perpendicular lines (axis) intersecting a single point. This point, called the center of gravity (CG), is the point at which all weight is considered to be concentrated and at which all forces are measured. Theoretically, the aircraft will balance if suspended at the CG. When in flight, the aircraft will rotate about the CG. The longitudinal axis (X) passes from the nose to the tail of the helicopter. Movement of the helicopter around the longitudinal axis is called roll, or lateral control. The lateral axis (Y) passes from the pilot's left to right. Movement of the helicopter around the lateral axis is called pitch, or longitudinal control. The vertical

axis (Z) passes vertically through the CG. Movement of the helicopter around the vertical axis is called yaw, or directional control. As the aircraft moves through the air, the axis system moves with it. Therefore the movement of the aircraft can be describes by the movement of its CG. [16]

A Helicopter is composed, mainly, of the following components (figure 3.1): 1-

- Rotor system: The rotor system, or more simply rotor, is the rotating part of a helicopter that generates lift

- Flight controls: The pilot must be given a large amount of control over the helicopter.

He must be able to control its height, its velocity in both horizontal axes, and its yaw rate. [16]
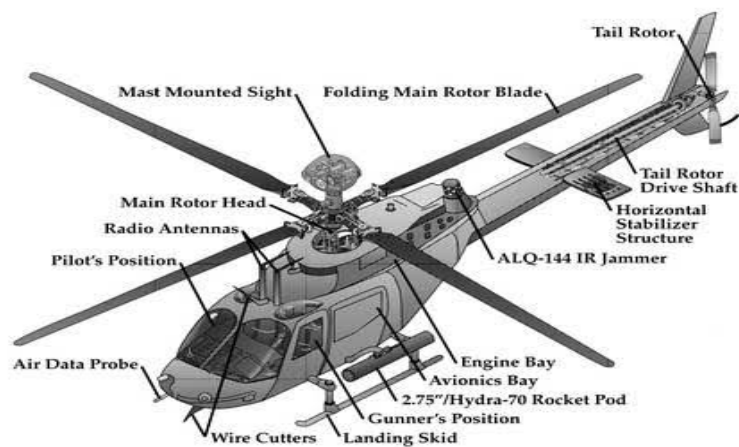
Figure 3.1: Helicopter components

3.2.1   Pitch Attitude

In a helicopter there are two types of pitch control (collective and cyclic) which are describes as follows:

3.2.1.1   Collective pitch control

The movement of the collective pitch control causes the stationary and rotating swashplate mounted centrally on the rotor shaft to translate vertically, i.e. move up or down (Figure 3.2), which causes a change in the pitch angle of all the main rotor blades collectively (i.e., all at the same time) and independently of their position. Therefore, if a collective input is made, all the blades change equally, and the result is the helicopter increases or decreases its total lift derived from the rotor.[17]

3.2.1.2   Cyclic pitch control

This primary flight control allows the pilot to fly the helicopter in any direction of travel: forward, backward, left, and right. The purpose of the cyclic pitch control is to tilt the tip-path plane in the direction of the desired horizontal direction. The cyclic controls

the rotor disk tilt versus the horizon, which directs the rotor disk thrust to enable the pilot to control the direction of travel of the helicopter. The rotor disk tilts in the same direction the cyclic pitch control is moved. If the cyclic is moved forward, the rotor disk tilts forward, and so on [17].

3.2.2   Roll Attitude

It is a rotation around the longitudinal axis.it is an axis drawn through the body of the vehicle from tail to nose, in the forward direction of flight, or in the direction of the pilots face. Even in roll control, we have to distinguish between the roll of fuselage, and the roll of the rotor disc. This latest is used to make the engine move to the left or the right.[18]

3.2.3   Yaw Attitude

It is a rotation about the vertical axis. It is an axis drawn from top to bottom, and perpendicular to the other axis.[18]

3.3   Rotor Operation

A discussion of rotor system control must begin with an understanding of degrees of freedom. A degree of freedom will be defined as the ability to move around an axis. A helicopter has three degrees of freedom with regards to fuselage attitude and it has another three degrees of freedom regarding to rotor blade actions. Each degree of freedom provided to the rotor head has a mechanical of effective (virtual) hinge associated with it.[16]
The three types of hinges, degrees of freedom, and their uses are describes as follows:

3.3.1   Flapping

Flapping is the upward and downward rotation of just a few degrees by a rotor blade about a horizontal hinge during rotation about the mast [16]. In order to manoeuver the helicopter, the rotor disk must be tilted. The rotor blades therefore must be allowed some vertical movement. Vertical blade movement is termed flapping.[5]

3.3.2   Lead and Lag

Hunting, or lead/lag, the third degree of freedom on a rotor blade, is the fore and aft movement of blades about a vertical hinge. Lead/lag is allowed in order to relieve stress forces caused by dissymmetry of drag and conservation of angular momentum [16]. While the rotors are turning and as the blade flaps, its center of mass moves with respect to the center of rotation. When the blade's center of mass is closer to the center of rotation it will tend to lead (move faster). If the blade's center of mass is farther away, it will tend to lag (move slower). [5]
A diagram view of typical hinge arrangement and the three movement of the blades is shown in figure(3.2):
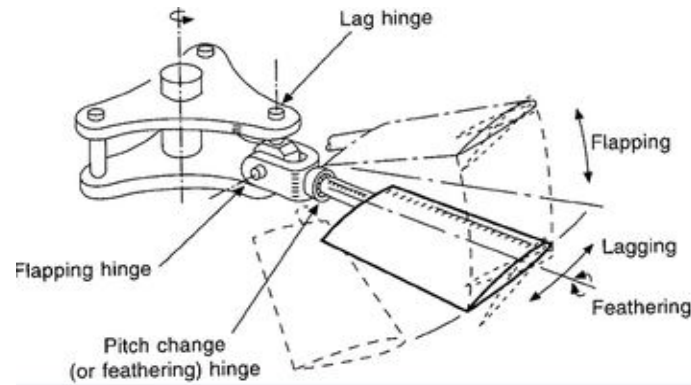
Figure 3.2: Typical hinge arrangement

3.3.3 Rotor Disc

It is the disc formed by the blades as they are rotating. It can be titled in the lateral or in the longitudinal direction, because the blades are allowed to flap. The rotor tilt is shown in figure(3.3).[18]
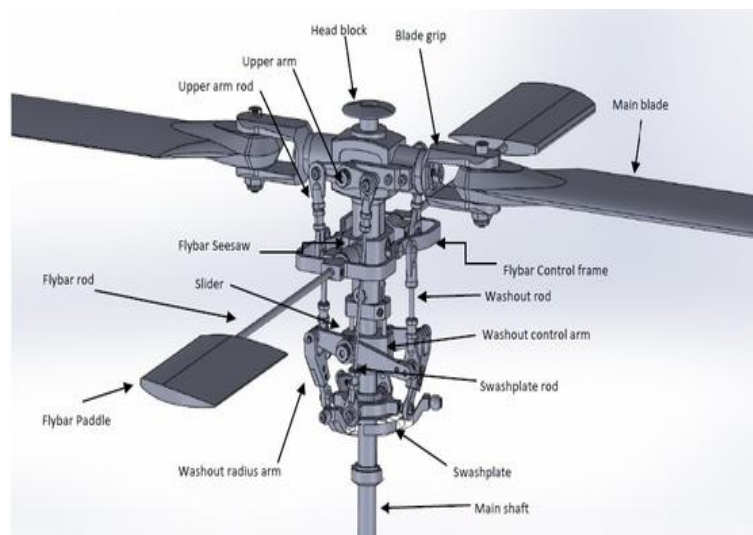
Figure 3.3: Schematic of an articulated rotor hub

3.4 Equation of Motion

A dynamic model of the system under study is indispensable to perform simulations and controller design. Therefore, in order for us to achieve our objectives, we need to model the helicopter and give its state space representation. The study of the dynamic response and the control of a helicopter is extremely complicated since each rotor blade has its own degrees of freedom in addition to those of the fuselage. Fortunately, for small perturbations we can consider only the rotor's motion as a physical entity and assume that its speed Ω is constant. In the following description of the equations, we are going to assume, small perturbations and body fixed axes system (see figure 3.1) such that the inertia terms can be linearized and the lateral and longitudinal motions

may be considered uncoupled and therefore can be treated independently. The helicopter itself is also assigned an axis system, which is the body fixed axis system whose origin is helicopter's center of gravity as shown in figure (3.4).[15]
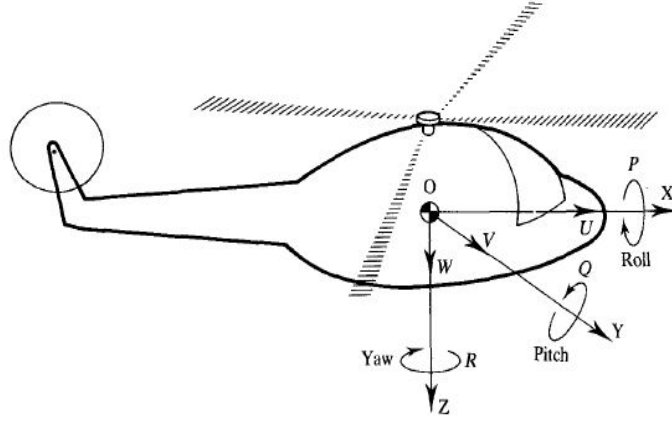
Figure 3.4: Conventional body fixed axis set for a helicopter

The mathematical description of a helicopter can ultimately be expressed in the following generic form:

$$\dot{x} = f(x, u, t) \tag{3.1}$$

Where $f(x, u, t)$ is a vector of functions, x is the vector of states, u is the vector of inputs and t denotes the time.

Angular velocity component:

$$\overrightarrow{w} = i.P + j.Q + k.R \tag{3.2}$$

Linear velocity component:

$$\overrightarrow{V_p} = i.U + j.V + k.W \tag{3.3}$$

For components of the vector $\overrightarrow{r}$, that locates mass particles dm inside XYZ

$$\overrightarrow{r} = i.x + j.y + k.z \tag{3.4}$$

Pitching, rolling and yawing moments L, M, N

$$\overrightarrow{M} = i.L + j.M + k.N \tag{3.5}$$

The time variable t is replaced by

$$\tau = \frac{\rho s \pi R^3 \Omega}{m} t \tag{3.6}$$

Where A is the blade area which given by: $A = s\pi R^2$, s is a solidity factor, R is the rotor blade radius, ρ is a material density, Ω is a rotor speed, m is the mass of the helicopter. In this analysis, it is customary to assume that the perturbations are small. For example:

$$U \triangleq U_0 + u \tag{3.7}$$

The trim or equilibrium values are denoted by a subscript 0 and the small perturbation values of a variable are denotes by the lower case letter. The force acting on the engine are X, Y and Z. They are the sum of both the aero dynamical and thrust forces ΔX, ΔY, ΔZ and the gravitational forces δX, δY, δZ.

3.4.1   Longitudinal Motion

To control longitudinal motion the two inputs δ_B and δ_{θ_0} are used and after some assumptions, non-dimensionalization of the equations and algebraic manipulations, the resulting longitudinal equations of motions are given as follows:

$$\frac{du}{d\tau} = x_u u + x_w w + x_q \frac{d\theta_F}{d\tau} - mg\theta_f \cos(\gamma) + x_{\delta_B}\delta_B + x_{\theta_{B_0}}\delta_{\theta_0} \tag{3.8}$$

$$\frac{dw}{d\tau} = z_u u + z_w w + (V + z_q)\frac{d\theta_F}{d\tau} - mg\theta \sin(\gamma) + z_{\delta_B}\delta_B + z_{\theta_{B_0}}\delta_{\theta_0} \tag{3.9}$$

$$\frac{dq}{d\tau} = m_u u + m_w w + m_{\dot{w}}\dot{w} + m_q q + m_{\delta_B}\delta_B + m_{\theta_{B_0}}\delta_{\theta_0} \tag{3.10}$$

$$\frac{d\theta_F}{d\tau} = q \tag{3.11}$$

$$x_u = \frac{X_u}{\rho A_{ref}\Omega R} \tag{3.12}$$

$$X_u = \frac{\partial X}{\partial u} \tag{3.13}$$

Where X_u is called the stability derivative, A_{ref} is the reference area given by: $A_{ref} \triangleq \pi R^2$, γ is the angle of climb, θ_F is the pitch attitude of the fuselage, δ_B is the cyclic pitch control term, δ_{θ_0} is the collective pitch control term.[20]

3.4.2   Lateral Motion

To control the lateral motion, we use the two inputs δ_A and δ_T. Following the same procedures as in the longitudinal motions we obtain the following lateral equations:

$$\frac{dv}{d\tau} = Y_v v + mg\phi\cos(\gamma) - \frac{Vd\phi_F}{d\tau} + mg\psi_F\sin(\gamma) + Y_{\delta_A}\delta_A + Y_{\delta_T}\delta_T \tag{3.14}$$

$$\frac{dp}{d\tau} = I_v v + I_p p + I_r r + \frac{i_x z}{i_x x}\dot{r} + I_{\delta_A}\delta_A + I_{\delta_T}\delta_T \tag{3.15}$$

$$\frac{dr}{d\tau} = n_v v + n_p p + n_r r + \frac{i_x z}{i_z z}\dot{r} + n_{\delta_A}\delta_A + n_{\delta_T}\delta_T \tag{3.16}$$

$$i_{xx} = \frac{I_{xx}}{mR^2} \tag{3.17}$$

$$i_{zz} = \frac{I_{zz}}{mR^2} \tag{3.18}$$

$$i_{xz} = \frac{I_{xz}}{mR^2}\dot{\phi}_F = p \tag{3.19}$$

where I_{xz}, I_{zz} and I_{xx} are moments of inertia, R is the rotor blade radius and m is the mass of the helicopter. [20]

3.4.3   Hovering Motion

When a helicopter hovers, the forward velocity V is almost zero and usually, x_w, x_q, m_m and $m_{\dot{w}}$ are negligible. Then, the hovering equations are given as follows:

$$\dot{u} = x_u u - mg\theta_F + x_{\delta_B}\delta_B + x_{\delta_{\theta_0}}\delta_{\theta_0} \tag{3.20}$$

$$\dot{w} = z_u u + z_w w + z_{\delta_B}\delta_B + z_{\delta_{\theta_0}}\delta_{\theta_0} \tag{3.21}$$

$$\dot{q} = m_u u + m_q q + m_{\delta_B}\delta_B + m_{\delta_{\theta_0}}\delta_{\theta_0} \tag{3.22}$$

$$\dot{\theta} = q \tag{3.23}$$

For more details about the equations of motion refer to [20] and [21].

3.5   Linearization methods

When the non-linear model is linearized, an operating point for all the state derivatives must be chosen. The best control would be obtained by choosing this operating point as being the equilibrium of the helicopter, defined as where the translator velocities are zero and the Euler angles are constant, this equilibrium would yield a suspended, but slightly tilting helicopter. Unfortunately, not all states equal zero when the helicopter is in this equilibrium. This is due to the fact that the tail rotor exerts a negative force on the helicopter along $^b y$, resulting in a constant negative acceleration along this axis. That is, to maintain the helicopter in the equilibrium, the Euler angle ϕ must be non-zero, such that the main-rotor thrust-component along $^b y$ cancels out the tail-rotor thrust.[22]

However, it is estimated that the tail-rotor thrust is small compared to the main-rotor thrust, why ϕ is close to zero?. Given this assumption, the operating point of Euler angles is chosen to be zero. This approximation simplifies the linearization, but introduces an error. This error is expected to be relatively small enough, why the controller should be able to compensate for it.[22]

From the approximation, also the operating point of flapping angles is chosen to be zero, and from the definition of the equilibrium, it follows that the operating point of the angular velocities are zero. Some of the states depend on the main-rotor thrust and main-rotor drag, why these expressions also need to be linearized. The main-rotor thrust equations introduces a problem when linearizing this equation, why the linearization is performed numerically. This is also the case for the main-rotor drag. Both of these equations have operating points different from zero.[22]

Different linearization methods are used throughout the linearization. The equations with operating points in zero, are linearized with a Taylor approximation, except for the flapping equations, which are linearized through a black-box method. Variables are divided into large-scale and small-scale values, are written with a 0 and δ respectively as subscript in front of the variable as shown in the notation below

$$q(t) = q_0(t) + q_\delta(t) \tag{3.24}$$

Where $^b y$ Indicates the force along y axis in the body fixed frame, $q_0(t)$ is the operating point, and $q_\delta(t)$ is the small-signal value.[22]

3.6   Helicopter Stability and Control

Helicopters are inherently unstable in hover; moreover, the response to control inputs are not immediate, and rotor systems produce their own gusty air. Harry Reasoner said: "A helicopter does not want to fly. It is maintained in the air by a variety of forces and controls working in opposite to each other. And if there is any disturbance in this delicate balance the helicopter stops flying immediately and disastrously."[16]. A proper balance is required between stability and controllability during the design of a helicopter, providing stability to make the flight predictable, but enough controllability to offer pilots the ability to maneuver the helicopter. As with fixed-wing aircraft, both static stability and dynamic stability contribute to the flying qualities of a helicopter. A statically unstable motion is also dynamically unstable but a statically stable motion may be either stable or unstable dynamically.[21]

Consider a helicopter hovering in still air when a gust of wind affects the rotor disc from the side. The disc will flap away from the wind and, if no corrective action is taken by the pilot, the helicopter will move away from the gust. The movement of the helicopter will result in it experiencing continual sideways changes in the airflow affecting the disc and, although it will be statically stable, because the amplitude of the oscillations will be dynamically unstable. The effect of a gust of wind from any direction will produce the same effect on the disc, therefore the helicopter is dynamically unstable in the pitching and rolling planes. Following the movement of the helicopter, it can be alternately yawing to port and starboard with each successive sideways movement of the helicopter. Therefore, when hovering, the helicopter is statically stable but dynamically unstable in the yawing plane.[21]

Lift and control in hovering flight are maintained by rotation of the wings to provide aerodynamic forces on the rotor blades. General vertical flight involves climb or descent with the rotor horizontal. A useful aircraft must be capable of translational flight as well. The helicopter accomplishes forward flight by keeping the rotor nearly horizontal, so that the rotor disk sees a relative velocity in its own plane in addition to the rotational velocity of the blades. The rotor continues to provide lift and control for the aircraft. It also provides the propulsive force to sustain forward flight, by means of a small forward tilt of the rotor thrust.[14]

3.6.1   Static Stability

Static stability is the initial tendency of a system, once disturbed, to return toward an equilibrium, or trimmed position.

a. Positive static stability: it is the initial tendency toward an equilibrium position after a disturbance.

b. Negative static stability: it is the initial tendency to continue moving away from equilibrium following a disturbance

c. Neutral static stability: it is the initial tendency to accept a displacement as the new equilibrium position. [16]

The term static stability is peculiar to the field of aeronautical engineering and should not be confused with stability of differential equations. Indeed, a statically stable vehicle may still produce divergent oscillations and thus be dynamically unstable. [2]

3.6.2 Dynamic Stability

Dynamic stability is the tendency of a system, over time, to return to an equilibrium or trimmed condition. Positive dynamic stability exists if the system eventually returns to equilibrium. A neutrally dynamically stable system does not get any closer or further from equilibrium after time goes by. A system with negative dynamic stability eventually continues in the direction of the disturbance. Dynamic stability concerns itself with the resulting motion in time. If a helicopter is disturbed from equilibrium, the time history of the resulting motion indicates the dynamic stability of the system [16].

3.7 Station Keeping System

In order to make the helicopter a viable operational aircraft, short-comings in stability and control characteristics generally have to be made good by use of automatic flight control systems. The helicopter is basically flyable but in absence of automatic aids, continuous correction by the pilot would be required, which is tiring process. The correction is to utilize some of the available control power to generate moments proportional to a given motion variable and there by correct the motion. Autob stabilization systems have in the past used mechanical devices integral to the rotor. Alternatively, devices may be electromechanical, operating on attitude or rate signals from helicopter motion sensors. Electric or electronic systems are the more flexible. [21]

Station Keeping System is used to enable a helicopter to maintain its position fixed in space or to keep its station, for quite long periods of time. (See Hall and Bryson(1973)for further discussion of this system.) Obviously, the situation requires that the flight is carried out at hover, or near hover, i.e. with forward speeds not greater than 1 m/s^{-l}. For a Sikorsky helicopter S-61, the following model is given, by taking which the blade dynamics are also taken into account. [20]
The state and control vectors are defined as:

$$x = [\theta_F \ \phi_F \ q_F \ p_F \ u \ v]^T \tag{3.25}$$

$$U = \begin{bmatrix} -\phi_R \\ \theta_R \end{bmatrix} \tag{3.26}$$

where states are: θ_F denotes the pitch attitude of fuselage, ϕ_F the roll angle of fuselage, q_F the pitch rate of fuselage, p_F the roll rate of fuselage, u the velocity along the x-axis of the fuselage, v the velocity along the y-axis of the fuselage. Inputs are ϕ_R the roll tilt angle of the rotor, θ_R the pitch tilt angle of the rotor.

3.8 Conclusion

This chapter opens with a brief introduction to the helicopter and its rotor systems. Some distinctive features of helicopter flight which give rise to particular control problems were indicated before proceeding to a development of appropriate equations of small perturbation motion for both longitudinal into the state and output equations which represent the dynamics of an aircraft. A special feature of this analysis is the impact which the inclusion of the dynamics of the main rotor in the equations of motion can make to the response obtained from the controlled helicopter.

Chapter 4

Simulation Results

4.1   Introduction

In order to compare between the two methods described in chapter one, block pole placement and general form, different cases are simulated using the helicopter model at lateral motion presented in the precedent chapter. For each case a number of parameters are calculated to serve as a basis of the comparative study. As stated in chapter one, the state feedback gain in MIMO control systems is not unique.

In order to take advantage of this non-unicity and see how the stability and the performance of a system are affected by the choice of the state feedback gain, this section will be devoted to carry out a comparative study of the different criteria listed below using the software package Matlab:

- Gain magnitudes

- System sensitivity and robustness

- Time response specifications

For this purpose, we will use a helicopter model at lateral motion which has the following state equation:

$$\dot{x(t)} = Ax(t) + Bu(t) \tag{4.1}$$

Where:

$$x(t)^T = [\theta_F \ \phi_F \ q_F \ p_F \ u \ \nu] \tag{4.2}$$

$$u^T = [-\phi_R \ \theta_R] \tag{4.3}$$

The helicopter under study is modeled by the following state equation:[20]

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -0.042 & 0.32 & 0.003 & 0.001 \\ 0 & 0 & -1.23 & -1.6 & 0.004 & -0.012 \\ -32.2 & 0 & 4.7 & -1.0 & -0.02 & -0.005 \\ 0 & 32.2 & -1.0 & -4.7 & 0.005 & -0.02 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -0.3 & 6.3 \\ -23 & -1.1 \\ 1.0 & -32.2 \\ -32.2 & 1.0 \end{pmatrix}$$

The state space equation can be written as:

$$
\begin{pmatrix} \dot{\theta}_F \\ \dot{\phi}_F \\ \dot{q}_F \\ \dot{p}_F \\ \dot{u} \\ \dot{\nu} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -0.042 & 0.32 & 0.003 & 0.001 \\ 0 & 0 & -1.23 & -1.6 & 0.004 & -0.012 \\ -32.2 & 0 & 4.7 & -1.0 & -0.02 & -0.005 \\ 0 & 32.2 & -1.0 & -4.7 & 0.005 & -0.02 \end{pmatrix} \begin{pmatrix} \theta_F \\ \phi_F \\ q_F \\ p_F \\ u \\ \nu \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -0.3 & 6.3 \\ -23 & -1.1 \\ 1.0 & -32.2 \\ -32.2 & 1.0 \end{pmatrix} \begin{pmatrix} -\phi_R \\ \theta_R \end{pmatrix}
$$

Where:

θ_F: the pitch attitude of fuselage.

ϕ_F: the roll angle of fuselage.

q_F: the pitch rate of fuselage.

p_F: the roll rate of fuselage.

u: the velocity along the x-axis of the fuselage.

ν: the velocity along the y-axis of the fuselage.

ϕ_R: the lateral tilt of the rotor disc.

θ_R: the longitudinal tilt.

System Stability

The eigenvalues of the open loop system are:

$$-1.4924, \quad -0.6390, \quad 0.0585 \pm j0.5049, \quad 0.1662 \pm j0.3751.$$

It can be clearly seen from the complex conjugate pair$(0.0585 \pm j0.5049, \quad 0.1662 \pm j0.3751)$, which have positive real part, we conclude that the system is unstable. In order to stabilize our system, and provided that it is controllable, we can use state feedback design to relocate all the eigenvalues to the desired eigenvalues.

The desired eigenvalues chosen for the helicopter stability, given in [20] are:

$$-0.6206 \pm j2.2816, \quad -1.729 \pm j3.8915, \quad -7.2387, \quad -8.1667.$$

For more detail we recommend the reader to see [20].

4.2 The Case of The Block Pole Placement

4.2.1 System Controllability

We have $n = 6$ and $m = 2$, then $l = n/m = 3$ is an integer, so the set of controllability indices is fixed and we have found it to be: $\{k_1 = 2, k_2 = 2, k_3 = 2\}$.

It follows that the desired matrix A_D can be in two forms:

- One block matrix of dimension 6×6.

- Three block poles of dimension 2×2.

Since the system is unstable, we want to design a state feedback controller such that the closed-loop system $(A - BK)$ has the following set of desired eigenvalues:

$$-0.6206 \pm j2.2816, \quad -1.729 \pm j3.8915, \quad -7.2387, \quad -8.1667.$$

Since we have the controllability matrix: $W_c = [B \ AB \ AB^2]$.

The rank of W_c is: $rank(W_c) = n = 6$, i.e: the controllability matrix has full rank, the pair (A, B) is block controllable. therefore the pair (A, B) can be converted into multivariable block controllable companion form (A_c, B_c).

Using the required similarity transformation T_c, the pair (A_c, B_c) is as follows:

$$A_c = T_c A T_c^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.3818 & 0.0216 & 0.0277 & -0.0010 & -1.5771 & 0.2709 \\ -0.1311 & -0.1012 & -0.0014 & 0.0134 & -1.2444 & -0.1049 \end{pmatrix} \tag{4.4}$$

$$B_c = T_c B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.5}$$

4.2.2 Finding The Feedback Gain Matrix

4.2.2.1 Block Poles Constructed in Diagonal Form

The state feedback gain K_c is to be selected so that: $A_c - B_c K_c = A_D$,

Where A_D is a desired closed-loop matrix whose eigenvalues are the set of desired eigenvalues. The desired block poles are given in diagonal form by:

$$R_1 = \begin{pmatrix} -7.2387 & 0 \\ 0 & -8.1667 \end{pmatrix}, R_2 = \begin{pmatrix} -0.6206 & 2.2816 \\ -2.2816 & 0.6206 \end{pmatrix}, R_3 = \begin{pmatrix} -1.7294 & 3.8915 \\ -3.8915 & -1.7294 \end{pmatrix} \tag{4.6}$$

And the coefficients of the desired characteristic polynomial are given by:

$$[D_{f3}\ D_{f2}\ D_{f1}] = -\begin{bmatrix} R_1^2 & R_2^2 & R_3^2 \end{bmatrix}.V_R^{-1} \tag{4.7}$$

Where V_R^{-1} is the inverse of the right block Vandermonde matrix.

Then from ΔA and the coefficients of the desired characteristic polynomial, K_C is given by:

$$K_C = \begin{pmatrix} -58.7010 & -52.7658 & 13.0569 & -53.5700 & 8.5745 & -5.4971 \\ 44.2760 & -62.6798 & 54.4904 & 6.9453 & 5.4360 & 9.8489 \end{pmatrix} \tag{4.8}$$

Computing the state feedback gain matrix, that places the block poles of the closed-loop system to the desired locations, in original coordinates, yields;

$$K = \begin{pmatrix} -9.7487 & -0.4031 & 0.2788 & -0.4815 & 0.2443 & 0.826 \\ -0.7261 & -2.7358 & 3.1686 & 0.1864 & 0.3187 & -0.0553 \end{pmatrix} \tag{4.9}$$

Finally, the closed-loop matrix will be:

$$A-BK = \begin{pmatrix} 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 11.6500 & 17.1145 & -19.9207 & 1.3498 & -1.9317 & 0.3742 \\ -225.0191 & -12.2809 & 8.6673 & -12.8796 & 5.9729 & 1.8277 \\ -45.8327 & -87.6890 & 106.4511 & -6.5201 & 9.9989 & -1.8686 \\ -313.1824 & 21.9556 & 4.8079 & -20.0180 & 7.5519 & 2.6960 \end{pmatrix} \tag{4.10}$$

4.2.2.2 Block Pole Constructed in Controllable Form

The desired block poles are given in controllable form by:

$$R_1 = \begin{pmatrix} 0 & 1.0000 \\ -18.1346 & -3.4588 \end{pmatrix}, R_2 = \begin{pmatrix} -1.2412 & -5.5908 \\ 1.0000 & 0 \end{pmatrix}, R_3 = \begin{pmatrix} -15.4054 & -59.1163 \\ 1.0000 & 0 \end{pmatrix}$$

K_c is given by:

$$K_C = \begin{pmatrix} 98.6452 & 330.5290 & 64.1371 & 60.4757 & 15.0695 & 19.9898 \\ -18.2657 & -0.1012 & -3.4602 & 18.1480 & -2.2444 & 3.3539 \end{pmatrix} \tag{4.11}$$

K is given by:

$$K = \begin{pmatrix} 16.4507 & -2.2320 & -5.0568 & -0.4431 & -1.5998 & -0.1540 \\ 2.8950 & 0.1783 & 0.5210 & 0.0563 & -0.0034 & 0.0246 \end{pmatrix} \tag{4.12}$$

The closed-loop matrix will be:

$$A - BK = \begin{pmatrix} 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ -13.3034 & -1.7926 & -4.8414 & -0.1674 & -0.4556 & -0.1999 \\ 381.5500 & -51.1408 & -116.9631 & -11.7302 & -36.7956 & -3.5280 \\ 44.5688 & 7.9718 & 26.5332 & 1.2546 & 1.4709 & 0.9399 \\ 526.8167 & -39.8499 & -164.3496 & -19.0251 & -51.5058 & -5.0048 \end{pmatrix}$$

$$\tag{4.13}$$

4.2.2.3 Block Pole Constructed in Observable Form

The desired block poles are given in observable form by:

$$R_1 = \begin{pmatrix} 0 & -18.1346 \\ 1.0000 & -3.4588 \end{pmatrix}, R_2 = \begin{pmatrix} -1.2412 & 1.0000 \\ -5.5908 & 0 \end{pmatrix}, R_3 = \begin{pmatrix} 0 & -59.1163 \\ 1.0000 & -15.4054 \end{pmatrix}$$

K_c is given by:

$$K_C = \begin{pmatrix} -124.8303 & 844.6265 & -96.3968 & 185.8503 & -20.9381 & 137.6682 \\ -42.0000 & 235.8922 & -30.9811 & 57.5134 & -8.2501 & 39.3615 \end{pmatrix} \qquad (4.14)$$

K is given by:

$$K = \begin{pmatrix} 48.9312 & 6.8855 & 0.5860 & 0.5635 & -4.1765 & 0.1126 \\ 14.5840 & 2.1130 & 0.3146 & 0.2464 & -1.1680 & 0.0410 \end{pmatrix} \qquad (4.15)$$

The closed-loop matrix will be:

$$A - BK = \begin{pmatrix} 0 & 0 & 0.0010 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0010 & 0 & 0 \\ -0.0772 & -0.0112 & -0.0018 & -0.0011 & 0.0061 & -0.0002 \\ 1.1415 & 0.1607 & 0.0126 & 0.0116 & -0.0973 & 0.0026 \\ 0.3885 & 0.0612 & 0.0142 & 0.0064 & -0.0335 & 0.0012 \\ 1.5610 & 0.2518 & 0.0176 & 0.0132 & -0.1333 & 0.0036 \end{pmatrix} \qquad (4.16)$$

4.2.3 The Gain Magnitude

The different norms of gain for three forms are given in table 4.1:

Norm/ Form	Diagonal form	Controllable form	Observable form
The 1-norm	10.4748	19.3457	63.5152
The 2-norm	9.8316	17.6221	51.7536
The Infinity norm	11.2390	25.9365	61.2552
The Frobenius norm	10.6662	17.6818	51.7539

Table 4.1: Gains magnitude for three forms.

Discussion:

From the table 4.1, corresponding to the gains magnitude for the three forms we remark:

- The gain matrix norms results corresponding to four types of norm in the diagonal form are less than what is found at the controllable form and the observable form.

- The diagonal form gives a best gain matrix norms results which means that the requirement of less control effort to stabilize the system.

4.2.4 Robust Stability

In order to study robust stability of our system, Using Matlab and the different formulas given in chapter 2 we compute the three measures M_1, M_2 and M_3. Before doing this, let's first compute the different norms and sensitivities of the closed-loop matrix on the three forms (diagonal, controllable and observable).

The norms of the eigenvectors matrix:

The results obtained for norm of the eigenvector matrix for three forms are given in table 4.2:

Norm / Form	Diagonal form	Controllable form	Observable form
$\| V \|_2$	2.0108	1.9407	2.2058
$\| T \|_2$	137.8586	1.7272×10^3	5.1570×10^5

Table 4.2: The norms of the eigenvectors matrix.

The norm of each column of the right eigenvector matrix $\| \nu_i \|_2$:

The results obtained for norm each column of the right eigenvector matrix for three forms are given in the table 4.3:

Diagonal form	Controllable form	Observable form
$\| \nu_1 \|_2= 1$; $\| \nu_4 \|_2= 1$	$\| \nu_1 \|_2= 1$; $\| \nu_4 \|_2= 1$	$\| \nu_1 \|_2= 1$; $\| \nu_4 \|_2= 1$
$\| \nu_2 \|_2= 1$; $\| \nu_5 \|_2= 1$	$\| \nu_2 \|_2= 1$; $\| \nu_5 \|_2= 1$	$\| \nu_2 \|_2= 1$; $\| \nu_5 \|_2= 1$
$\| \nu_3 \|_2= 1$; $\| \nu_6 \|_2= 1$	$\| \nu_3 \|_2= 1$; $\| \nu_6 \|_2= 1$	$\| \nu_3 \|_2= 1$; $\| \nu_6 \|_2= 1$

Table 4.3: The norm of each column of the right eigenvector matrix for three forms.

The norm of each row of the left eigenvector matrix $\| t_i \|_2$:

The results obtained for norm each row of the left eigenvector matrix for three forms are given in table 4.4:

Diagonal form	Controllable form	Observable form
$\| t_1 \|_2= 66.8021$	$\| t_1 \|_2= 1.1796 \times 10^3$	$\| t_1 \|_2= 3.5763 \times 10^3$
$\| t_2 \|_2= 66.8021$	$\| t_2 \|_2= 1.2597 \times 10^3$	$\| t_2 \|_2= 3.7113 \times 10^3$
$\| t_3 \|_2= 59.9940$	$\| t_3 \|_2= 0.0120 \times 10^3$	$\| t_3 \|_2= 0.1229 \times 10^3$
$\| t_4 \|_2= 59.9940$	$\| t_4 \|_2= 0.0120 \times 10^3$	$\| t_4 \|_2= 0.1229 \times 10^3$
$\| t_5 \|_2= 41.3611$	$\| t_5 \|_2= 0.0497 \times 10^3$	$\| t_5 \|_2= 0.0081 \times 10^3$
$\| t_6 \|_2= 660.3477$	$\| t_6 \|_2= 0.0497 \times 10^3$	$\| t_6 \|_2= 0.0081 \times 10^3$

Table 4.4: The norm of each row of the left eigenvector matrix for three forms.

Discussion:

From the tables 4.2, 4.3 and 4.4, we can notice that:

- A small norm of right eigenvector matrix $\| V \|_2$ is found in the case of controllable form. However, it is almost equal in the diagonal and observable forms, by against the norm of the left eigenvector matrix $\| T \|_2$ is in observable form bigger than what it is found in the controllable form and diagonal form.

- The norm of each column of the right eigenvector matrices $\| \nu_i \|_2$ for three forms is 1, but The norm of each row of the left eigenvector matrices $\| t_i \|_2$ for $(i = 1, 2 \cdots 6)$, is different for three forms. The diagonal form gives the best result.

The sensitivity of the eigenvalues:

$$s(\Lambda) = \| V \|_2 \, . \, \| T \|_2 \tag{4.17}$$

The sensitivity of all the eigenvalues for three forms is given in table 4.5:

	Diagonal form	Controllable form	Observable form
$s(\Lambda)$	277.2022	3.3520×10^3	1.1375×10^4
The inverse of $s(\Lambda)$	0.0036	2.9833×10^{-4}	8.7909×10^5

Table 4.5: The sensitivity of the eigenvalues for three forms.

Discussion:

From the tables 4.5, we see that:

- The sensitivity of all eigenvalues for the three forms with different input sequences, we can clearly see that the diagonal form gives the smallest sensitivity $s(\Lambda)$.

- By against, the inverse of $s(\Lambda)$ in the observable form is more larger than in two other forms.

- The sensitivity is affected by both input sequence ordering and eigenvalues arrangement.

The sensitivity of each eigenvalues:

$$s(\lambda_i) = \| \nu_i \|_2 \, . \, \| t_i \|_2; \quad (i = 1, 2, \cdots, 6) \tag{4.18}$$

We have computed the sensitivity of each eigenvalue for three forms which is given in the following tables:

The sensitivity of each eigenvalue $s(\lambda_i)$	Diagonal form
$s(\lambda_1 = -1.7294 + 3.8915i)$	66.8021
$s(\lambda_2 = -1.7294 - 3.8915i)$	66.8021
$s(\lambda_3 = -0.6206 + 2.2816i)$	59.9940
$s(\lambda_4 = -0.6206 - 2.2816i)$	59.9940
$s(\lambda_5 = -8.1667)$	41.3611
$s(\lambda_6 = -7.2387)$	60.3477

Table 4.6: The sensitivity of each eigenvalue for block diagonal form.

The sensitivity of each eigenvalue $s(\lambda_i)$	Controllable form
$s(\lambda_1 = -1.7294 + 3.8915i)$	1.1796×10^3
$s(\lambda_2 = -1.7294 - 3.8915i)$	1.2597×10^3
$s(\lambda_3 = -0.6206 + 2.2816i)$	0.0120×10^3
$s(\lambda_4 = -0.6206 - 2.2816i)$	0.0120×10^3
$s(\lambda_5 = -8.1667)$	0.0497×10^3
$s(\lambda_6 = -7.2387)$	0.0497×10^3

Table 4.7: The sensitivity of each eigenvalue for block controllable form.

The sensitivity of each eigenvalue $s(\lambda_i)$	Observable form
$s(\lambda_1 = -1.7294 + 3.8915i)$	3.5763×10^3
$s(\lambda_2 = -1.7294 - 3.8915i)$	3.7113×10^3
$s(\lambda_3 = -0.6206 + 2.2816i)$	0.1229×10^3
$s(\lambda_4 = -0.6206 - 2.2816i)$	0.1229×10^3
$s(\lambda_5 = -8.1667)$	0.0081×10^3
$s(\lambda_6 = -7.2387)$	0.0081×10^3

Table 4.8: The sensitivity of each eigenvalue for block observable form.

Discussion:

From the tables 4.6, 4.7 and 4.8, we can see that:

- The sensitivity of each eigenvalues for the three forms with the same eigenvalues, we notice that the diagonal form gives the smallest sensitivity $s(\lambda)$.

Robust stability measures:

Now, we are able to compute the three measures for diagonal, controllable and observable forms:

The measure M_1 is given by:

$$M_1 = \min_{1 \leq \omega \leq \infty} \sigma(A_{new} - j\omega I) \tag{4.19}$$

Where: $A_{new} = A - BK$, A_{new} is an $(n \times n)$ real matrix.

The measure M_2 is given by:

$$M_2 = s(\Lambda)^{-1} \mid Re(\lambda_n) \mid ; \quad (\mid Re(\lambda_n) \mid \leq \cdots \leq \mid Re(\lambda_1) \mid) \tag{4.20}$$

The results obtained for measures M_1 and M_2 for three forms are given in table 4.9:

Measure/Form	Diagonal form	Controllable form	Observable form
Measure M_1	0.0140	0.0161	0.0285
Measure M_2	0.0022	1.8515×10^{-4}	5.4555×10^5

Table 4.9: Measures M_1 and M_2 for the three forms

To deduce the stability measure M_3, we need to compute M_3 for each eigenvalue then, set the minimum as the stability measure M_3 of the corresponding matrix $(A - BK)$.

The measure M_3 is given by:

$$M_3 = \min_{1 \leq i \leq n} s(\lambda_i)^{-1} \mid Re(\lambda_i) \mid \tag{4.21}$$

The results obtained for measure M_3 of each eigenvalue for three forms are given in the following tables:

The eigenvalues λ_i	Diagonal form
$-1.7294 + 3.8915i$	0.0259
$-1.7294 - 3.8915i$	0.0259
$-0.6206 + 2.2816i$	0.0103
$-0.6206 - 2.2816i$	0.0103
-8.1667	0.1974
-7.2387	0.1199

Table 4.10: Robust stability measure M_3 of each eigenvalue for block diagonal form.

The eigenvalues λ_i	Controllable form
$-1.7294 + 3.8915i$	0.0015
$-1.7294 - 3.8915i$	0.0014
$-0.6206 + 2.2816i$	0.0518
$-0.6206 - 2.2816i$	0.0518
-8.1667	0.1645
-7.2387	0.1458

Table 4.11: Robust stability measure M_3 of each eigenvalue for block controllable form.

The eigenvalues λ_i	Observable form
$-1.7294 + 3.8915i$	0.0005
$-1.7294 - 3.8915i$	0.0005
$-0.6206 + 2.2816i$	0.0051
$-0.6206 - 2.2816i$	0.0051
-8.1667	1.0027
-7.2387	0.8888

Table 4.12: Robust stability measure M_3 of each eigenvalue for block observable form.

From the table 4.12, we conclude that:

Measure/Form	Diagonal form	Controllable form	Observable form
The Measure M_3	0.0103	0.0014	4.6598×10^{-4}

Table 4.13: The measure M_3 for three forms.

Discussion:

From the tables 4.9 and 4.13, we can see that:

- The measures M_1 and M_3 are small in the diagonal form and observable form respectively, by against the measure M_2 is greater in the observable form.

- The robust stability measures are very small due to the eigenvalues near the origin. However, better measures M_2 and M_3 are obtained in the observable form and for measure M_1 is obtained in the diagonal form.

4.2.5 Robust Performance

In order to compute the relative change of the eigenvalues of the closed-loop matrix $(A - BK)$, we use the command "rand" in Matlab to generate a random perturbation matrix ΔA which is given for the three forms as follows:

$$\Delta A = \begin{pmatrix} 0.0040 & 0.0042 & 0.0034 & 0.0024 & 0.0058 & 0.0004 \\ 0.0008 & 0.0005 & 0.0090 & 0.0040 & 0.0006 & 0.0017 \\ 0.0024 & 0.0090 & 0.0037 & 0.0010 & 0.0023 & 0.0065 \\ 0.0012 & 0.0094 & 0.0011 & 0.0013 & 0.0035 & 0.0073 \\ 0.0018 & 0.0049 & 0.0078 & 0.0094 & 0.0082 & 0.0065 \\ 0.0024 & 0.0049 & 0.0039 & 0.0096 & 0.0002 & 0.0045 \end{pmatrix} \tag{4.22}$$

The above matrix may be assumed to represent the variation of the element of the matrix $(A - BK)$, due to disturbances caused by air turbulence. The new closed-loop matrix after perturbation is:

Diagonal form:

$$A + \Delta A - BK = \begin{pmatrix} 0.0055 & 0.0037 & 1.0049 & 0.0082 & 0.0035 & 0.0021 \\ 0.0030 & 0.0063 & 0.0044 & 1.0079 & 0.0094 & 0.0030 \\ 1.6574 & 17.1223 & -19.9163 & 1.3562 & -1.9230 & 0.3789 \\ -225.0172 & -12.2801 & 8.6704 & -12.8758 & 5.9784 & 1.8300 \\ -45.8258 & -87.6798 & 106.4562 & -6.5120 & 10.0051 & -1.8601 \\ -313.1806 & 21.9633 & 4.8131 & -20.0127 & 7.5578 & 2.6980 \end{pmatrix} \tag{4.23}$$

Controllable form:

$$A + \Delta A - BK = \begin{pmatrix} 0.0040 & 0.0042 & 1.0034 & 0.0024 & 0.0058 & 0.0004 \\ 0.0008 & 0.0005 & 0.0090 & 1.0040 & 0.0006 & 0.0017 \\ -13.3010 & -1.7836 & -4.8377 & -0.1664 & -0.4533 & -0.1934 \\ 381.5512 & -51.1314 & -116.9620 & -11.7289 & -36.7921 & -3.5207 \\ 44.5706 & 7.9767 & 26.5410 & 1.2640 & 1.4791 & 0.9464 \\ 526.8191 & -39.8450 & -164.3457 & -19.0155 & -51.5056 & -5.0003 \end{pmatrix} \tag{4.24}$$

Observable form:

$$A + \Delta A - BK = 10^3 \times \begin{pmatrix} 0.0000 & 0.0000 & 0.0010 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0010 & 0.0000 & 0.0000 \\ -0.0772 & -0.0112 & -0.0018 & -0.0011 & 0.0061 & -0.0002 \\ 1.1415 & 0.1607 & 0.0126 & 0.0116 & -0.0973 & 0.0026 \\ 0.3885 & 0.0612 & 0.0143 & 0.0064 & -0.0334 & 0.0012 \\ 1.5610 & 0.2518 & 0.0176 & 0.0132 & -0.1333 & 0.0036 \end{pmatrix}$$

(4.25)

The relative change:

The relative change of the eigenvalues of the closed-loop matrix due to the perturbation may be computed using the formula :

$$r_i = \frac{|\lambda_i - \lambda_i'|}{|\lambda_i|} = \frac{|\Delta \lambda_i|}{|\lambda_i|} \; ; \qquad (i = 1, 2, \cdots, n).$$

(4.26)

Where:

λ_i are the eigenvalues of the closed-loop systems $(A - BK)$.

λ_i' are the eigenvalues of the closed-loop systems $(A + \Delta A - BK)$.

The relative change for each eigenvalue:

The following tables show the eigenvalues of the disturbed closed-loop systems matrix $(A + \Delta A - BK)$, for the three forms are given by:

Diagonal form:

λ_i **of** $(A - BK)$	λ_i' **of** $(A + \Delta A - BK)$	**Relative change** r_i
$-1.7294 + 3.8915i$	$-1.9005 + 3.9204i$	0.0408
$-1.7294 - 3.8915i$	$-1.9005 - 3.9204i$	0.0408
$-0.6206 + 2.2816i$	$-0.6339 + 2.2769i$	0.0060
$-0.6206 - 2.2816i$	$-0.6339 - 2.2769i$	0.0060
-8.1667	-8.0973	0.0085
-7.2387	-6.9111	0.0453

Table 4.14: The relative change of each eigenvalue for block diagonal form.

Controllable form:

λ_i **of** $(A - BK)$	λ_i' **of** $(A + \Delta A - BK)$	**Relative change** r_i
$-1.7294 + 3.8915i$	$-7.8130 + 1.5196i$	0.1911
$-1.7294 - 3.8915i$	$-7.8130 - 1.5196i$	0.2244
$-0.6206 + 2.2816i$	$-1.7048 + 3.8413i$	0.0131
$-0.6206 - 2.2816i$	$-1.7048 - 3.8413i$	0.0131
-8.1667	$-0.5238 + 2.3533i$	0.0510
-7.2387	$-0.5238 - 2.3533i$	0.0510

Table 4.15: The relative change of each eigenvalue for block controllable form.

Observable form:

λ_i of $(A - BK)$	λ_i' of $(A + \Delta A - BK)$	Relative change r_i
$-1.7294 + 3.8915i$	$-9.7699 + 0.0000i$	0.1978
$-1.7294 - 3.8915i$	$-5.9573 + 0.0000i$	0.1781
$-0.6206 + 2.2816i$	$-1.5717 + 3.9941i$	0.0443
$-0.6206 - 2.2816i$	$-1.5717 - 3.9941i$	0.0443
-8.1667	$-0.6063 + 2.2583i$	0.0116
-7.2387	$-0.6063 - 2.2583i$	0.0116

Table 4.16: The relative change of each eigenvalue for block observable form.

Discussion:

From the tables 4.14, 4.15 and 4.16, we can see that:

- The relative change increases as the eigenvalues approaches the origin.

- The smallest relative change is obtained by block diagonal form. Hence less sensitive system perturbation is obtained using this arrangement.

4.2.6   Time Domain Performances

Taking $x_0 = [1; 1; 1; 1; 1; 1]$ as the initial conditions, the performances of the states of the helicopter model for the different input sequences for the three forms are given by:
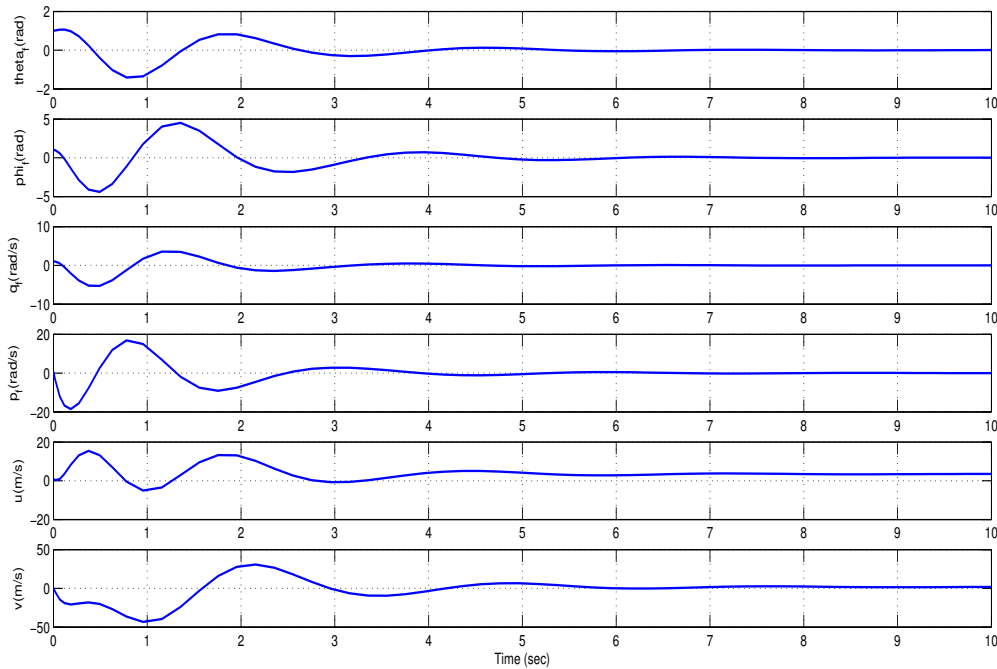
Diagonal form:

Figure 4.1: The time response for block diagonal form.

The previous figure 4.1 demonstrates that the eigenvalues of the system are all negative real parts. which implies that the system is stabilized. Observing that the impulse response will converge with a time response is equal 5 seconds.

From the graphical results shown in the figure 4.1, the initial conditions response of the closed-loop system for diagonal form is summarized in the following table:

The states	Maximum overshoot	Percent overshoot	Falling time	Settling time
θ_F	7.1416×10^4	2.3356×10^6	0.1354	6.3282
ϕ_F	1.9582×10^4	1.1055×10^5	0.0058	7.1916
q_F	4.8511×10^4	7.7373×10^5	0.0292	6.8283
p_F	37.9309	109.4330	1.2378	6.3431
u	2.6162×10^6	3.9457×10^6	0.0320	6.7484
ν	28.7884	5.5755	0.9644	6.2602

Table 4.17: The response to initial conditions of the closed-loop matrix for diagonal form.

Controllable form:
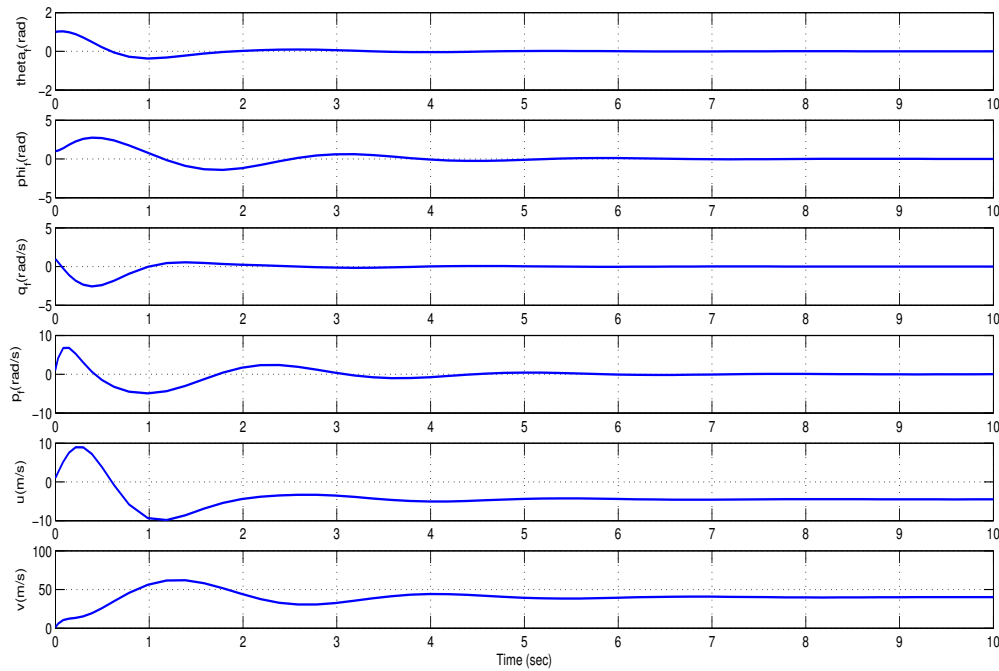
Figure 4.2: The time response for block controllable form.

The same thing as before, the graphical results shown in the figure 4.2 demonstrates the high quality of convergence and stability. Also we observe that the convergence rate(which is the time response) is equal to 4 seconds.

From the graphical results shown in the figure 4.2, the initial conditions response of the closed-loop system for diagonal form is summarized in the following table:

The states	Maximum overshoot	Percent overshoot	Falling time	Settling time
θ_F	1.7893×10^5	1.7359×10^7	0.3451	4.4414
ϕ_F	3.2248×10^4	1.1713×10^6	0.1696	6.3892
q_F	4.1258×10^4	1.6129×10^6	0.0528	4.8428
p_F	5.7359×10^5	8.4418×10^6	0.0372	6.7585
u	118.8833	1.2066×10^3	0.1351	4.5479
ν	54.3274	87.6632	0.6224	5.9831

Table 4.18: The response to initial conditions of the closed-loop matrix for controllable form.
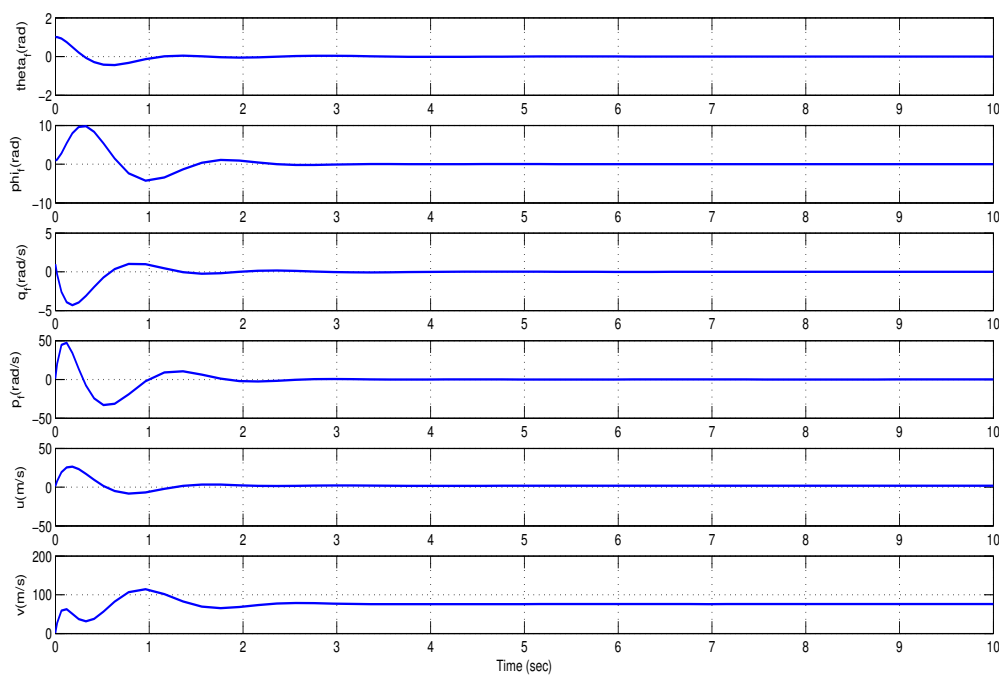
Observable form:

Figure 4.3: The time response for block observable form.

In this case, the notable thing which is remarque it is very high speed of convergence (time response is equal to 2 seconds) with maximum overshoot.

From the graphical results shown in the figure 4.3 , the initial conditions response of the closed-loop system for diagonal form is summarized in the following table:

The states	Maximum overshoot	Percent overshoot	Falling time	Settling time
θ_F	3.7902×10^5	3.7685×10^7	0.2044	3.2914
ϕ_F	1.0561×10^5	1.0771×10^6	0.0308	2.8005
q_F	1.4058×10^5	3.2653×10^6	0.0111	2.6474
p_F	6.8929×10^6	1.4585×10^7	0.0029	2.4797
u	1.3556×10^3	5.1263×10^3	0.0016	1.9894
ν	50.4800	44.1864	0.5630	2.9165

Table 4.19: The response to initial conditions of the closed-loop matrix for observable form.

4.2.7 Algorithm

A. Block Pole Placement:

step(1): Given matrix $A \in \Re(6 \times 6)$, $B \in \Re(6 \times 2)$ of the helicopter.

step(2):

- Compute controllability index l, where: $l = n \backslash m$.

- Check if l is an integer.

step(3): Construct the block controllability matrix W_c:

$$W_c = [B \ , AB \ , A^2 B, \cdots , A^{(l-1)} B] \, , \qquad \text{Where: } l = n \backslash m.$$

- Check the non singularity of W_c: $rank(W_c) = n$.

- Compute the inverse of the matrix W_c: $W_c^{-1} = inv(W_c)$.

step(4): Construct the similarity transformation of matrix T_c:

- Form the matrix T_{c1}: $T_{c1} = [0_m \ \ 0_m \cdots I_m] W_c^{-1}$.

- Construct the matrix T_c:

$$T_c = [T_{c1}^T, \ (T_{c1} A)^T, \ (T_{c1} A^2)^T, \cdots , (T_{c1} A^{(l-1)})^T)]^T.$$

- Compute the inverse of the matrix T_c: $T_c^{-1} = inv(T_c)$

step(5): Compute (A_c, B_c):

- $A_c = T_c \, A \, T_c^{-1}$.

- $B_c = T_c \, B$.

step(6):

- Given the desired block pole in diagonal form, where the desired eigenvalues chosen for the helicopter stability are:

$$-1.729 \pm j3.8915, -0.6206 \pm j2.2816, -8.1667, -7.2387.$$

- Given the desired block pole in controllable form, where the desired eigenvalues chosen for the helicopter stability are:

$$-1.729 \pm j3.8915, -0.6206 \pm j2.2816, -8.1667, -7.2387.$$

- Given the desired block pole in observable form, where the desired eigenvalues chosen for the helicopter stability are:

$$-1.729 \pm j3.8915, -0.6206 \pm j2.2816, -8.1667, -7.2387.$$

step(7): Construct the right block Vandermonde matrix V_R, where:

$$V_R = [I \ I \cdots I; R_1 \ R_2 \cdots R_l; R_1^2 \ R_2^2 \cdots R_l^2; \cdots ; R_1^{(l-1)} \ R_2^{(l-1)} \cdots R_l^{(l-1)}].$$

- Compute the inverse of the matrix V_R: $V_R^{-1} = inv(V_R)$.

step(8): Compute the desired charactiristic polynomial D_f, where:

$$D_f = -[R_1^l \ R_2^l \cdots R_l^l]V_R^{-1}.$$

step(9): Compute the state feedback gain matrix $K = K_c T_c$.

step(10): Compute the closed loop matrix $(A - BK)$.

step(11): Compute the different norms of K.

B. Robust stability:

step(1): Compute the eigenvectors and the eigenvalues of $(A - BK)$.

step(2):

- Compute the norms of right eigenvectors matrix $\| V \|$.

- Compute the norms of left eigenvectors matrix $\| T \|$, where: $\| T \| = inv(\| V \|)$.

step(3):

- Compute the norms of each column of the right eigenvectors matrix $\| \nu_i \|$.

- Compute the norms of each row of the left eigenvectors matrix $\| t_i \|$.

step(4): Compute the sensitivity of the eigenvalues $s(\Delta)$, where: $s(\Delta) = \| V \| . \| T \|$.

- Compute the inverse of the matrix $s(\Delta)$: $s(\Delta)^{-1} = inv(s(\Delta))$.

step(5): Compute the sensitivity of each eigenvalue $s(\lambda_i)$, where: $s(\lambda_i) = \parallel \nu_i \parallel . \parallel t_i \parallel$.

step(6): Compute the three measures M_1, M_2 and M_3.

C. Robust performance:

step(1): Given the rand closed-loop matrix ΔA.

step(2): Compute the new closed-loop matrix after perturbation $(A + \Delta A - BK)$.

step(3): Compute the relative change r_i, where:

- λ_i are the eigenvalues of the closed-loop system $(A - BK)$.

- λ'_i are the eigenvalues of the closed-loop system $(A + \Delta A - BK)$.

D. The time response for block pole:

step(1): Compute the time response of the closed-loop matrix,taking $x_0 = [1; 1; 1; 1; 1; 1]$ as the initial conditions.

step(2): Given Maximum overshoot M_p, Percent overshoot (POS)%, Falling time T_r en second (s), Settling time T_s en second (s).

Remark:

- We use the same algorithm, but we change the desired block pole form for the three forms (diagonal, controllable and observable).

- And,we can use the left block Vandermonde matrix V_L in the case right block Vandermonde placement rather we use V_R. For more details, we refer the reader to see [1].

See appendix A.

4.3 The Case of The General Form

The pair(A, B) is controllable, since rank $(W_c) = 6$ and can be converted into multivariable controllable companion form (A_c, B_c).

The controllability indices are: $k_1 = 3$, $k_2 = 3$.

Using the required similarity transformation T_c, the pair$(A_c, B_c)$is as follows:

$$A_c = T_c A T_c^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -0.3818 & 0.0277 & -1.5771 & 0.0216 & -0.0010 & 0.2709 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ -0.1311 & -0.0014 & -1.2444 & -0.1012 & 0.0134 & -0.1049 \end{pmatrix}$$

(4.27)

$$B_c = T_c B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(4.28)

4.3.1 Finding The Feedback Gain Matrix

The state feedback gain K_c is to be selected so that: $A_c - B_c K_c = A_c - E_{bc} k_c = A_d$,

Where: A_d is a desired closed-loop matrix with the given set of desired eigenvalues.

$K_c = D_G^{-1} k_c$, and the original feedback gain matrix is given by $K = K_c T_c$.

$$
E_{bc} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.29}
$$

And the matrix D_G is computed such that $B_c = E_{bc} D_G$.

The matrix D_G is found to be:

$$
D_G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.30}
$$

Since the controllability indices are $k_1 = 3$, $k_2 = 3$, then the desired characteristic polynomial and the closed-loop matrix A_d are as follows:

$$
\Delta(s) = (s^3 + 9.4079 s^2 + 15.7274 s + 45.6587)(s^3 + 10.6975 s^2 + 43.1718 s + 131.2709) \tag{4.31}
$$

And:

$$
A_d = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-45.6587 & -15.7274 & -9.4079 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & -131.2709 & -43.1718 & -10.6975
\end{pmatrix} \tag{4.32}
$$

Computing the state feedback gain matrix, that places the general controllable form of the closed-loop system to the desired locations, in original coordinates, yields;

$$
K = \begin{pmatrix}
-0.1230 & -0.8478 & 0.0621 & -0.2556 & 0.0106 & -0.0609 \\
9.7589 & 0.3206 & -1.5972 & 0.0587 & -0.6437 & -0.0084
\end{pmatrix} \tag{4.33}
$$

Finally, the closed-loop matrix will be:

$$
A - BK = \begin{pmatrix}
0 & 0 & 1.0000 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.0000 & 0 & 0 \\
-61.5182 & -2.2744 & 10.0387 & -0.1264 & 4.0616 & 0.0356 \\
7.9069 & -19.1462 & -1.5585 & -7.4132 & -0.4608 & -1.4220 \\
282.1607 & 11.1723 & -46.7904 & 1.1455 & -20.7582 & -0.2143 \\
-13.7181 & 4.5809 & 2.5968 & -12.9876 & 0.9893 & -1.9726
\end{pmatrix} \tag{4.34}
$$

4.3.2 The Gain Magnitude

The different norms of gain for general controllable form are given in the table 4.20:

Type of norm	The value of K
The 1-norm	9.8819
The 2-norm	9.9164
The Infinity norm	12.3875
The Frobenius norm	9.9557

Table 4.20: Gains magnitude for general controllable form.

4.3.3 Robust Stability

Let's compute the three measures M_1, M_2 and M_3. But we have to compute first the different norms and sensitivities of closed-loop matrix:

The norms of the eigenvectors matrix:

The results obtained for norm of the eigenvector matrix for general controllable form are given in the table 4.21 :

Norm/Form	Controllable form
$\parallel V \parallel_2$	1.7375
$\parallel T \parallel_2$	57.6718

Table 4.21: The norms of the eigenvectors matrix for general controllable form.

The norm of each column of the right eigenvector matrix $\parallel \nu_i \parallel_2$:

The results obtained for norm each column of the right eigenvector matrix for general controllable form are given in the table 4.22:

Norm/Form	Controllable form
$\parallel \nu_i \parallel_2$	$\parallel \nu_1 \parallel_2 = 1$; $\parallel \nu_4 \parallel_2 = 1$
	$\parallel \nu_2 \parallel_2 = 1$; $\parallel \nu_5 \parallel_2 = 1$
	$\parallel \nu_3 \parallel_2 = 1$; $\parallel \nu_6 \parallel_2 = 1$

Table 4.22: The norm of each column of the right eigenvector matrix for general controllable form.

The norm of each row of the left eigenvector matrix $\parallel t_i \parallel_2$:

The results obtained for norm each row of the left eigenvector matrix for general controllable form are given in the table 4.23:

Norm/Form	General controllable form
$\| t_i \|_2$	$\| t_1 \|_2 = 24.4675$
	$\| t_2 \|_2 = 24.4675$
	$\| t_3 \|_2 = 8.3591$
	$\| t_4 \|_2 = 8.3591$
	$\| t_5 \|_2 = 4.9107$
	$\| t_6 \|_2 = 46.6570$

Table 4.23: The norm of each row of the left eigenvector matrix for general controllable form.

The sensitivity of all the eigenvalues for general controllable form is given in table 4.24:

	General controllable form
$s(\Lambda)$	100.2053
The inverse of $s(\Lambda)$	0.0100

Table 4.24: The sensitivity of the eigenvalues for general controllable form.

Then,we have computed the sensitivity of each eigenvalue for general controllable form is given in the following table:

The sensitivity of each eigenvalue $s(\lambda_i)$	General controllable form
$s(\lambda_1 = -1.7294 + 3.8915i)$	24.4675
$s(\lambda_2 = -1.7294 - 3.8915i)$	24.4675
$s(\lambda_3 = -0.6206 + 2.2816i)$	8.3591
$s(\lambda_4 = -0.6206 - 2.2816i)$	8.3591
$s(\lambda_5 = -8.1667)$	4.9107
$s(\lambda_6 = -7.2387)$	46.6570

Table 4.25: The sensitivity of each eigenvalue for general controllable form.

Robust stability measures:

Now, we are able to compute the three measures for general controllable form:

The results obtained for measures M_1 and M_2 for general controllable form are given in the table 4.26:

The measure	General controllable form
$\mathbf{M_1}$	0.0782
$\mathbf{M_2}$	0.0062

Table 4.26: Measures M_1 and M_2 for general controllable form.

The results obtained for measure M_3 of each eigenvalue for general controllable form are given in the following table:

The eigenvalue λ_i	General controllable form
$-1.7294 + 3.8915i$	0.0707
$-1.7294 - 3.8915i$	0.0707
$-0.6206 + 2.2816i$	0.0742
$-0.6206 - 2.2816i$	0.0742
-8.1667	1.6630
-7.2387	0.1551

Table 4.27: Robust stability measure M_3 of each eigenvalue for general controllable form.

From the table 4.27, we conclude that:

The Measure	General controllable form
M_3	0.0707

Table 4.28: The measure M_3 for general form.

4.3.4   Robust Performance

In order to compute the relative change of the eigenvalues of the closed-loop matrices $(A - BK)$, for the general controllable form case, we use the command "rand" in Matlab to generate a random perturbation matrix ΔA which is given as follows:

$$\Delta A = \begin{pmatrix} 0.0006 & 0.0029 & 0.0037 & 0.0005 & 0.0042 & 0.0070 \\ 0.0040 & 0.0043 & 0.0020 & 0.0074 & 0.0098 & 0.0067 \\ 0.0053 & 0.0002 & 0.0049 & 0.0027 & 0.0030 & 0.0018 \\ 0.0042 & 0.0098 & 0.0034 & 0.0042 & 0.0070 & 0.0013 \\ 0.0066 & 0.0017 & 0.0095 & 0.0055 & 0.0067 & 0.0100 \\ 0.0063 & 0.0011 & 0.0092 & 0.0094 & 0.0054 & 0.0017 \end{pmatrix} \tag{4.35}$$

The same perturbation as in the previous case, is used. The new closed-loop matrix after perturbation is:

$$A + \Delta A - BK = \begin{pmatrix} 0.0006 & 0.0029 & 1.0037 & 0.0005 & 0.0042 & 0.0070 \\ 0.0040 & 0.0043 & 0.0020 & 1.0074 & 0.0098 & 0.0067 \\ -61.5129 & -2.2742 & 10.0436 & -0.1238 & 4.0646 & 0.0374 \\ 7.9111 & -19.1363 & -1.5551 & -7.4090 & -0.4538 & -1.4207 \\ 282.1673 & 11.1740 & -46.7809 & 1.1510 & -20.7516 & -0.2043 \\ -13.7118 & 4.5820 & 2.6060 & -12.9782 & 0.9947 & -1.9709 \end{pmatrix} \tag{4.36}$$

The relative change for each eigenvalue:

The following tables show the eigenvalues of the disturbed closed-loop system matrix $(A + \Delta A - BK)$ for general controllable form are given by:

λ_i of $(A - BK)$	λ_i' of $(A + \Delta A - BK)$	Relative change r_i
$-1.7294 + 3.8915i$	-7.3867	1.6124
$-1.7294 - 3.8915i$	-8.1049	1.7540
$-0.6206 + 2.2816i$	$-1.6614 + 3.8870$	0.8091
$-0.6206 - 2.2816i$	$-1.6614 - 3.8870i$	0.8091
-8.1667	$-0.6343 + 2.2318i$	0.9620
-7.2387	$-0.6343 - 2.2318i$	0.9631

Table 4.29: The relative change of each eigenvalue for general controllable form.

4.3.5   Time Domain Performances

From the result graphs, the response to initial conditions of the closed-loop matrix for general controllable form is summarized in the following tables:

General controllable form:
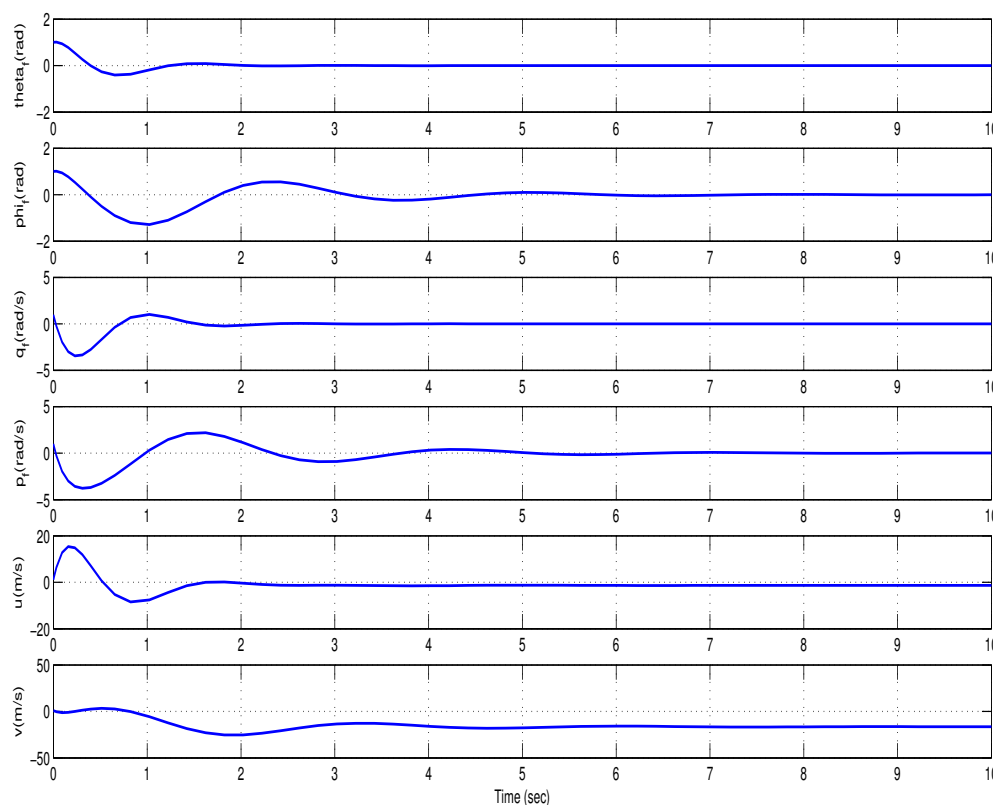
Figure 4.4: The time response for general controllable form.

The graphical results shown in the figure 4.4 demonstrates the high quality of convergence and stability. Also we observe that the convergence rate(which is the time response) is equal to 4 seconds.

From the graphical results shown in the figure 4.4 , the initial conditions response of the closed-loop system for diagonal form is summarized in the following table:

The states	Maximum overshoot	Percent overshoot	Falling time	Settling time
θ_F	1.1284×10^7	1.1159×10^9	0.2551	1.9455
ϕ_F	5.8110×10^4	4.5244×10^6	0.2390	6.8922
q_F	6.7220×10^5	4.9329×10^4	0.0204	2.1774
p_F	1.8656×10^4	1.9552×10^7	0.0195	6.2239
u	534,1004	3.4799×10^3	0.0435	2.2570
ν	52.5723	208,3410	1.2399	6.5289

Table 4.30: The time response for general controllable form.

4.3.6   Algorithm

A. General Controllable form:

step(1): Given matrix $A \in \Re(6 \times 6)$, $B \in \Re(6 \times 2)$ of the helicopter.

step(2):

- Compute controllability index l, where: $l = n \backslash m$.

- Check if l is not an integer.

step(3): Construct the block controllability matrix P:

$P = [B_1, \quad AB_1, \quad , A^2B_1, \cdots A^{(k_1-1)}B_1, \quad B_2 \ AB_2, \quad A^2B_2, \cdots, A^{(k_2-1)}B_2, \cdots B_m \ , AB_m, \quad A^2B_m, \cdots,$
$A^{(k_m-1)}B_m]$, where: $k_i = \{k_1 \ \ k_2 \ \ \cdots k_m\}$.

- Check the non singularity of P: $\sum_{i=1}^{m} k_i = n$.

- Compute the inverse of the matrix P : $P^{-1} = inv(P)$.

step(4): Construct the similarity transformation of matrix T_c:

- Construct the matrix T_c :

$$T_c = [P_1; \ \ P_1A; \ \ P_1A^2; P_2; \ \ P_2A; \ \ P_2A^2; \cdots ; P_m; \ \ P_mA; \ \ P_mA^{(m-1)}].$$

- Compute the inverse of the matrix T_c: $T_c^{-1} = inv(T_c)$

step(5): Compute (A_c, B_c):

- $A_c = T_c \, A \, T_c^{-1}$.

- $B_c = T_c \, B$.

step(6): Given the closed-loop matrix A_d.
step(7): Compute the state feedback gain matrix $K = K_cT_c$.
step(8): Compute the closed loop matrix $(A - BK)$.
step(9): Compute the different norms of K.

B. Robust stability:

step(1): Compute the eigenvectors and the eigenvalues of $(A - BK)$.

step(2):

- Compute the norms of right eigenvectors matrix $\parallel V \parallel$.

- Compute the norms of left eigenvectors matrix $\parallel T \parallel$, where: $\parallel T \parallel = inv(\parallel V \parallel)$.

step(3):

- Compute the norms of each column of the right eigenvectors matrix $\parallel \nu_i \parallel$.

- Compute the norms of each row of the left eigenvectors matrix $\parallel t_i \parallel$.

step(4): Compute the sensitivity of the eigenvalues $s(\Delta)$, where: $s(\Delta) = \parallel V \parallel . \parallel T \parallel$.

- Compute the inverse of the matrix $s(\Delta)$: $s(\Delta)^{-1} = inv(s(\Delta))$.

step(5): Compute the sensitivity of each eigenvalue $s(\lambda_i)$, where: $s(\lambda_i) = \parallel \nu_i \parallel . \parallel t_i \parallel$.

step(6): Compute the three measures M_1, M_2 and M_3.

C. Robust performance:

step(1): Given the rand closed-loop matrix ΔA.

step(2): Compute the new closed-loop matrix after perturbation $(A + \Delta A - BK)$.

step(3): Compute the relative change r_i, where:

- λ_i are the eigenvalues of the closed-loop system $(A - BK)$.

- λ_i' are the eigenvalues of the closed-loop system $(A + \Delta A - BK)$.

D. The time response for general form:

step(1): Compute the time response of the closed-loop matrix, taking $x_0 = [1; 1; 1; 1; 1; 1]$ as the initial conditions.

step(2): Given Maximum overshoot M_p, Percent overshoot (POS)%, Falling time T_r en second (s), Settling time T_s en second (s).

See appendix B.

4.4 Comparison of Results

4.4.1 Norm of Feedback Gain

Block diagonal	Block controllable	Block observable	General controllable
$\|K\|_1$=10.4748	$\|K\|_1$=19.3457	$\|K\|_1$=63.5152	$\|K\|_1$=9.8819
$\|K\|_2$=9.8316	$\|K\|_2$=17.6221	$\|K\|_2$=51.7536	$\|K\|_2$=9.9164
$\|K\|_\infty$=11.2390	$\|K\|_\infty$=25.9365	$\|K\|_\infty$=61.2552	$\|K\|_\infty$=12.3875
$\|K\|_F$=10.6662	$\|K\|_F$=17.6818	$\|K\|_F$=51.7539	$\|K\|_F$=9.9557

Table 4.31: The different norms of feedback gain for block and general forms.

The feedback gain matrix norm represents the control effort needed to drive the system to the desired behavior, thus, according to previous results, the feedback gain in the block diagonal form is the best result which means that the requires of less control effort to stabilize the system.

4.4.2 Robust Stability

Method	Form	M_1	M_2	M_3
Block pole	**Diagonal form**	0.0140	0.0022	0.0103
	Controllable form	0.0161	1.8515×10^{-4}	0.0014
	Observable form	0.0285	5.4555×10^{5}	4.6598×10^{-4}
General form	**Controllable form**	0.0782	0.0062	0.0707

Table 4.32: The different measures for block and general forms.

From the two measures M_1 and M_3 and the different sensitivities computed before, we can say that using the block diagonal form, results in a better system response, in the terms of robustness stability.

And the measure M_2 gives the best results in the block observable form.

4.4.3   Robust Performance

Form	λ_i	$\lambda_i^{'}$	r_i
Block diagonal form	$-1.7294 + 3.8915i$	$-1.9005 + 3.9204i$	0.0408
	$-1.7294 - 3.8915i$	$-1.9005 - 3.9204i$	0.0408
	$-0.6206 + 2.2816i$	$-0.6339 + 2.2769i$	0.0060
	$-0.6206 - 2.2816i$	$-0.6339 - 2.2769i$	0.0060
	-8.1667	-8.0973	0.0085
	-7.2387	-6.9111	0.0453
Block controllable form	$-1.7294 + 3.8915i$	$-7.8130 + 1.5196i$	0.1911
	$-1.7294 - 3.8915i$	$-7.8130 - 1.5196i$	0.2244
	$-0.6206 + 2.2816i$	$-1.7048 + 3.8413i$	0.0131
	$-0.6206 - 2.2816i$	$-1.7048 - 3.8413i$	0.0131
	-8.1667	$-0.5238 + 2.3533i$	0.0510
	-7.2387	$-0.5238 - 2.3533i$	0.0510
Block observable form	$-1.7294 + 3.8915i$	$-9.7699 + 0.0000i$	0.1978
	$-1.7294 - 3.8915i$	$-5.9573 + 0.0000i$	0.1781
	$-0.6206 + 2.2816i$	$-1.5717 + 3.9941i$	0.0443
	$-0.6206 - 2.2816i$	$-1.5717 - 3.9941i$	0.0443
	-8.1667	$-0.6063 + 2.2583i$	0.0116
	-7.2387	$-0.6063 - 2.2583i$	0.0116
General controllable form	$-1.7294 + 3.8915i$	-7.3867	1.6124
	$-1.7294 - 3.8915i$	-8.1049	1.7540
	$-0.6206 + 2.2816i$	$-1.6614 + 3.8870$	0.8091
	$-0.6206 - 2.2816i$	$-1.6614 - 3.8870i$	0.8091
	-8.1667	$-0.6343 + 2.2318i$	0.9620
	-7.2387	$-0.6343 - 2.2318i$	0.9631

Table 4.33: The relative change for block and general forms.

The smallest relative change is obtained by the block diagonal form, it is furthermore less sensitive system to model perturbation.

4.4.4 Time Response

The following tables show the initial value response performances for the different outputs for block placement pole and general form:

Method Block Pole Placement:

The states	Characteristic	Diagonal form	Controllable form	Observable form
θ_F	M_P	7.1416×10^4	1.7893×10^5	3.7902×10^5
	$POS\%$	2.3356×10^6	17359×10^7	3.7685×10^7
	$T_f(s)$	0.1354	0.3451	0.2044
	$T_s(s)$	6.3282	4.4414	3.2914
ϕ_F	M_P	4.8511×10^4	3.2248×10^4	1.0561×10^5
	$POS\%$	7.7373×10^5	1.1713×10^6	1.0771×10^6
	$T_f(s)$	0.0292	0.1696	0.0308
	$T_s(s)$	6.8283	6.3892	2.8005
q_F	M_P	2.6162×10^6	4.1258×10^4	1.4058×10^5
	$POS\%$	3.9457×10^6	1.6129×10^6	3.2653×10^6
	$T_f(s)$	0.0320	0.0528	0.0111
	$T_s(s)$	6.7484	4.8428	2.6474
p_F	M_P	1.9582×10^4	5.7359×10^5	1.4585×10^7
	$POS\%$	1.1055×10^5	8.4418×10^6	6.8929×10^6
	$T_f(s)$	0.0058	0.0372	0.0029
	$T_s(s)$	7.1916	6.7585	2.4797
u	M_P	37.9309	118.8833	1.3556×10^3
	$POS\%$	109.4330	1.2066×10^3	5.1263×10^3
	$T_f(s)$	1.2378	0.1351	0.0016
	$T_s(s)$	6.3431	4.5479	1.9894
ν	M_P	28.7884	54.3274	50.4800
	$POS\%$	5.5755	87.6632	44.1864
	$T_f(s)$	0.9644	0.6224	0.5630
	$T_s(s)$	6.2602	5.9831	2.9165

Table 4.34: The time response for block pole placement.

Method General Form:

The states	Characteristic	Controllable form
θ_F	M_P	1.1284×10^7
	$POS\%$	1.1159×10^9
	$T_f(s)$	0.2551
	$T_s(s)$	1.9455
ϕ_F	M_P	5.8110×10^4
	$POS\%$	4.5244×10^6
	$T_f(s)$	0.2390
	$T_s(s)$	6.8922
q_F	M_P	6.7220×10^5
	$POS\%$	4.9329×10^4
	$T_f(s)$	0.0204
	$T_s(s)$	2.1774
p_F	M_P	1.8656×10^4
	$POS\%$	1.9552×10^7
	$T_f(s)$	0.0195
	$T_s(s)$	6.2239
u	M_P	534,1004
	$POS\%$	3.4799×10^3
	$T_f(s)$	0.0435
	$T_s(s)$	2.2570
ν	M_P	52.5723
	$POS\%$	208,3410
	$T_f(s)$	1.2399
	$T_s(s)$	6.5289

Table 4.35: The time response for general form

From the tables 4.34 and 4.35, we notice that there is a trade-off, such that the block diagonal form ensures the best results concerning maximum overshoot and percent overshoot. On the other hand, the block observable form gives a better falling time and a better settling time.

4.5   Conclusion

In this chapter, we studied the application of the state feedback controller to the helicopter. All the simulation results presented in this chapter were carried out in Matlab-Simulink software. We started with applying two methods in order to design state feedback controller to the dynamics of the helicopter at lateral motion. Then we compared between the block pole placement and general form constructed in diagonal, controllable and observable forms. The comparative study between the two methods involved the different robustness criteria. Second, we talk about the robustness, time response and control effort, where latent structure assignment provides robustness. Overall, from the conducted comparative study, we concluded that the block diagonal form gives the best results were obtained in terms of robustness and control effort, but we found that the best time response results are in the block observable form.

GENERAL CONCLUSION

GENERAL CONCLUSION

The purpose of this thesis is to compare and evaluate between two methods, based on canonical form to a MIMO system with an application to a lateral motion of a helicopter, and select the feedback gain matrix that leads to the best gain magnitude, time response characteristics and system robustness.

Throughout the comparative study, we have observed that the feedback gain magnitude of the closed-loop matrix is significantly affected by the arrangement of the eigenvalues in the desired matrix A_D, and the arrangement given in the block diagonal form gives the best results especially when combined with other forms.

We have also noticed that the robust stability measures of the closed loop matrices $(A - BK)$ depend strongly on both the input sequence and the arrangement of the eigenvalues, meaning that good results were obtained for measures M_2 and M_3 are obtained in block observable form and measure M_1 is obtained in block diagonal form.

Through this analysis, we wanted to see how the system robustness is affected by the choice of the feedback gain matrix by computing the sensitivity of eigenvalues, robustness stability measures and the relative change, and from the obtained results, we have concluded that the block diagonal form gives the best results which means that the requirement of less control effort to stabilize the system.

We know that the time response of a closed-loop system, the better (higher speed) is in block observable form with smallest settling time and smallest falling time. Our comparative study shows that smallest maximum overshoot and smallest percent overshoot are obtained in block diagonal form.

Based on the simulation results, we have observed that the block diagonal form, led to the best case for all design criteria; except for the time response. Indeed, gave the best results in terms of robustness and control effort, meaning that it is less sensitive to small random perturbations.

We can conclude that no theory can demonstrate the existence of specific form that can deals with the best performance of MIMO system. Therefore each system has a form that gives the best characteristic in the study for the design criteria.

As further work we may suggest the following:

- Study each criteria for all possible forms in the observer design.

- Comparative study between the general controller canonical form and observer design.

APPENDICES

Appendix-A

The Case of Block Pole Placement

Program:

clear all

clc

% A. Given matrices A ,B ,C and D of the helicopter:

$A = [0\ 0\ 1\ 0\ 0\ 0; 0\ 0\ 0\ 1\ 0\ 0; 0\ 0\ -0.042\ 0.32\ 0.003\ 0.001; 0\ 0\ -1.23\ -1.6\ 0.004\ -0.012; -32.2\ 0\ 4.7\ -1.0\ -0.02\ -0.005; 0\ 32.2\ -1.0\ -4.7\ 0.005\ -0.02]$; % matrix A.

$B = [0\ 0; 0\ 0; -0.3\ 6.3; -23\ -1.1; 1.0\ -32.2; -32.2\ 1.0]$; % matrix B.

$C = eye(6,6)$; % matrix C.; $D = zeros(6,2)$; % matrix D.

$dA = 0.01 * rand(6,6)$; % compute the rand closed-loop matrix ΔA.

$[nA, mA] = size(A)$; $[nB, mB] = size(B)$; % Size of matrices A , B :

%Check size of matrices:

if $nA\ = mA$

$display('Matrix\ A\ is\ not\ square!')$

return, **else if** $mA\ = nB$

$display('Inner\ matrix\ dimension\ (A,B)\ does\ not\ agree!')$

return, **end**

$l = nA/mB$; %Compute controllability index l

%Check if l is an integer:

if $mod(nA, mB)\ = 0$

$y = input('l\ is\ not\ an\ integer\ enter\ Y\ if\ you\ want\ to\ use\ general\ controller\ canonical\ form')$;

if $y ==' Y'$; $S = input('enter\ selection\ sequence\ S')$;

$[A_d, B_d, C_d, k] = GCCF(A, B, C, S)$;

end, **return**, **else**

% Contruct the block controllability matrix W_c

$W_c = B$; $temp = B$;

for $i = 1 : l - 1$

$temp = A * temp$; $W_c = [W_c\ temp]$;

end, **end** % $W_c = [B\ AB\ A^2B \cdots A^{(l-1)}B]$

% Check non singularity of W_c

if $rank(W_c)\ = nA$

$display('System\ block\ is\ not\ controllable!')$

return, else

% construct the matrix t_{c1}

$W_i = inv(W_c);$ % Compute w_{di} inverse of W_c

$t_{c1} = W_i(nA - mB + 1 : end, :);$ **end**

% Check non singularity of W_c

if $rank(W_c)\ = nA$

$display('System\ block\ is\ not\ controllable!')$

return, else

% construct the matrix t_{c1}

$W_i = inv(W_c);$ % Compute W_{ci} inverse of W_c

$t_{c1} = W_i(nA - mB + 1 : end, :);$ **end**

% Construct similarity transformation matrix T_c

$T_c = t_{c1}; \quad temp = t_{c1};$

for $i = 1 : (l - 1)$

$temp = temp * A; \quad T_c = [T_c; temp];$

end% $T_c = [t_{c1}; t_{c1} * A; \cdots ; t_{ci}^{(l-1)} * A]$

$t_{ci} = inv(T_c);$ % Compute t_{ci} inverse of T_c

$A_c = roundn(T_c * A * t_{ci}, -4);$ % Compute A_c

$B_c = roundn(T_c * B, -4);$ % Compute B_c

% The desired block pole in controllable form:

$R_1 = [0\ \ 1.0000; -18.1346\ \ -3.4588]; \quad R_2 = [-1.2412\ \ -5.5908; 1.0000\ \ 0];$

$R_3 = [-15.4054\ \ -59.1163; 1.0000\ \ 0];$

$dA = 0.01 * rand(6, 6);$ % compute the rand closed-loop matrix ΔA.

$I = eye(2, 2);$

$V_R = [I\ \ I\ \ I; R_1\ \ R_2\ \ R_3; R_1^2\ \ R_2^2\ \ R_3^2];$ % Construct the right block Vandermonde matrix V_R.

$D_f = -[R_1^3\ \ R_2^3\ \ R_3^3] * inv(V_R);$ % Compute the desired charactiristic polynomial D_f.

$D_{f1} = D_f(:, 5 : 6);\ \ D_{f2} = D_f(:, 3 : 4);\ \ D_{f3} = D_f(:, 1 : 2);$

$K_{c1} = D_{f1} + A_c(5 : 6, 5 : 6);\ \ K_{c2} = D_{f2} + A_c(5 : 6, 3 : 4);\ \ K_{c3} = D_{f3} + A_c(5 : 6, 1 : 2);$

$K_c = [K_{c3}\ \ K_{c2}\ \ K_{c1}];$

$K = K_c * T_c'$; % Compute the state feedback gain matrix $K = K_c T_c$.

$A_{new} = A - B * K$; % Compute the closed loop matrix $A - BK$.

% The different norms of K:

$K_1 = norm(K, 1)$; % the 1-norm.

$K_2 = norm(K, 2)$; % the 2-norm.

$K_{inf} = norm(K, inf)$; % the Infinity norm.

$K_{fro} = norm(K,' fro')$; % the Frobenius norm.

% B. Robust stability:

$\lambda = eig(A_{new})$; % compute the eigenvalues of $A - BK$.

$[V_c, D_c] = eig(A_{new})$; % compute the eigenvectors and eigenvalues of $A - BK$.

$right\ norm = norm(V_c, 2)$; % Compute the norms of right eigenvectors matrix.

% The norms of each column of the right eigenvectors matrix:

$v_{c1} = norm(V_c(:, 1), 2)$; $v_{c2} = norm(V_c(:, 2), 2)$; $v_{c3} = norm(V_c(:, 3), 2)$;

$v_{c4} = norm(V_c(:, 4), 2)$; $v_{c5} = norm(V_c(:, 5), 2)$; $v_{c6} = norm(V_c(:, 6), 2)$;

$norms\ V_c = [v_{c1}\ v_{c2}\ v_{c3}\ v_{c4}\ v_{c5}\ v_{c6}]$;

$T_c = inv(V_c)$; % compute the inverse of V_c.

$left\ norm = norm(T_c, 2)$; % Compute the norm of left eigenvectors matrix.

% The norms of each row of the left eigenvectors matrix:

$t_{c1} = norm(T_c(1, :), 2)$; $t_{c2} = norm(T_c(2, :), 2)$; $t_{c3} = norm(T_c(3, :), 2)$;

$t_{c4} = norm(T_c(4, :), 2)$; $t_{c5} = norm(T_c(5, :), 2)$; $t_{c6} = norm(T_c(6, :), 2)$;

$norms\ T_c = [t_{c1}\ t_{c2}\ t_{c3}\ t_{c4}\ t_{c5}\ t_{c6}]$;

$S_c\ total = right\ norm * left\ norm$; % Compute the sensitivity of the eigenvalues $S(\Lambda)$.

$S_c = inv(S_c\ total)$; % the inverse of $S(\Lambda)$.

% Compute the sensitivity of each eigenvalue $S(\lambda_i)$:

$s_{c1} = v_{c1} * t_{c1}$; $s_{c2} = v_{c2} * t_{c2}$; $s_{c3} = v_{c3} * t_{c3}$;

$s_{c4} = v_{c4} * t_{c4}$; $s_{c5} = v_{c5} * t_{c5}$; $s_{c6} = v_{c6} * t_{c6}$;

$S_c\ total = [s_{c1}\ s_{c2}\ s_{c3}\ s_{c4}\ s_{c5}\ s_{c6}]$;

% Compute the measure M_1:

%————————————————————————

% $M_1 = \min \sigma * ((A - BK) - i * \omega * I)$; $where : (1 < \omega < inf)$

%————————————————————————

$I_1 = eye(6)$; $\omega = 0 : 0.01 : 100$; $n = length(\omega)$; $Y = [\ ]$;

for $k = 1 : n$

$X = svd(A_{new} - j * \omega(k) * I_1);$

end

$M_1 = \min(Y);$

% Compute the measure M_2:

%——

% $M_2 = s(\Lambda)^{-1} * \mid Re(\lambda_n) \mid; \; where : (\mid Re(\lambda_n) \mid < \cdots < \mid Re(\lambda_1) \mid);$

%——

$l = [\;];$

for $i = 1 : 6$

$l_i = abs(real(\lambda(i))); \qquad l = [l \quad l_i];$

end

$Mm = \min(l); \quad M_2 = S_c * Mm;$

%Compute the measure M_3:

%——

%$M_3 = \min s(\lambda_i)^{-1} * \mid Re(\lambda_i) \mid; \; where : (1 < i < n)$

%——

$\lambda_1 = complex(-1.7294, 3.8915); \quad \lambda_2 = complex(-1.7294, -3.8915);$

$\lambda_3 = complex(-0.6206, 2.2816); \quad \lambda_4 = complex(-0.6206, -2.2816);$

$\lambda_5 = -8.1667; \quad \lambda_6 = -7.2387;$

$s = [s_{c1} \; s_{c2} \; s_{c3} \; s_{c4} \; s_{c5} \; s_{c6}]; \quad \Lambda = [\lambda_1 \; \lambda_2 \; \lambda_3 \; \lambda_4 \; \lambda_5 \; \lambda_6]; \quad m = [\;];$

for $i = 1 : 6$

$M_i = inv(s(i)) * abs(real(\Lambda(i))); \qquad m = [m, M_i];$

end

$M_3 = min(m)$

% C. Robust performance:

$norm \, \Delta A = norm(dA, 2); \;$ % the 2-norm of ΔA.

$\Delta A_{new} = A + dA - B * K; \;$ % the new closed-loop matrix after perturbation $A + \Delta A - BK$.

%——

%Compute the relative change $r_i = \mid \lambda_i - \lambda_i' \mid / \mid \lambda_i \mid; \; where \, i = 1, 2, \cdots, 6.$

%——

$eig_1 = eig(A - B * K); \;$ % compute the eigenvalues of $A - BK$.

$eig_2 = eig(A + dA - B * K); \;$ % compute the eigenvalues of $A + \Delta A - BK$.

$relative = [\;];$

for **i=1:6**

$relative_{11} = abs((eig_1(i) - eig_2(i))/eig_1(i));$

$relative_1 = [relative_{11}]; \quad relative = [relative; relative_1];$

end

$relative\ change = relative\,';$

% **D. Time response:**

% **plot the time response for the states:** θ_F , ϕ_F , q_F , p_F , u , ν **and find Maximum overshoot** M_P, **Percent overshoot** $(POS)\%$, **Falling time** T_r **en** (s), **Delay time** T_d **en** (s), **Settling time** T_s **en** (s) **of the states:** θ_F , ϕ_F , q_F , p_F , u , ν, **using Simulink**

$sim('Time\ dom')$

$S_1 = stepinfo(\theta_F, t_1); \quad S_2 = stepinfo(\phi_F, t_1); \quad S_3 = stepinfo(q_F, t_1);$

$S_4 = stepinfo(p_F, t_1); \quad S_5 = stepinfo(u, t_1); \quad S_6 = stepinfo(\nu, t_1);$

figure;

subplot(611);plot$(t_1, \theta_F,' -b','linewidth', 1.5); ylabel('theta_f(rad)');$**grid on;hold on**

subplot(612);plot$(t_1, \phi_F,' -b','linewidth', 1.5); ylabel('phi_f(rad)');$**grid on; hold on**

subplot(613);plot$(t_1, q_F,' -b','linewidth', 1.5); ylabel('q_f(rad/s)');$**grid on; hold on**

subplot(614);plot$(t_1, p_F,' -b','linewidth', 1.5); ylabel('p_f(rad/s)');$**grid on; hold on**

subplot(615);plot$(t_1, u,' -b','linewidth', 1.5); ylabel('u(m/s)');$**grid on; hold on**

subplot(616);plot$(t_1, \nu,' -b','linewidth', 1.5); ylabel('v(m/s)');$

$xlabel('Time(sec)');$**grid on; hold on**

Appendix-B

The Case of General Controllable Canonical Form

Program:

clear all

clc

% **A. Given matrices** A ,B ,C **and** D **of the helicopter:**

$A = [0\ 0\ 1\ 0\ 0\ 0; 0\ 0\ 0\ 1\ 0\ 0; 0\ 0\ -0.042\ 0.32\ 0.003\ 0.001; 0\ 0\ -1.23\ -1.6\ 0.004\ -0.012; -32.2\ 0\ 4.7\ -1.0\ -0.02\ -0.005; 0\ 32.2\ -1.0\ -4.7\ 0.005\ -0.02];$ % **matrix** A.

$B = [0\ 0; 0\ 0; -0.3\ 6.3; -23\ -1.1; 1.0\ -32.2; -32.2\ 1.0];$ % **matrix** B.

$C = eye(6, 6);$ % **matrix** C.; $D = zeros(6, 2)$ % **matrix** D.

$dA = 0.01 * rand(6, 6);$ % **compute the rand closed-loop matrix** ΔA.

% **Size of matrices** A , B :

$[nA, mA] = size(A); [nB, mB] = size(B);$

% **Check size of matrices:**

if $nA\ = mA$

$display('Matrix\ A\ is\ not\ square!')$

return,else if $mA\ = nB$

$display('Inner\ matrix\ dimension\ (A, B)\ does\ not\ agree!')$

return,end

% **compute** $l = n/m$

$l = nA/mB;$

if $mod(nA, mA)\ = 0$

$display('lisnotanintegerweusegeneralcontrollercanonicalform');$

end

% **General controllable**

$k = 0;$ **for** $i = 1 : mB$

if $sum(k') == nA$

$display('systemiscompleteleycontrollable')$

else if $sum(k')\ = nA$

$display('systemispartiallycontrollable')$

end, **end**

% **construct matrix** p

$B_1 = B(:, 1);\quad B_2 = B(:, 2);$

$P = [(B_1) \ (A * B_1) \ (A^2 * B_1) \ (B_2) \ (A * B_2) \ (A^2 * B_2)]; \quad PI = inv(P);$

% construct similarity transformation matrix T_c

$P_1 = PI(3,:); \ \%K_1 = 3 \quad P_2 = PI(6,:); \ \%K_1 + K_2 = 6$

$T_c = [(P_1); (P_1 * A); (P_1 * A * A); (P_2); (P_2 * A); (P_2 * A * A)]$

% new state space

$A_c = T_c * A * inv(T_c); \ \%$ Compute $A_c.$

$B_c = T_c * B; \ \%$ Compute $B_c.$

$A_d = [0 \ 1 \ 0 \ 0 \ 0 \ 0; \cdots$

$0 \ 0 \ 1 \ 0 \ 0 \ 0; \cdots$

$-45.6587 \ -15.7274 \ -9.4079 \ 0 \ 0.0000 \ 0.0000; \cdots$

$0 \ 0 \ 0 \ 0 \ 1 \ 0; \cdots$

$0 \ 0 \ 0 \ 0 \ 0 \ 1; \cdots$

$0.0000 \ -0.0000 \ 0.0000 \ -131.2709 \ -43.1718 \ -10.6975]; \ \%$ Given the closed-loop matrix $A_d.$

$K_{CG} = pinv(B_c) * (A_c - A_d) * T_c; \ \%$ Compute the state feedback gain matrix $K = K_c T_c;$

$A_{CG} \ new = A - B * K_{CG}; \ \%$ Compute the closed-loop matrix $(A - BK).$

% The different norms of K:

$K_1 = norm(K_{CG}, 1); \ \%$ the 1-norm.

$K_2 = norm(K_{CG}, 2); \ \%$ the 2-norm.

$K_{inf} = norm(K_{CG}, inf); \ \%$ the Infinity norm.

$K_{fro} = norm(K_{CG}, 'fro'); \ \%$ the Frobenius norm.

% B. Robust stability:

$\lambda = eig(A_{CG} \ new); \ \%$ compute the eigenvalues of $A - BK.$

$[V_{CG}, D_{CG}] = eig(A_{CG} \ new); \ \%$ compute the eigenvectors and eigenvalues of $A - BK.$

$right \ norm = norm(V_{CG}, 2); \ \%$ Compute the norms of right eigenvectors matrix.

%The norms of each column of the right eigenvectors matrix:

$v_{CG1} = norm(V_{CG}(:,1), 2); \ v_{CG2} = norm(V_{CG}(:,2), 2); \ v_{CG3} = norm(V_{CG}(:,3), 2);$

$v_{CG4} = norm(V_{CG}(:,4), 2); \ v_{CG5} = norm(V_{CG}(:,5), 2); \ v_{CG6} = norm(V_{CG}(:,6), 2);$

$norms \ V_{CG} = [v_{CG1} \ v_{CG2} \ v_{CG3} \ v_{CG4} \ v_{CG5} \ v_{CG6}];$

$T_{CG} = inv(V_{CG}); \ \%$ compute the inverse of $V_{CG}.$

$left \ norm = norm(T_{CG}, 2); \ \%$ Compute the norm of left eigenvectors matrix.

% The norms of each row of the left eigenvectors matrix:

$t_{CG1} = norm(T_{CG}(1,:), 2); \ t_{CG2} = norm(T_{CG}(2,:), 2); \ t_{CG3} = norm(T_{CG}(3,:), 2);$

$t_{CG4} = norm(T_{CG}(4,:),2);\ t_{CG5} = norm(T_{CG}(5,:),2);\ t_{CG6} = norm(T_{CG}(6,:),2);$

$norms\ T_{CG} = [t_{CG1}\ t_{CG2}\ t_{CG3}\ t_{CG4}\ t_{CG5}\ t_{CG6}];$

$S_{CG}\ total = right\ norm * left\ norm;\ \%$ Compute the sensitivity of the eigenvalues $S(\Lambda)$.

$S_{CG} = inv(S_{CG}\ total);\ \%$ the inverse of $S(\Lambda)$.

$\%$ Compute the sensitivity of each eigenvalue $S(\lambda_i)$:

$s_{CG1} = v_{CG1} * t_{CG1};\ s_{CG2} = v_{CG2} * t_{CG2};\ s_{CG3} = v_{CG3} * t_{CG3};$

$s_{CG4} = v_{CG4} * t_{CG4};\ s_{CG5} = v_{CG5} * t_{CG5};\ s_{CG6} = v_{CG6} * t_{CG6};$

$S_{CG}\ total = [s_{CG1}\ s_{CG2}\ s_{CG3}\ s_{CG4}\ s_{CG5}\ s_{CG6}];$

$\%$ Compute the measure M_1:

$\%$————————————————————————————————-

$\%\ M_1 = \min \sigma * ((A - BK) - i * \omega * I);\ where : (1 < \omega < inf)$

$\%$————————————————————————————————-

$I_1 = eye(6);\quad \omega = 0 : 0.01 : 100;\quad n = length(\omega);\quad Y = [\ ];$

for $k = 1 : n$

$X = svd(A_{CG}\ new - j * \omega(k) * I_1);$

end

$M_1 = \min(Y);$

$\%$Compute the measure M_2:

$\%$————————————————————————————————

$\%M_2 = s(\Lambda)^{-1} * |\ Re(\lambda_n)\ |;\ where : (|\ Re(\lambda_n)\ | < \cdots < |\ Re(\lambda_1)\ |);$

$\%$————————————————————————————————

$l = [\ ];$

for $i = 1 : 6$

$l_i = abs(real(\lambda(i)));\quad l = [l\ l_i];$

end

$Mm = \min(l);\quad M_2 = S_{CG} * Mm;$

$\%$Compute the measure M_3:

$\%$————————————————————————————————

$\%M_3 = \min s(\lambda_i)^{-1} * |Re(\lambda_i)|;\ where : (1 < i < n)$

$\%$————————————————————————————————

$\lambda_1 = complex(-1.7294, 3.8915);\ \lambda_2 = complex(-1.7294, -3.8915);$

$\lambda_3 = complex(-0.6206, 2.2816);\ ; \lambda_4 = complex(-0.6206, -2.2816);$

$\lambda_5 = -8.1667;\ \lambda_6 = -7.2387;$

$s = [s_{CG1}\ s_{CG2}\ s_{CG3}\ s_{CG4}\ s_{CG5}\ s_{CG6}];\quad \Lambda = [\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6];\quad m = [\ ];$

for $i = 1 : 6$

$M_i = inv(s(i)) * abs(real(\Lambda(i)));\ m = [m, M_i];$

end

$M_3 = min(m)$

% **C. Robust performance:**

$norm\ \Delta A = norm(dA, 2);$ % the 2-norm of ΔA.

$\Delta A_{CG}\ new = A + dA - B * K_G;$ % **the new closed-loop matrix after perturbation** $A + \Delta A - BK$.

%————————————————————————————————-

%**Compute the relative change** $r_i = \mid \lambda_i - \lambda_i' \mid / \mid \lambda_i \mid;\ \ where\ i = 1, 2, \cdots, 6.$

%————————————————————————————————-

$eig_1 = eig(A - B * K_G);$ % compute the eigenvalues of $A - BK$.

$eig_2 = eig(A + dA - B * K_G);$ % compute the eigenvalues of $A + \Delta A - BK$.

$relative = [\ ];$

for **i=1:6**

$relative_{11} = abs((eig_1(i) - eig_2(i))/eig_1(i));$

$relative_1 = [relative_{11}];$

$relative = [relative; relative_1];$

end

$relative\ change = relative';$

% **D. Time response:**

%plot the time response for the states: θ_F, ϕ_F, q_F, p_F, u, ν **and find Maximum overshoot** M_P, **Percent overshoot** $(POS)\%$ **en, Falling time** T_r **en** (s), **Delay time** T_d **en** (s), **Settling time** $T_{(s)}$ **en** s **of the states:** θ_F, ϕ_F, q_F, p_F, u, ν, **using Simulink**

$sim('Time\ dom')\quad S_1 = stepinfo(\theta_F, t_1);\ S_2 = stepinfo(\phi_F, t_1);$

$S_3 = stepinfo(q_F, t_1);\ S_4 = stepinfo(p_F, t_1);$

$S_5 = stepinfo(u, t_1);\ S_6 = stepinfo(\nu, t_1);$

figure;

subplot(611);plot$(t_1, \theta_F, '-b', 'linewidth', 1.5);ylabel('theta_f(rad)');$**grid on; hold on**

subplot(612);plot$(t_1, \phi_F, '-b', 'linewidth', 1.5);ylabel('phi_f(rad)');$**grid on; hold on**

subplot(613);plot$(t_1, q_F, '-b', 'linewidth', 1.5);ylabel('q_f(rad/s)');$**grid on; hold on**

subplot(614);plot$(t_1, p_F, '-b', 'linewidth', 1.5);ylabel('p_f(rad/s)');$**grid on; hold on**

subplot(615);plot$(t_1, u, '-b', 'linewidth', 1.5);ylabel('u(m/s)');$**grid on; hold on**

subplot(616);plot$(t_1, \nu, '-b', 'linewidth', 1.5);ylabel('v(m/s)');xlabel('Time(sec)');$

grid on; hold on

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