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Theme :

NON-COMMUTATIVE SURFACES FORMALISM AND APPLICATION TO 2D GRAVITY

By :

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Before the jury composed of:

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Abstract

In this project, the main goal is to be familiarized with the formalism of non-commutative geometry. This work develops non-commutative deformations of Riemannian geometry in the light of Whitney theorem. Two steps have been covered toward this goal:

1. Introducing the Moyal algebra $\mathcal{A}$, which is a non-commutative deformation of the algebra of smooth functions on a region of $\mathbb{R}^2$,and

2. Development of the non-commutative Riemannian geometry for two dimensional surfaces embedded in three dimensional space. Application is made for spherical surfaces to obtain the elements of the $2D$ non-commutative gravity, i.e. metric, left connection, Riemannian tensor, Ricci tensor and scalar curvature.

Key words : Non-commutative surfaces; Connection; Riemannian and Ricci tensors.

Résumé

Dans ce projet, l'objectif principal est de se familiariser avec le formalisme de la géométrie non commutative. Ce travail développe des déformations non commutatives de la géométrie Riemannienne à la lumière du théorème de Whitney. Deux étapes doivent être accomplit pour cet objectif :

1. Introduction de l'algèbre de Moyal $\mathcal{A}$, qui est une déformation non commutative de l'algèbre des fonctions differentielles sur une région de $\mathbb{R}^2$.

2. Suivi du développement de la géométrie Riemannienne non commutative pour des surfaces à deux dimensions dans un espace tridimentionel. Une Application est faite aux surfaces sphériques pour obtenir les éléments $2D$ de la gravitation non commutative, c'est-à-dire, métrique, connexion à gauche, tenseur de Riemann, tenseur de Ricci et courbure scalaire.

Mots clés : Surfaces non commutatives, Connexion, Tenseurs de Riemann et de Ricci.

الملخص

في هذا المشروع ، الهدف الرئيسي هو التعرف على شكليات الهندسة غير التبادلية. هذا العمل يطور تشوهات غير تبادلية للهندسة الريمانية في ضوء نظرية ويتني. تم تغطية خطوتين لتحقيق هذا الهدف

1 .تقديم **Moyal algebra A** وهو تشوه غير تبادلي لجبر الوظائف السلسة على منطقة **R x R**

2.تطوير الهندسة الريمانية غير التبادلية للبعدين ،اسطح مدمجة في الفضاء ثلاثي الابعاد. يتم اجراء التطبيقات على الاسطح الكروية للحصول على عناصر الجاذبية غير التبادلية ثنائية الابعاد.

أي: متري ، اتصال يسار ، موتر ريماني ، موتر ريتشي و انحناء عددي.

الكلمات الاساسية:اسطح غير قابلة للتبادل، اتصال، موتر ريمانيان و ريتشي.

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Dedication

To the symbol of love and giving, to the source of tenderness and kindness, the apple of my eye, my dear mother.

To my support in life, my strength and the light of my heart, my dear father.

To the candles and light of my life, my brothers and sisters.

To all my friends who have supported me in my academic journey

Finally to the person who encouraged me and supported me in my weakness and was by my side the whole time my fiancé “Thank you”

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Introduction

Quantum mechanics appears, in its first formulation proposed by Heisenberg, as a modification of classical Hamiltonian mechanics in which space coordinates are replaced by operators that do not commute with each other [1]. The origin of non-commutative geometry (NCG) goes back to quantum mechanics, which played an important role in the mid-twentieth century in the development of functor algebra theory, where in the transition from classical mechanics to quantum mechanics, the commutative algebra of functions over the phase space area change to the non-commutative operators algebra acting on Hilbert space. The main purpose of the non-commutative geometry is to generalize the duality between geometric space and algebra to the more general case where the algebra is no longer commutative [2–8]. NCG could be also an excellent solution to many problems of quantum gravity or non-commutative Riemannian geometry [9–15]. Applications in physics are very large, spanning from M-theory to quantum Hall effects and including non-commutative standard model and non-commutative quantum field theory and others.

In this brief overview, we are interested in knowing the formalism of non-commutative geometry within the framework of developing Riemannian geometry by following several steps. The most important of which is the non-commutative deformation of the algebra by introducing Moyal Algebra and the development of non-commutative Riemannian geometry for two dimensional surfaces embedded in $3D$-space, by applying to spherical surfaces to obtain the elements of gravity [16]. Our present work or manuscript contains three chapters:

The first chapter: introduces, on a background level, the basic concepts of Riemannian geometry. The first section, the properties of curves and surfaces are shown, in the neighborhood of some point where methods of calculus are the appropriate tools to be applied. While the second section focuses on differential manifolds, Riemannian manifolds, which are smooth surfaces with 2 dimensions ($2D$), are evoked.

In the second chapter, the formalism of non-commutative geometry is identified. Here the non-commutative deformations of Riemannian geometry are developed through the introduction of the Moyal algebra $\mathcal{A}$, which is a non-commutative deformation of the algebra of smooth functions on the R^2 region, followed by the development of the non-commutative geometry of the two-dimensional spherical space.

The last chapter is entirely devoted to the work of our master. We will use tools described in the first two chapters and apply them to spherical surfaces to obtain non-commutative geometric object, namely the metric, left connection, Riemann tensor, Ricci tensor and scalar curvature.

Chapter 1

Riemannian geometry

Riemannian geometry is the branch of differential geometry that studies Riemannian manifolds, smooth manifold with a Riemannian metric, i.e with an inner product on the tangent space at each point that varies smoothly from point to point. This gives, in particular, local notions of angle, length of curves, surface area and volume [9], which we will consider in below. Generalization to higher dimensions, though very important to physics application, is not considered in this chapter. The matter of subsection (1.1.2) and (1.1.3) in this chapter is taken from [18].

1.1 Description of curves and surfaces

There are two aspects to the differential geometry of curves and surfaces. Roughly speaking, classical differential geometry is the study of the local properties of curves and surfaces. By local properties we mean those properties that depend only on the behavior of the curve or surface near the point. A method that has proven adequate to study these properties is the calculus method. Therefore, the curves and surfaces considered in differential geometry are defined by functions that can be differentiated to a certain degree. On the other hand is the so-called global differential geometry. Here the influence of local properties on the behavior of the entire curve is examined on surface. The matter in this subsection (1.1.1) is taken from [17].

1.1.1 Parametrized curves

A parametrized curve is a differentiable map $\alpha : I \longrightarrow \mathbb{R}^3$ of an open interval $I =]a, b[$ of the real line $\mathbb{R}$ into $\mathbb{R}^3$.

Example 1:

The parametrized differentiable curve given by $\alpha{=}(a\cos t, a\sin t, bt)$ has its path in $\mathbb{R}^3$ as a helix of pitch $2\pi b$ on the cylinder $x^2 + y^2 = a^2$. The parameter t here measures the angle which the x axis makes with the line joining the origin O to the projection of the point $\alpha(t)$ over the (xy) plane (see figure 1.1).

Example 2 :

The map α: $\mathbb{R} \longrightarrow \mathbb{R}^2$ given by $\alpha(t){=}\ (t^3 - 4t, t^2 - 4)$, with $t \in \mathbb{R}$, is a parametrized differentiable curve in $\mathbb{R}^2$ (see figure 1.1). Notice that $\alpha(2){=}\alpha(-2){=}(0, 0)$; that is, the map α has self-intersection and is then not a one-to-one map.

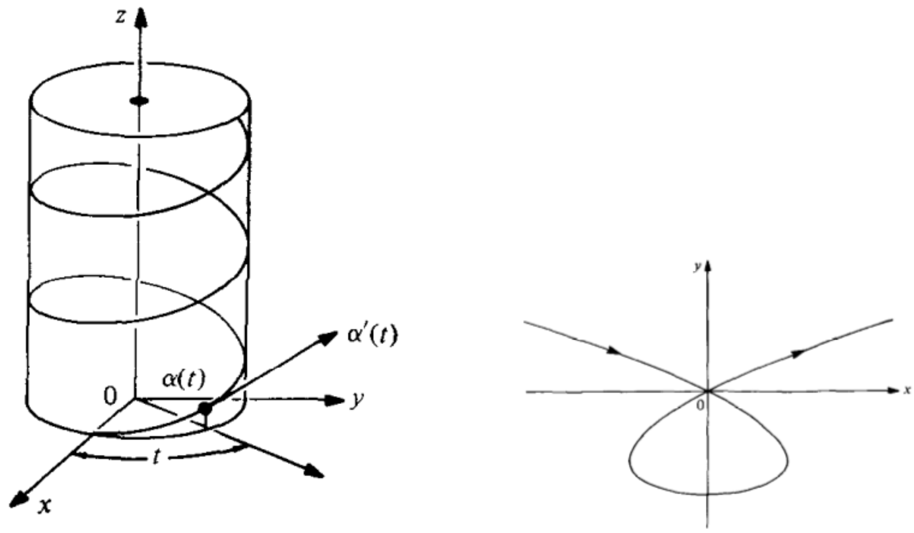

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Figure 1.1: (Left):Helix curve [17]. (Right) Curve of self-intersection [17]

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1.1.2 Quadratic form

We call a quadratic form at a point $(x_0^1, \ldots, x_0^n)$ a collection of numbers $g_{ij}, i, j = 1, \ldots, n$, such that $g_{ij} = g_{ji}$, referred to the coordinate system $(x^1, \ldots, x^n)$. If there is a change of coordinates $x = x(z)$, by which the coordinates $(x^1, \ldots, x^n)$ are transformed into $(z^1, \ldots, z^n)$ with $x^i(z_0^1, \ldots, z_0^n) = x_0^i, i = 1, \ldots, n$, this same quadratic form is defined in the new coordinates $(z^1, \ldots, z^n)$ by a collection of numbers $h_{kl}, k, l = 1, \ldots, n$, such that $h_{kl} = h_{lk}$, linked to the previous one by the formula

$$g_{ij} = \left(\frac{\partial x^k}{\partial z^j}\right)\Bigg|_{z^s=z_0^s} h_{kl} \left(\frac{\partial x^l}{\partial z^i}\right)\Bigg|_{z^s=z_0^s} \tag{1.1}$$

In matrix form, this equality is written $G = A^T HA$. If the quadratic form g_{ij} defined at the point P is transformed according to the formula (1.1) when a change of coordinate is carried out, one defines, on the tangent vector over P, a quadratic function (or bi-linear function) of two vectors ξ and η by imposing

$$\langle \xi, \xi \rangle = g_{ij}\xi^i\xi^j, \qquad \langle \xi, \eta \rangle = g_{ij}\xi^i\eta^j. \tag{1.2}$$

By virtue of the transformation law (1.1), the functions thus defined do not depend on the choice of the coordinate system but only on the point P and the vector ξ. We use quadratic form to define the length of a differentiable curve $x^i{=}x^i(t)$ in the flat space of dimension n equipped with coordinates $(x^1, \ldots, x^n)$ as follow

$$l = \int \sqrt{\langle v, v \rangle} dt = \int \sqrt{g_{ij}v^iv^j} dt, \tag{1.3}$$

where $v^i(t) = \frac{\partial x^i(t)}{\partial t}$. There is an alternative definition of quadratic form called Riemannian metric

$$\langle \xi, \eta \rangle \geqslant 0 \qquad \forall \xi, \eta \in T_p(s),$$

where $T_p(s)$ is the tangent space to the manifold at point p.

DEFINITION 1 : By Riemannian metric in a domain of the space $\mathbb{R}^n$, we mean a positive quadratic form which is defined on the tangent vectors at each point and which represents a differentiable function of the point. Taking up the definition of the quadratic form proposed in the previous paragraph, we will give a somewhat modified statement of the Riemannian metric

DEFINITION 2 : By Riemannian metric in a domain of space, equipped with any coordinates$(z^1, \ldots, z^n)$, we mean a collection of functions $g_{ij} = g_{ij}(z^1, \ldots, z^n)$, $i, j = 1, \ldots, n$, such that the matrix (g_{ij}) is positive definite. By introducing in the same domain a new system of coordinates $(y^1, \ldots, y^n)$, such as $z^i = z^i(y^1, \ldots, y^n), i = 1, \ldots, n$, the Riemannian metric is defined in new coordinates by a collection of functions $g'_{ij} = g'_{ji}(y^1, \ldots, y^n), ij = 1, \ldots, n$, such that

$$g'_{ij} = \frac{\partial z^k}{\partial y^i} g_{kl} \frac{\partial z^l}{\partial y^j}. \tag{1.4}$$

The fact that the matrix (g_{ij}) is positively defined implies $g_{ij}\xi^i\xi^j > 0$ for all non zero vectors ξ. Given the Riemannian metric, the length of a curve $z^i = z^i(t)$ is expressed by the formula

$$L = \int_a^b \sqrt{g_{ij}(z(t)) \frac{\partial z^i}{\partial t} \frac{\partial z^j}{\partial t}} \tag{1.5}$$

For two curves $z^i = f^i(t), z^i = h^i(t)$, which intersect for $t = t_0$, the angle formed by these curves at the intersection point is given by the quantity ϕ such that $(0 \leqslant \phi < \pi)$ and

$$\cos(\phi) = \frac{\langle \xi, \eta \rangle}{|\xi||\eta|} \tag{1.6}$$

where $\langle \xi, \eta \rangle = g_{ij}\xi^i\eta^j, |\xi| = \sqrt{\langle \xi, \xi \rangle}$, and ξ, η being the velocity vectors at the intersection point $(t = t_0)$.

Example :

Euclidean metric
a) In two dimensions space $n = 2$, we have, in Euclidean coordinates $x^1 = x, x^2 = y$,

$$g_{ij} = \delta_{ij} = \begin{cases} 1 & for \quad i = j \\ \\ 0 & for \quad i \neq j \end{cases} \tag{1.7}$$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.8}$$

To obtain this metric in the polar coordinates system (r, φ) we use

$$x^1 = r\cos\varphi, \qquad x^2 = r\sin\varphi$$

After sitting $r = z^1, \varphi = z^2$, and using the equation

$$g_{ij} = \sum_{k=1}^{n} \frac{\partial x^k}{\partial z^i} \frac{\partial x^k}{\partial z^j},$$

we obtain the different components

$$\begin{aligned} g_{11} &= \cos^2\varphi + \sin^2\varphi = 1 \\ g_{12} &= -r\cos\varphi\sin\varphi + r\sin\varphi\cos\varphi = 0 \\ g_{21} &= -\sin\varphi\cos\varphi + \cos\varphi\sin\varphi = 0 \\ g_{22} &= r^2(\sin^2\varphi + \cos^2\varphi) = r^2. \end{aligned} \tag{1.9}$$

Under matrix form the metric in polar coordinates reads

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{1.10}$$

1.1.3 Surfaces

A surface in three-dimensional space is the simplest object possessing what is called intrinsic geometry. There are three different ways to define a surface in three-dimensional space

1. The simplest way is to define it as a graph of a given function f:

$$z = f(x, y)$$

2. A more general method consists in writing the equation of the surface as constraint on a function F of the x, y and z variables :

$$F(x, y, z) = 0$$

3. A surface can finally be defined para-metrically, like a curve

$$\vec{r} = \vec{r}(u, v),$$

   or, in a more developed form,

$$x = X(u, v), \quad y = Y(u, v), \quad z = Z(u, v),$$

   where u, v are parameters that cover a domain defined on the plane (u, v).

By definition, we will say that the equation $F(x, y, z) = 0$ defines a non-singular surface at the point $P = (x_0, y_0, z_0)$, where $F(x_0, y_0, z_0) = 0$, if the gradient of the function F is non-zero at P:

$$\vec{\nabla} F|_{x_0,y_0,z_0} = \frac{\partial F}{\partial x}\vec{e}_1 + \frac{\partial F}{\partial y}\vec{e}_2 + \frac{\partial F}{\partial z}\vec{e}_3 \neq 0,$$

for $x = x_0, y = y_0, z = z_0$.
Theorem :
If a surface is defined parametrically and the point $P = (u_0, v_0)$ is non-singular, the surface can be defined, in the vicinity of this point, by the equation $F(x, y, z) = 0$ with $F(x_0, y_0, z_0) = 0$ and $(\vec{\nabla} F)_{x_0,y_0,z_0} \neq 0$. Staying in the neighborhood of a non-singular point $P = (x_0, y_0, z_0)$ of the surface, the three modes of (local) definition of the surfaces (by means of differentiable functions) are equivalent.

Examples :

1. ***Ellipsoid :*** This surface is described by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

   where a, b and c are non zero reals. This surface has no singular point and it is not possible to give global graphic representation (locally is possible). Also, the parametric definition is globally impossible (so as to have all non-singular points),

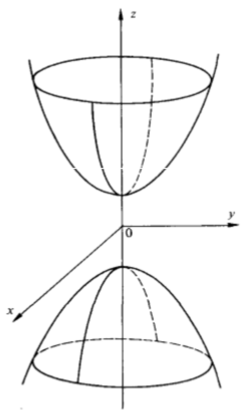

Figure 1.2: Non connected surfaces [17]

2. ***Hyperbolic with one sheet :*** This surface is defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

for which the graph representation is globally impossible. The parametric definition is possible globally by choosing as parameters $u = z, v = \phi$, where ϕ is the polar angle.

3. ***Hyperbolic with two sheets :***This surface is defined by

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

One of the sheets can be defined both in the graphic form or in the parametric form. All the points being non-singular (see figure 1.2).

1.1.4 Tangent plane

The tangent plane to a surface $\mathcal{S}$, defined by equation $f(x, y, z) = 0$, at a regular point (x_0, y_0, z_0) is an approximation of the surface in the neighborhood of that point by an affine plane. The surface and the tangent plane have the same normal vector. If we have this regular vector, then the equation for the tangent plane in $\mathbb{R}^3$ passing through the regular point $(x_0, y_0, z_0) \in \mathcal{S}$ is given by [19]

$$(x - x_0)\frac{\partial f}{\partial x} + (y - y_0)\frac{\partial f}{\partial y} + (z - z_0)\frac{\partial f}{\partial z} = 0. \tag{1.11}$$

Example : Let $\mathcal{S}$ be the surface of equation $x^2 + y^2 + z^2 - 3 = 0$. The tangent plane to $\mathcal{S}$ at the point$(1, -1, 1)$ admit for equation $x - y + z = 3$ [4] (see figure 1.3).

1.1.5 The metric of the sphere

To understand how the metric can differ from Euclidian case, let us consider geometry on a spherical surface. The sphere $S^2 \subset \mathbb{R}^3$ of radius R centered at the origin of the coordinates system has the equation [20]

$$x^2 + y^2 + z^2 = R^2. \tag{1.12}$$

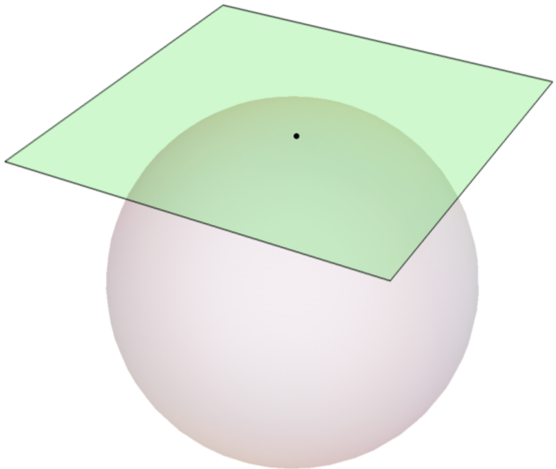

Figure 1.3: Tangent plane of the surface

If we work in cylindrical coordinates system (such that $x^2 + y^2 = r_\perp^2$), then the equation for the sphere can be rewritten as

$$r_\perp^2 + z^2 = R^2. \tag{1.13}$$

In spherical coordinates r, θ, ϕ, this sphere has the simple equation $r = R$, with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi[$. The parameters (θ, ϕ) can be used as local coordinates on the sphere, except in its north and south poles (in which we have $\theta = 0, \theta = \pi$; these are the singular points of the spherical coordinates system). We know that the $3D$ Euclidean metric $ds^2 = dx^2 + dy^2 + dz^2$ is written in spherical coordinates as follows [20]

$$ds^2 = dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{1.14}$$

We can induce the metric of our embedded sphere from the metric of 3D Euclidean space in cylindrical coordinates system. To do this, we use Eq.(1.13) to relate $dr_\perp$ and dz when we only move on the surface of the sphere. This relation is given by $zdz = -r_\perp dr_\perp$. Plugging this later into the metric of the $3D$ Euclidean space [20]

$$ds^2 = dr_\perp^2 + r_\perp^2 d\phi^2 + dz^2, \tag{1.15}$$

we obtain

$$\begin{aligned} ds^2 &= dr_\perp^2 + r_\perp^2 d\phi^2 + \frac{r_\perp^2}{z^2} dr_\perp^2 \\ &= \left(1 + \frac{r_\perp^2}{z^2}\right) dr_\perp^2 + r_\perp^2 d\phi^2 \\ &= \left(1 + \frac{r_\perp^2}{R^2 - r_\perp^2}\right) dr_\perp^2 + r_\perp^2 d\phi^2, \end{aligned} \tag{1.16}$$

where in the last line we used Eq.(1.13). At last, the metric takes the form

$$ds^2 = \frac{dr_\perp^2}{1 - \left(\frac{r_\perp^2}{R^2}\right)^2} + r_\perp^2 d\phi^2 \tag{1.17}$$

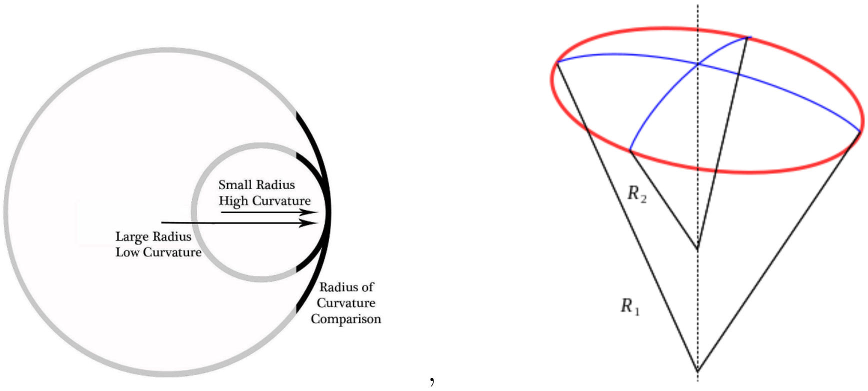

Figure 1.4: (Left) Comparison of Radius of Curvature showing small radius with high curvature vs large radius with low curvature [29]. (Right) Radius of curvature.

1.2 Differentiable Manifold

A differentiable manifold is a type of manifold that is "locally" similar to a vector space. Formally, a differentiable manifold is a topological manifold with a globally defined differential structure. Any topological manifold can carry a differential structure locally by using the homeomorphisms in its atlas and the standard differential structure on a vector space. Differentiable manifolds are very important in physics. Special kinds of differentiable manifolds form the basis for physical theories such as classical mechanics, general relativity, and Yang–Mills theory. It is possible to develop a calculus for differentiable manifolds. The study of calculus on differentiable manifolds is known as differential geometry [21].

1.2.1 Riemannian manifolds as metric spaces

A riemannian manifold M defined as a smooth manifold with some positive definite quadratic form $\langle \quad , \quad \rangle$ on the tangent bundle TM could be turned into a metric space. If $\gamma : [a, b] \in \mathbb{R} \longrightarrow M$ is a continuously differentiable curve in the riemannian manifold M, then the length L(γ) is defined by

$$L_{[a,b]}(\gamma) = \int_a^b \langle \gamma', \gamma' \rangle dt,$$

where $\gamma'(t)$ is an element of the tangent bundle TM at point $\gamma(t)$. With this definition of length, every connected riemannian manifold M becomes a metric space in a natural way: the distance $d(x, y)$ between the points such $x = \gamma(a)$ and $y = \gamma(b)$ of M, is defined as

$$d(x, y) = inf\{L_{[a,b]}(\gamma)\}$$

where γ is any continuously differentiable curve joining x and y [21].

1.2.2 Curvature

The curvature of a curve is, somehow, the rate at which that curve is turning. There are two precision needed for this definition. First, the curvature should be a geometric property of the curve and not be changed by the way (the speed) one moves along it. Thus curvature is defined to be the absolute value of the rate at which the tangent line is turning when one moves along the curve at a speed of one unit per second. Second, the curvature doesn't depend on the concavity sens of the curve. For instance if one looks at a circle, the top is concave down and the bottom is concave up, but clearly one wants the curvature of a circle to be positive all the way round. Negative curvature simply doesn't make sense for curves [22].

Radius of curvature

The radius R of curvature in differential geometry is the inverse of the curvature. Normally the formula of curvature is given by

$$R = \frac{1}{k},$$

here k is the curvature value. The radius of the circular arc is the best value which approximate the curve at that point. The radius of curvature tells us how curved a curve is. For surfaces, the radius of the curvature is the radius of the circle that fits best in a *normal section* [23] (see figure 1.4).

Types of Curvature

In general, there are two types of curvature namely extrinsic curvature and intrinsic curvature. Extrinsic curvature is defined for submanifold of a manifold that depends on its particular inserting. Intrinsic curvature is a curvature such as Gaussian curvature that is detectable to 2-D by the inside of a surface and not just outside [23].

Chapter 2

Formalism of non-commutative geometry

The matter of this chapter is taken from [24].

2.1 Introduction

In the two last decades there has been great progress in developing the theory of non-commutative geometry and exploring its applications in physics [25]. The mathematical foundation of non-commutative geometry began to appear in the 1980's by Alain Connes works. The roots of non-commutative geometry lie in quantum mechanics which was the impetus for an important development in algebraic theory. Quantum physics provides mathematical tools that make it possible to understand geometry in Plank's scale as it applies to a pair of conjugate variables x_i and p_j in quantum mechanics such

$$[x_i, p_j] = i\hbar\delta_{ij}.$$

By analogy with non-commutative space-time coordinates, x^μ are replaced by non-commuta tive coordinates satisfying

$$[x^\mu, x^\nu] = i\theta^{\mu\nu},$$

where $\theta^{\mu\nu}$ are real anti-symmetric coefficients. Algebra that is created by substituting classical fields with non-commutative fields, and the normal product with a non-commutative product is called Weyl algebra and plays an important role in quantum mechanics [26].

There is another way to give the same algebra by using only functions, not using operators. For the product of those functions, one could introduce an associative product $\star$ instead of operators product. The star product is different from the usual multiplication of functions, but is given as a deformation of the usual multiplication [27].

2.2 Moyal product, the metric and tangent bundles.

The star product is a particularly useful way to deal with non-commutative geometries, because one can continue to work with ordinary functions. It is sufficient to keep in mind that they obey a modified product rule in the algebra. With this, one can build non-commutative quantum theories of fields by replacing the normal products of the fields in all expressions by the star products. We can extend the isomorphism between vector spaces to an isomorphism between algebras by constructing a new product, noted $\star$ [1].

In mathematics, the Moyal product is an example of a phase-space star product [28]. It is an associative, non-commutative product. For the example of functions f and g

depending on two variables t_1 and t_2, the star product, or more precisely the Moyal product, is defined by

$$(f \star g)(t_1, t_2) = \lim_{t' \to t} \exp\left[\overline{h}\left(\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2'} - \frac{\partial}{\partial t_2}\frac{\partial}{\partial t_1'}\right)\right] f(t_1, t_2) g(t_1', t_2'), \tag{2.1}$$

where $\theta^{12} = \overline{h}$ is the deformation parameter, and the exponential is to be understood as a power series in the differential operator $\left(\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2'} - \frac{\partial}{\partial t_2}\frac{\partial}{\partial t_1'}\right)$.

The theory of non-commutative surfaces to be developed extends to more general star products over algebras of smooth functions. Noting ∂_i for derivative $\partial/\partial t_i$, one can easily check that for smooth functions f and g,

$$\partial_i(f \star g) = (\partial_i f) \star g + f \star (\partial_i g). \tag{2.2}$$

Let consider $A^3 = A \oplus A \oplus A$ so that there is a natural two-sided structure of the type A on $A^3 \otimes A^3$, defined for each of the $a, b \in A$ and $X \otimes Y \in A^3 \otimes A^3$ by $a \star (X \otimes Y) \star b = a \star X \otimes Y \star b$. We define the map,

$$A^3 \otimes A^3 \longrightarrow A, \quad (a, b, c) \otimes (f, g, h) \longrightarrow a \star f + b \star g + c \star h, \tag{2.3}$$

which is denoted by big dot "$\bullet$" . It is a map of the two-sided A-modules i.e. for $X, Y \in A^3$ and $a, b \in A$.

$$(a \star X) \bullet (Y \star b) = a \star (X \bullet Y) \star b.$$

We will refer to this map as the point product. Let $X = (X^1, X^2, X^3)$ be an element of A^3. We set

$$\partial_i X = (\partial_i X^1, \partial_i X^2, \partial_i X^3)$$

and define the following 2×2 matrix

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad g_{ij} = \partial_i X \bullet \partial_j X \tag{2.4}$$

The element $X \in A^3$ is called non-commutative embedding in A^3, and define a non-commutative surface if the matrix $g^0 = g|_{\overline{h}=0}$ is reversible. In this case, we call g a non-commutative surface scale. For the non-commutative surface of X with the metric g there is a unique right inverse matrix 2×2, i.e.,

$$g_{ij} \star g^{jk} = \delta_i^k.$$

The associativity of multiplication of matrices over any associative algebra ensures that the left and right inverses of g are equal. Given a non-commutative surface X, we take

$$E_i = \partial_i X \tag{2.5}$$

and define the left and right tangent bundles TX and $\tilde{T}X$ respectively, by

$$TX = \{a \star E_1 + b \star E_1 | a, b \in A\}.$$

$$\tilde{T}X = \{E_1 \star a + E_1 \star b | a, b \in A\}.$$

The metric g induces the isomorphism $g : TX \otimes \tilde{T}X \to A$. Back to the inverse metric, we can define the basis with upper index

$$E^i = g^{ij} \star E_j.$$

$$\tilde{E}^i = E_j \star g^{ji}.$$

2.3 Riemannian connection (Levi-Civita connection)

We introduce connections of the tangent bundles by following the standard procedure in the theory of surfaces. Let the operator $\nabla_i : TX \longrightarrow TX$, where $i = 1, 2$, be the Levi-Civita affine connection. It is also called the covariant derivative and is defined by requiring that $\nabla_i Z$ is equal to the left tangent component of $\partial_i Z$ for all $Z \in TX$. Similarly we define

$$\tilde{\nabla}_i : \tilde{T}X \longrightarrow \tilde{T}X$$

by requesting that $\tilde{\nabla}_i \tilde{Z}$ is equal to the right tangent component of $\partial_i \tilde{Z}$ for all $\tilde{Z} \in \tilde{T}X$. For all $Z \in TX, W \in \tilde{T}X$, and $f \in A$, both connections verify

$$\nabla_i(f \star Z) = \partial_i f \star Z + f \star \nabla_i Z, \qquad \tilde{\nabla}_i(W \star f) = W \star \partial_i f + \tilde{\nabla}_i W \star f. \tag{2.6}$$

To describe the connections more accurately, we note the existence of Γ^k_{ij} and $\tilde{\Gamma}^k_{ij}$ in A such that

$$\nabla_i E_j = \Gamma^k_{ij} \star E_k \tag{2.7}$$

$$\tilde{\nabla}_i E_j = E_k \star \tilde{\Gamma}^k_{ij} \tag{2.8}$$

Since the metric is invertible, the elements Γ^k_{ij} and $\tilde{\Gamma}^k_{ij}$ are uniquely identified from equations (2.7) and (2.8):

$$\Gamma^k_{ij} = \partial_i E_j \bullet E^k \tag{2.9}$$

$$\tilde{\Gamma}^k_{ij} = E^k \bullet \partial_i E_j \tag{2.10}$$

Obviously, Γ^k_{ij} and $\tilde{\Gamma}^k_{ij}$ have symmetric indices i and j. The following expressions are very useful for later purposes:

$$\Gamma_{ijk} = \partial_i E_j \bullet E_k$$

$$\tilde{\Gamma}_{ijk} = E_k \bullet \partial_i E_j$$

In contrast to the commutative case, Γ^k_{ij} and $\tilde{\Gamma}^k_{ij}$ do not agree, in general. In fact, we have

$$\Gamma^k_{ij} = {}_c\Gamma_{ijl} \star g^{lk} + Y_{ijl} \star g^{lk}, \tag{2.11}$$

$$\tilde{\Gamma}^k_{ij} = g^{kl} \star {}_c\Gamma_{ijl} - g^{kl} \star Y_{ijl}, \tag{2.12}$$

where

$${}_c\Gamma_{ijl} = \frac{1}{2}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ji}), \tag{2.13}$$

$$Y_{ijl} = \frac{1}{2}(\partial_i E_j \bullet E_l - E_l \bullet \partial_i E_j). \tag{2.14}$$

We call Y_{ijl} the non-commutative torsion of a non-commutative surface. The left and right connections consist of two parts. The ${}_c\Gamma_{ijl}$ part depends only on the metric, and the torsion part contains more information about the non-commutative surface embedded in A^3. The non-commutative torsion depends explicitly on the embedding. In the classical limit with $\bar{h} = 0$, Y^k_{ij} vanishes and both Γ^k_{ij} and $\tilde{\Gamma}^k_{ij}$ reduce to the standard Levi–Civita connection.

The operator ∇_i of Levi-Civita satisfies the following property:

$$\partial_i g(Z, \tilde{Z}) = g(\nabla_i Z, \tilde{Z}) + g(Z, \tilde{\nabla}\tilde{Z}) \qquad \forall Z \in TX, \quad \tilde{Z} \in \tilde{T}X \tag{2.15}$$

This means that the connections are metric compatible. For the special case where $Z = E_j$, $\tilde{Z} = E_k$, the equation (2.15) is written

$$\partial_i g(E_j, E_k) = \partial_i(E_j \bullet E_k) = \partial_i E_j \bullet E_k + E_j \bullet \partial_i E_k = g(\nabla_i E_j, E_k) + g(E_j, \tilde{\nabla}_i E_k).$$

This is equivalent to

$$\partial_i g_{jk} - \Gamma_{ijk} - \tilde{\Gamma}_{ijk} = 0. \tag{2.16}$$

In the commutative case equation (2.16) is not uniquely sufficient to define the Γ_{ijk} and $\tilde{\Gamma}_{ijk}$ connections.

2.4 The second fundamental form and curvatures

The basic theory of Riemannian geometry states that, in any Riemannian manifold there exist an unique affine connection, that is torsion-free and metric-compatible. Defining the commutators

$$[\nabla_i, \nabla_j] = \nabla_i\nabla_j - \nabla_j\nabla_i$$

and

$$[\tilde{\nabla}_i, \tilde{\nabla}_j] = \tilde{\nabla}_i\tilde{\nabla}_j - \tilde{\nabla}_j\tilde{\nabla}_i.$$

The following computations show us that, for $f \in A$

$$[\nabla_i, \nabla_j](f \star Z) = f \star [\nabla_i, \nabla_j]Z, \qquad Z \in TX,$$

$$[\tilde{\nabla}_i, \tilde{\nabla}_j](W \star f) = W[\tilde{\nabla}_i, \tilde{\nabla}_j] \star f, \qquad W \in \tilde{T}X,$$

The right-hand side of the first equation belongs to TX, while the right-hand side of the second equation belongs to $\tilde{T}X$. According to this remark, the following maps have a sense

$$[\nabla_i, \nabla_j] : TX \longrightarrow TX, \qquad [\tilde{\nabla}_i, \tilde{\nabla}_j] : \tilde{T}X \longrightarrow \tilde{T}X.$$

Since TX is generated by E_1 and E_2 as left A-module (and so on for $\tilde{T}X$), the previous maps act on the basis E_k, and the result is written

$$[\nabla_i, \nabla_j]E_k = R^l_{kij} \star E_l, \qquad [\tilde{\nabla}_i, \tilde{\nabla}_j]E_k = E_l \star \tilde{R}^l_{kij}, \tag{2.17}$$

where R^l_{kij} and $\tilde{R}^l_{kij}$ belong to A. We refer to R^l_{kij} and $\tilde{R}^l_{kij}$ respectively, as Riemannian curvature of the left and right tangent bundels of the non-commutative surface X. The second fundamental form in classical surface theory plays an important role. It is a quadratic form on the tangent plane of a smooth surface in three-dimensional Euclidean space, and measures the change in unit normal direction from one point to another on the surface. A Similar idea exists for non-commutative surfaces. We define the left and right second basic form for non-commutative surfaces X by

$$h_{ij} = \partial_i E_j - \Gamma^k_{ij} \star E_k,$$

$$\tilde{h}_{ij} = \partial_i E_j - E_k \star \tilde{\Gamma}^k_{ij}.$$

According to relation (2.7) we have

$$h_{ij} \bullet E_k = 0, \quad E_k \bullet \tilde{h}_{ij} = 0 \tag{2.18}$$

The Riemann curvatures are uniquely determined by the relations (2.17). In fact, we have

$$R^l_{kij} = g([\nabla_i, \nabla_j]E_k, E^l), \qquad \tilde{R}^l_{kij} = g(\tilde{E}^l, [\tilde{\nabla}_i, \tilde{\nabla}_j]E_k), \tag{2.19}$$

And after some simple calculations one got the following result

$$\begin{aligned} R^l_{kij} &= \partial_j \Gamma^l_{ik} - \Gamma^p_{ik} \star \Gamma^l_{jp} + \partial_i \Gamma^l_{jk} + \Gamma^p_{jk} \star \Gamma^l_{ip}, \\ \tilde{R}^l_{kij} &= \partial_j \tilde{\Gamma}^l_{ik} - \tilde{\Gamma}^l_{jp} \star \tilde{\Gamma}^p_{ik} + \partial_i \tilde{\Gamma}^l_{jk} + \tilde{\Gamma}^l_{ip} \star \tilde{\Gamma}^p_{jk}. \end{aligned} \tag{2.20}$$

It was proven in [24] that

$$R_{lkij} = R^p_{kij} \star g_{pl} = -g_{kp} \star R^p_{lij} = \tilde{R}_{lkij},$$

and

$$R_{lkij} = h_{jk} \bullet \tilde{h}_{il} - h_{ik} \bullet \tilde{h}_{jl}, \tag{2.21}$$

which represents the generalized Gauss equation. The Ricci curvature and the scalar curvature of the non-commutative surface are determined by the sectional curvatures of the Riemann manifold by choosing the Riemannian metric. We denote them respectively by:

$$R_{ij} = R^p_{ipj}, \quad R = g^{ji} \star R_{ij}. \tag{2.22}$$

Chapter 3

Application to non commutative spherical surfaces

In this section we consider a concrete example of non-commutative surfaces [24] : the non-commutative sphere with specific local map, called spherical coordinates. The north and south poles are not considered to eliminate singularities. We will use short notation of trigonometric functions :

$$\cos(\theta) = C_\theta, \qquad \cos(\phi) = C_\phi,$$
$$\sin(\theta) = S_\theta, \qquad \sin(\phi) = S_\phi.$$

Also, we will use the star product (2.1) with tensor product symbol $\otimes$ instead of the limit process definition.

3.1 Non-commutative sphere

The sphere of radius R embedded in three-dimensional space can be described by the angular coordinates. Let $U =]0, \pi[\times[0, 2\pi[$, and $t_1 = \theta$ and $t_2 = \phi$, respectively. U is mapped in three dimensions by the application $X(\theta, \phi)$ [13], given by

$$X(\theta, \phi) = \left(\frac{S_\theta C_\phi}{\cosh \overline{h}}, \frac{S_\theta S_\phi}{\cosh \overline{h}}, \frac{\sqrt{\cosh 2\overline{h}} C_\theta}{\cosh \overline{h}} \right), \tag{3.1}$$

We can prove that $X \bullet X$ satisfies the following relationship :

$$X \bullet X = X^1 \star X^1 + X^2 \star X^2 + X^3 \star X^3 = 1. \tag{3.2}$$

where X^1, X^2, and X^3 are the components of (3.1). Omitting the factor $\frac{1}{\cosh \overline{h}}$, let compute the first term $X^1 \star X^1$, which gives

$$\begin{aligned}
S_\theta C_\phi \star S_\theta C_\phi &= \exp\left(\overline{h} \left(\frac{\partial}{\partial\theta} \otimes \frac{\partial}{\partial\phi} - \frac{\partial}{\partial\phi} \otimes \frac{\partial}{\partial\theta} \right) \right) (S_\theta C_\phi \otimes S_\theta C_\phi) \\
&= \sum_{n=0}^{\infty} \frac{\overline{h}^n}{n!} \left(\frac{\partial}{\partial\theta} \otimes \frac{\partial}{\partial\phi} - \frac{\partial}{\partial\phi} \otimes \frac{\partial}{\partial\theta} \right)^n (S_\theta C_\phi \otimes S_\theta C_\phi) \\
&= \sum_{n=0}^{\infty} \frac{\overline{h}^n}{n!} \sum_{p=0}^{n} (-1)^{n-p} \binom{n}{p} \left(\frac{\partial^p}{\partial\theta^p} S_\theta \frac{\partial^{n-p}}{\partial\phi^{n-p}} C_\phi \otimes \frac{\partial^p}{\partial\phi^p} C_\phi \frac{\partial^{n-p}}{\partial\theta^{n-p}} S_\theta \right) \\
&= \sum_{n=0}^{\infty} \frac{\overline{h}^n}{n!} \sum_{p=0}^{n} (-1)^{n-p} \binom{n}{p} \left(S_{\theta+p\frac{\pi}{2}} C_{\phi+(n-p)\frac{\pi}{2}} \otimes C_{\phi+p\frac{\pi}{2}} S_{\theta+(n-p)\frac{\pi}{2}} \right),
\end{aligned} \tag{3.3}$$

where we have used series expansion of the exponential function in second line, and the Newton binomial formula in the third line. The last line comes from the trivial results

$$\frac{\partial^q}{\partial x^q}\cos x = \cos\left(x + q\frac{\pi}{2}\right), \qquad \frac{\partial^q}{\partial x^q}\sin x = \sin\left(x + q\frac{\pi}{2}\right).$$

Applying the following decomposition of trigonometric functions in (3.3), i.e.

1. $S_{\theta+p\frac{\pi}{2}} = S_\theta \cos p\frac{\pi}{2} + C_\theta \sin p\frac{\pi}{2}$,
2. $C_{\phi+(n-p)\frac{\pi}{2}} = C_\phi \cos((n-p)\frac{\pi}{2}) - S_\phi \sin((n-p)\frac{\pi}{2})$,
3. $C_{\phi+p\frac{\pi}{2}} = C_\phi \cos p\frac{\pi}{2} - S_\phi \sin p\frac{\pi}{2}$,
4. $S_{\theta+(n-p)\frac{\pi}{2}} = S_\theta \cos((n-p)\frac{\pi}{2}) + C_\theta \sin((n-p)\frac{\pi}{2})$,

we obtain the expression

$$\begin{aligned} S_\theta C_\phi \star S_\theta C_\phi &= \sum_{n=0}^{\infty}\frac{(-\hbar)^n}{n!}\sum_{p=0}^{n}(-1)^p\begin{pmatrix} n \\ p \end{pmatrix}\Big\{\cos^2(p\frac{\pi}{2})\cos^2((n-p)\frac{\pi}{2})T_1 \\ &\quad -\cos^2(p\frac{\pi}{2}\sin^2((n-p)\frac{\pi}{2})T_2 - \sin^2(p\frac{\pi}{2}\cos^2((n-p)\frac{\pi}{2})T_3 \\ &\quad +\sin^2(p\frac{\pi}{2}\sin^2((n-p)\frac{\pi}{2})T_4\Big\}, \end{aligned} \tag{3.4}$$

where

$$T_1 = (S_\theta C_\phi \otimes S_\theta C_\phi), \qquad T_2 = (S_\theta S_\phi \otimes C_\theta C_\phi), \tag{3.5}$$

$$T_3 = (C_\theta C_\phi \otimes S_\theta S_\phi), \qquad T_4 = (C_\theta S_\phi \otimes C_\theta S_\phi). \tag{3.6}$$

Using

$$\cos^2(p\frac{\pi}{2}) = \frac{1}{2}(1+\cos(p\pi)) = \frac{1}{2}(1+(-1)^p),$$

$$\sin^2(p\frac{\pi}{2}) = \frac{1}{2}(1-\cos(p\pi)) = \frac{1}{2}(1-(-1)^p),$$

$$\cos^2((n-p)\frac{\pi}{2}) = \frac{1}{2}(1+(-1)^{n-p})$$

$$\sin^2((n-p)\frac{\pi}{2}) = \frac{1}{2}(1-(-1)^{n-p}),$$

equation (3.4) reads

$$\begin{aligned} &S_\theta C_\phi \star S_\theta C_\phi = \sum_{n=0}^{\infty}\frac{(-\hbar)^n}{n!}\frac{1}{4}\sum_{p=0}^{n}(-1)^p\begin{pmatrix} n \\ p \end{pmatrix}\{(1+(-1)^n+(-1)^p+(-1)^{n-p})T_1 \\ &-(1-(-1)^n+(-1)^p-(-1)^{n-p})T_2 - (1-(-1)^n-(-1)^p+(-1)^{n-p})T_3 \\ &+(1+(-1)^n-(-1)^p-(-1)^{n-p})T_4\} \end{aligned}$$

From the binomial expression: $(a+b)^n = \sum_{p=0}^{n}\begin{pmatrix} n \\ p \end{pmatrix} a^p b^{n-p}$, we have $\sum_{p=0}^{n}\begin{pmatrix} n \\ p \end{pmatrix} = 2^n$ if $a = b = 1$ and $\sum_{p=0}^{n}(-1)^p\begin{pmatrix} n \\ p \end{pmatrix} = \delta_{n,0}$ if $-a = +b = 1$. Using these later relations we obtain

$$S_\theta C_\phi \star S_\theta C_\phi = \sum_{n=0}^{\infty} \frac{(-\overline{h})^n}{n!}\frac{1}{4}\{(\delta_{n,0} + (-1)^n\delta_{n,0} + 2^n + (-1)^n 2^n)T_1$$
$$-(\delta_{n,0} - (-1)^n\delta_{n,0} + 2^n - (-1)^n 2^n)T_2 - (\delta_{n,0} - (-1)^n\delta_{n,0} - 2^n + (-1)^n 2^n)T_3$$
$$+(\delta_{n,0} + (-1)^n\delta_{n,0} - 2^n - (-1)^n 2^n)T_4\}.$$

With the help of $\sum_{n=0}^{\infty} \frac{(\pm 2\overline{h})^n}{n!} = e^{\pm 2\overline{h}}$ and $\sum_{n=0}^{\infty} \frac{(\pm\overline{h})^n}{n!}\delta_{n,0} = 1$, and the hypertrigonometric functions, the last expression gives

$$\begin{aligned} S_\theta C_\phi \star S_\theta C_\phi &= \frac{1}{2}(1+\cosh(2\overline{h}))S_\theta^2 C_{\phi^2} + \frac{1}{2}(1-\cosh(2\overline{h}))C_\theta^2 S_{\phi^2} \\ &= \cosh^2(\overline{h})S_\theta^2 C_\phi^2 - \sinh^2(\overline{h})C_\theta^2 S_\phi^2, \end{aligned}$$

After restoring the factor $\frac{1}{\cosh^2(\overline{h})}$ the first term of (3.2) becomes

$$X^1 \star X^1 = \frac{1}{\cosh^2\overline{h}}(\cosh^2(\overline{h})S_\theta^2 C_\phi^2 - \sinh^2(\overline{h})C_\theta^2 S_\phi^2). \tag{3.7}$$

We do similar computation for $X^2 \star X^2$ and $X^3 \star X^3$, we get:

$$X^2 \star X^2 = \frac{1}{\cosh^2\overline{h}}(\cosh^2(\overline{h})S_\theta^2 S_\phi^2 - \sinh^2(\overline{h})C_\theta^2 C_\phi^2), \tag{3.8}$$

$$X^3 \star X^3 = \frac{\cosh(2\overline{h})}{\cosh^2\overline{h}}\cos\theta^2. \tag{3.9}$$

Adding up the three terms, we get

$$\begin{aligned} X^1 \star X^1 + X^2 \star X^2 + X^3 \star X^3 &= \frac{1}{\cosh^2\overline{h}}\{\cosh^2(\overline{h})S_\theta^2 C_\phi^2 - \sinh^2(\overline{h})C_\theta^2 S_\phi^2\} + \\ &\frac{1}{\cosh^2\overline{h}}\{\cosh^2(\overline{h})S_\theta^2 S_\phi^2 - \sinh^2(\overline{h})C_\theta^2 C_\phi^2\} + \frac{\cosh(2\overline{h})}{\cosh^2\overline{h}}\cos^2\theta \\ &= \frac{1}{\cosh^2\overline{h}}\{\cosh^2(\overline{h})S_\theta^2 - \sinh^2(\overline{h})C_\theta^2 + \cosh(2\overline{h})C_\theta^2\}. \end{aligned}$$

Using $\cosh(2\overline{h}) = \cosh^2(\overline{h}) - \sinh^2(\overline{h})$, our expression simplifies to give the final result

$$X^1 \star X^1 + X^2 \star X^2 + X^3 \star X^3 = S_\theta^2 + C_\theta^2 = 1.$$

3.2 The non-commutaive metric components of the sphere

Thus we may regard the non-commutative surface defined by X as an analog of the sphere S^2. We shall denote it by $S^2_{(\overline{h})}$ and refer to it as a non-commutative sphere. We have from equation (2.5)

$$\begin{aligned} E_1 &= \left(\frac{\cos\theta\cos\phi}{\cosh\overline{h}}, \frac{\cos\theta\sin\phi}{\cosh\overline{h}}, -\frac{\sqrt{\cosh 2\overline{h}}\sin\theta}{\cosh\overline{h}}\right), \\ E_2 &= \left(\frac{-\sin\theta\sin\phi}{\cosh\overline{h}}, \frac{\sin\theta\cos\phi}{\cosh\overline{h}}, 0\right). \end{aligned}$$

The four components $g_{ij} = E_i \bullet E_j$ of the metric g on $S^2(\overline{h})$ can now be calculated in similar way to previous computation of $X \bullet X$.

◇ For $g_{11} = E_1 \bullet E_1$, we have

$$g_{11} = \frac{1}{\cosh^2 \overline{h}} \left(C_\theta C_\phi \star C_\theta C_\phi + C_\theta S_\phi \star C_\theta S_\phi + \cosh 2\overline{h} S_\theta^2 \right).$$

The two first terms inside the braces can be obtained from $X^1 \star X^1 + X^2 \star X^2$ by the angle shift

$$\theta \longrightarrow \theta + \frac{\pi}{2},$$

and the substitution

$$C_\theta \longrightarrow S_\theta, \qquad S_\theta \longrightarrow -C_\theta,$$

The further simplifications :

$$\begin{aligned}
g_{11} &= \frac{1}{\cosh^2 \overline{h}} \left(\cosh^2(\overline{h}) C_\theta^2 C_\phi^2 - \sinh^2(\overline{h}) S_\theta^2 S_\phi^2 + \cosh^2(\overline{h}) C_\theta^2 S_\phi^2 - \sinh^2(\overline{h}) S_\theta^2 C_\phi^2 + \cosh 2\overline{h} S_\theta^2 \right) \\
&= \frac{1}{\cosh^2 \overline{h}} \left(\cosh^2(\overline{h}) C_\theta^2 - \sinh^2(\overline{h}) S_\theta^2 + \cosh 2\overline{h} S_\theta^2 \right) \\
&= \frac{1}{\cosh^2 \overline{h}} \left(\cosh^2(\overline{h}) C_\theta^2 - \sinh^2(\overline{h}) S_\theta^2 + (2\sinh^2 \overline{h} + 1) S_\theta^2 \right) \\
&= \frac{1}{\cosh^2 \overline{h}} \left(\cosh^2(\overline{h}) C_\theta^2 + \sinh^2(\overline{h}) + S_\theta^2 \right) \\
&= \frac{1}{\cosh^2 \overline{h}} \left(\cosh^2(\overline{h}) C_\theta^2 + (\cosh^2(\overline{h}) - 1) S_\theta^2 + S_\theta^2 \right) \\
&= \frac{1}{\cosh^2 \overline{h}} \left(\cosh^2(\overline{h})(C_\theta^2 + S_\theta^2) - S_\theta^2 + S_\theta^2 \right),
\end{aligned}$$

lead to

$$g_{11} = 1.$$

◇ for $g_{22} = E_2 \bullet E_2$, we have:

$$\begin{aligned}
g_{22} &= \frac{1}{\cosh^2 \overline{h}} \left(S_\theta S_\phi \star S_\theta S_\phi + S_\theta C_\phi \star S_\theta C_\phi \right) \\
&= \frac{1}{\cosh^2 \overline{h}} \left(\cosh^2(\overline{h}) S_\theta^2 S_\phi^2 - \sinh^2(\overline{h}) C_\theta^2 C_\phi^2 + \cosh^2(\overline{h}) S_\theta^2 C_\phi^2 - \sinh^2(\overline{h}) C_\theta^2 S_\phi^2 \right) \\
&= \frac{1}{\cosh^2 \overline{h}} \left(\cosh^2(\overline{h}) S_\theta^2 - \sinh^2(\overline{h}) C_\theta^2 \right),
\end{aligned}$$

which lead to

$$g_{22} = \sin^2 \theta - \tanh^2(\overline{h}) \cos^2 \theta.$$

◇ For $g_{12} = E_1 \bullet E_2$, we need to compute the two terms of

$$\frac{1}{\cosh^2 \overline{h}} \left(-C_\theta C_\phi \star S_\theta S_\theta + C_\theta S_\phi \star S_\theta C_\phi \right).$$

The first one is

$$C_\theta C_\phi \star S_\theta S_\phi = \widehat{\Sigma}_{n,p} \left(\frac{\partial^p}{\partial \theta^p} C_\theta \frac{\partial^{n-p}}{\partial \phi^{n-p}} C_\phi \otimes \frac{\partial^p}{\partial \phi^p} S_\phi \frac{\partial^{n_p}}{\partial \theta^{n-p}} S_\theta \right),$$

which is equal to

$$\widehat{\Sigma}_{n,p} \left(C_\theta C_{(p\frac{\pi}{2})} - S_\theta S_{(p\frac{\pi}{2})} \right) \left(C_\phi C_{(n-p)\frac{\pi}{2}} - S_\phi S_{(n-p)\frac{\pi}{2}} \right) \otimes$$

$$\left(S_\phi C_{(p\frac{\pi}{2})} + C_\phi S_{(p\frac{\pi}{2})} \right) \left(S_\theta C_{(n-p)\frac{\pi}{2}} + C_\theta S_{(n-p)\frac{\pi}{2}} \right),$$

where $\widehat{\Sigma}_{n,p}$ is the summation operator with adequate factor. Making the substitutions (3.5) and (3.6), we obtain

$$\begin{aligned}
C_\theta C_\phi \star S_\theta S_\phi &= \frac{1}{4}\{(1+1+e^{-2\overline{h}})T_3 - (+e^{2\overline{h}} - e^{2\overline{h}})T_4 - (-e^{-2\overline{h}} + e^{2\overline{h}})T_1 + \\
&\quad (1+1-e^{-2\overline{h}} - e^{-2\overline{h}})T_2)\} \\
&= \frac{1}{2}\Big((1+\cosh(2\overline{h}))T_3 + (\sinh(2\overline{h}))T_4 - (\sinh(2\overline{h}))T_1 + (1-\cosh(2\overline{h}))T_2)\Big) \\
&= \frac{1}{2}(2T_3 + \sinh(2\overline{h}))(T_1 - T_2) \\
&= -C_\theta C_\phi S_\theta S_\phi - \frac{1}{2}\sinh(2\overline{h})(C_\theta^2 S_\phi^2 - S_\theta^2 C_\phi^2).
\end{aligned}$$

The second term is obtained following the same steps. Then

$$\begin{aligned}
g_{12} &= \frac{1}{\cosh^2(\overline{h})}\Big(-C_\theta C_\phi S_\theta S_\phi - \frac{1}{2}\sinh(2\overline{h})(C_\theta^2 S_\phi^2 + S_\theta^2 C_\phi^2)\Big) + \\
&\quad \Big(C_\theta S_\phi S_\theta C_\phi - \frac{1}{2}\sinh(2\overline{h})(C_\theta^2 C_\phi^2 - S_\theta^2 S_\phi^2)\Big) \\
&= \frac{\sinh\overline{h}}{\cosh\overline{h}}(\sin^2\theta - \cos^2\theta). \qquad (3.10)
\end{aligned}$$

Finally we observe that $g_{21} = -g_{12}$ and from all these results the metric is expressed or:

$$g_{ij} = \begin{pmatrix} 1 & \frac{\sinh\overline{h}}{\cosh\overline{h}}(\sin^2\theta - \cos^2\theta) \\ & \\ -\frac{\sinh\overline{h}}{\cosh\overline{h}}(\sin^2\theta - \cos^2\theta) & \sin^2\theta - \frac{\sinh^2\overline{h}}{\cosh^2\overline{h}}\cos^2\theta \end{pmatrix}. \qquad (3.11)$$

◇ For $\overline{h} \longrightarrow\longrightarrow 0$, then:

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ & \\ 0 & \sin^2\theta \end{pmatrix}. \qquad (3.12)$$

The components of this metric commute with each-other as they depend only on θ. Thus it makes sense to consider the usual determinant D of g. We have

$$D = \sin^2\theta + \tanh^2\overline{h}(\cos^2 2\theta - \cos^2\theta)$$

The components of the inverse metric are given by

$$g^{11} = \frac{\sin^2\theta - \tanh^2\overline{h}\cos^2\theta}{\sin^2\theta + \tanh^2\overline{h}(\cos^2 2\theta - \cos^2\theta)},$$

$$g^{22} = \frac{1}{\sin^2\theta + \tanh^2\overline{h}(\cos^2 2\theta - \cos^2\theta)},$$

$$g^{12} = -g^{21} = \frac{\tanh\overline{h}(\sin^2\theta - \cos^2\theta)}{\sin^2\theta + \tanh^2\overline{h}(\cos^2 2\theta - \cos^2\theta)}.$$

◇ For $\overline{h} \longrightarrow 0$,then:

$$g^{11} = 1, \qquad g^{22} = \frac{1}{\sin^2\theta}, \qquad g^{12} = -g^{21} = 0.$$

3.3 The Christofell symbols of the first and second kind for non-commutative sphere

Now we determine the connection symbols of the first kind, $\Gamma_{ijk} = \partial_i E_j \bullet E_K$, of the non-commutative sphere.

$$\begin{aligned}\Gamma_{111} &= \partial_1 E_1 \bullet E_1 \\ &= \left(-\frac{\sin\theta\cos\phi}{\cosh\overline{h}}, -\frac{\sin\theta\sin\phi}{\cosh\overline{h}}, -\frac{\sqrt{\cosh 2\overline{h}}\sin\theta}{\cosh\overline{h}}\right) \bullet \left(\frac{\cos\theta\cos\phi}{\cosh\overline{h}}, \frac{\cos\theta\sin\phi}{\cosh\overline{h}}, -\frac{\sqrt{\cosh 2\overline{h}}\cos\theta}{\cosh\overline{h}}\right) \\ &= \frac{1}{\cosh^2\overline{h}}\left(-S_\theta C_\phi \star C_\theta C_\phi - S_\theta S_\phi \star C_\theta C_\phi + \cosh 2\overline{h} S_\theta C_\theta\right). \qquad (3.13)\end{aligned}$$

The first term is

$$\begin{aligned}&(S_\theta C_\phi \star C_\theta C_\phi) = \widehat{\Sigma}_{n,p}\left(\frac{\partial^p}{\partial\theta^p} S_\theta \frac{\partial^{n-p}}{\partial\phi^{n-p}} C_\phi \otimes \frac{\partial^p}{\partial\phi^p} C_\phi \frac{\partial^{n_p}}{\partial\theta^{n-p}} C_\theta\right) = \\ &\widehat{\Sigma}_{n,p}\left(S_{(\theta+p\frac{\pi}{2})} C_{\phi+(n-)p)\frac{\pi}{2}} \otimes C_{(\theta+p\frac{\pi}{2}} C_{\phi+(n-p)\frac{\pi}{2}}\right) = \\ &\widehat{\Sigma}_{n,p}\left(S_\theta C_{(p\frac{\pi}{2})} + C_\theta S_{(p\frac{\pi}{2})}\right)\left(C_\phi C_{(n-p)\frac{\pi}{2}} - S_\phi S_{(n-p)\frac{\pi}{2}}\right) \otimes \\ &\left(C_\phi C_{(p\frac{\pi}{2})} - S_\phi S_{(p\frac{\pi}{2})}\right)\left(C_\theta C_{(n-p)\frac{\pi}{2}} - S_\theta S_{(n-p)\frac{\pi}{2}}\right)\end{aligned}$$

Using the short notation

$$\begin{aligned}(S_\theta C_\phi \otimes C_\phi C_\theta) = T_1', &\qquad (S_\theta S_\phi \otimes C_\phi S_\theta) = T_2' \\ (C_\theta C_\phi \otimes S_\phi C_\theta) = T_3', &\qquad (C_\theta S_\phi \otimes S_\phi S_\theta) = T_4',\end{aligned}$$

we obtain

$$\begin{aligned}(S_\theta C_\phi \star C_\theta C_\phi) &= \sum_{n=0}^{\infty}\frac{(-h)^n}{4n!}\sum_{p=0}^{n}(-1)^p\begin{pmatrix} n \\ p\end{pmatrix}\{(1+(-1)^n+(-1)^p+(-1)^{n-p})T_1' + \\ &(1-(-1)^n+(-1)^p-(-1)^{n-p})T_2' - (1-(-1)^n-(-1)^p+(-1)^{n-p})T_3' - \\ &(1+(-1)^n-(-1)^p-(-1)^{n-p})T_4'\} \\ &= \frac{1}{4}\{(1+1+e^{-2h}+e^{+2h})T_1' + (1-1+e^{-2h}+e^{+2h})T_2' - \\ &(1-1-e^{-2h}+e^{+2h})T_3' - (1+1-e^{-2h}-e^{+2h})T_4'\}.\end{aligned}$$

Removing the tensor product $\otimes$ and after some simplifications we obtain

$$-S_\theta C_\phi \star C_\theta C_\phi = -\left[\left(\cosh^2\overline{h} C_\phi^2 + \sinh^2\overline{h} S_\phi^2\right) - \frac{1}{2}\sinh(2\overline{h}) C_\phi S_\phi\right].$$

and, with similar computations, the expression of the second term :

$$-S_\theta S_\phi \star C_\theta C_\phi = -\left[(\cosh^2\overline{h} C_\phi^2 + \sinh^2\overline{h} S_\phi^2) + \frac{1}{2}\sinh(2\overline{h}) C_\phi S_\phi\right],$$

leading then to the final expression of (3.13):

$$\Gamma_{111} = \frac{1}{\cosh^2\overline{h}}\left[-\frac{1}{2}\cosh 2\overline{h} - \frac{1}{2}\cosh 2\overline{h} + \cosh 2\overline{h}\right] S_\theta C_\theta = 0.$$

Doing the same calculation for the remained symbols of the connection of non-commutative sphere, we get:

$$\begin{aligned}
\Gamma_{112} &= \sin 2\theta \tanh \overline{h}, \\
\Gamma_{121} &= -\sin 2\theta \tanh \overline{h}, \\
\Gamma_{122} &= \frac{1}{2} \sin 2\theta (1 + \tanh^2 \overline{h}), \\
\Gamma_{211} &= \sin 2\theta \tanh \overline{h}, \\
\Gamma_{212} &= \frac{1}{2} \sin 2\theta (1 + \tanh^2 \overline{h}), \\
\Gamma_{221} &= -\frac{1}{2} \sin 2\theta (1 + \tanh^2 \overline{h}), \\
\Gamma_{222} &= \sin 2\theta \tanh \overline{h}.
\end{aligned}$$

For the connection symbols of second kind, we use :

$$\Gamma_{ij}^{k} =_c \Gamma_{ijl} * g^{lk} + Y_{ijl} * g^{lk}$$

where ${}_c\Gamma_{ijl}$ and Y_{ijl} are, respectively, the classical connection and torsion given by (2.13) and (2.14). Setting $\alpha = \tanh \overline{h}$ and $\partial_1 = \partial_\theta, \partial_2 = \partial_\phi$, and using the metric (3.11), simple calculations will give the non vanishing components of ${}_c\Gamma_{ijl}$:

$${}_c\Gamma_{122} =_c \Gamma_{212} = -{}_c\Gamma_{221} = (\alpha^2 + 1) \sin\theta \cos\theta. \tag{3.14}$$

With the remark

$$E_l \bullet \partial_i E_j = \tilde{\Gamma}_{ijl} = \Gamma_{ijl}(\overline{h} \to -\overline{h}),$$

we can determine the non vanishing components of the torsion :

$$Y_{112} = -Y_{121} = Y_{211} = Y_{222} = 2\alpha \sin\theta \cos\theta \tag{3.15}$$

At this stage, the calculation for connection doesn't need the star product. The results are straightforward

$$\begin{aligned}
\Gamma_{11}^{1} &= \frac{\alpha^2(\cos 3\theta + \cos\theta)\csc\theta}{2\alpha^2 \cos 2\theta + \alpha^2 - 1}, \\
\Gamma_{11}^{2} &= -\frac{2\alpha \cot\theta}{2\alpha^2 \cos 2\theta + \alpha^2 - 1}, \\
\Gamma_{12}^{1} &= \frac{\alpha(1 - \alpha^2)\cot\theta}{2\alpha^2 \cos 2\theta + \alpha^2 - 1}, \\
\Gamma_{12}^{2} &= \frac{(2\alpha^2 \cos 2\theta - \alpha^2 - 1)\cot\theta}{2\alpha^2 \cos 2\theta + \alpha^2 - 1}, \\
\Gamma_{21}^{1} &= \frac{\alpha(1 - \alpha^2)\cot\theta}{2\alpha^2 \cos 2\theta + \alpha^2 - 1}, \\
\Gamma_{21}^{2} &= \frac{(2\alpha^2 \cos 2\theta - \alpha^2 - 1)\cot\theta}{\alpha^2 \cos 2\theta + \alpha^2 - 1}, \\
\Gamma_{22}^{1} &= \frac{(1 - \alpha^2)(1 + \alpha^2 + (-1 + \alpha^2)\cos 2\theta)\cot\theta}{2(2\alpha^2 \cos 2\theta + \alpha^2 - 1)}, \\
\Gamma_{11}^{2} &= \frac{\alpha((1 + \alpha^2)\cos 2\theta - 2)\cot\theta}{2\alpha^2 \cos 2\theta + \alpha^2 - 1}.
\end{aligned}$$

3.4 Riemannian curvature components and Ricci tensors R_{ij} and R

We calculate the Riemannian curvature using the definition

$$R^l_{kij} = \partial_j \Gamma^l_{ik} - \Gamma^p_{ik} \star \Gamma^l_{jp} + \partial_i \Gamma^l_{jk} + \Gamma^p_{jk} \star \Gamma^l_{ip},$$

where R^l_{kij} is the Riemannian curvature of the left tangent bundle of non-commutative surface. The explicit non vanishing components of R^l_{kij} are

$$\begin{aligned}
R^1_{112} &= \frac{\alpha(1-\alpha^4)(2+\cos 2\theta)}{(2\alpha^2\cos 2\theta + \alpha^2 - 1)^2}, \\
R^2_{112} &= -\frac{3\alpha^4 + 4\alpha^2 + 1}{(2\alpha^2\cos 2\theta + \alpha^2 - 1)^2}, \\
R^1_{121} &= \frac{\alpha(1-\alpha^4)(2+\cos 2\theta)}{(2\alpha^2\cos 2\theta + \alpha^2 - 1)^2}, \\
R^2_{121} &= \frac{1 + 4\alpha^2 + 3\alpha^4}{(2\alpha^2\cos 2\theta + \alpha^2 - 1)^2}, \\
R^1_{212} &= \frac{(1-\alpha^4)(1+3\alpha^2 + (-1+3\alpha^2)\cos 2\theta}{2(2\alpha^2\cos 2\theta + \alpha^2 - 1)^2}, \\
R^2_{212} &= -\frac{\alpha(1+\alpha^2)(2(1+\alpha^2) + (-1+\alpha^2)\cos 2\theta}{(2\alpha^2\cos 2\theta + \alpha^2 - 1)^2}, \\
R^1_{221} &= \frac{(\alpha^4-1)(1+3\alpha^2 + (-1+3\alpha^2)\cos 2\theta}{2(2\alpha^2\cos 2\theta + \alpha^2 - 1)^2}, \\
R^2_{221} &= \frac{\alpha(1+\alpha^2)(2(1+\alpha^2) + (-1+\alpha^2)\cos 2\theta}{(2\alpha^2\cos 2\theta + \alpha^2 - 1)^2}.
\end{aligned}$$

We can also compute the Ricci tensor $R_{ij} = R^p_{ipj}$:

$$\begin{aligned}
R_{11} &= \frac{1 + 4\alpha^2 + 3\alpha^4}{(2\alpha^2\cos 2\theta + \alpha^2 - 1)^2}, \\
R_{12} &= \frac{\alpha(1-\alpha^4)(2+\cos 2\theta)}{(2\alpha^2\cos 2\theta + \alpha^2 - 1)^2}, \\
R_{21} &= \frac{\alpha(\alpha^2+1)(2(1+\alpha^2) + (-1+\alpha^2)\cos 2\theta)}{(2\alpha^2\cos 2\theta + \alpha^2 - 1)^2}, \\
R_{22} &= \frac{(1-\alpha^4)(1+3\alpha^2 + (-1+3\alpha^2)\cos 2\theta)}{2(2\alpha^2\cos 2\theta + \alpha^2 - 1)^2},
\end{aligned}$$

and finally we deduce the scalar curvature

$$R = g^{ji} R_{ij} = \frac{2(\alpha^4-1)(3\alpha^2 + 2\alpha^2\cos 2\theta + 1)}{(2\alpha^2\cos 2\theta + \alpha^2 - 1)^3}.$$

This results is different from the commutative case ($\bar{h} = 0$) of scalar curvature where $R_0 = 2$. In the extreme case $\alpha = \pm 1$, the scalar curvature vanishes and the quantum sphere becomes flat everywhere on the map of the sphere.

Conclusion

In this master's thesis, we conducted a study on the formalism of non-commutative geometry and explored its applications in physics, by developing Riemann geometry for non-commutative surfaces, which is considered a first step towards constructing non-commutative static gravity. Our work begins with the construction of non-commutative Riemann geometry for two-dimensional surfaces embedded in three-dimensional space, and working on the associative algebra $\mathcal{A}$, which is a distortion of smooth functions in a region of R^2. On A^3 we define a dot product similar to the usual three-space Euclidean scalar product. An embedding of X from the non-commutative surface is defined as an element of A^3. The standard partial derivatives of X furnish a basis over A of the tangent bundles of the non-commutative surface.

We also applied it on the surface of the sphere, where it is considered the basic example of surfaces, and we obtained non-commutative curved geometric elements for two dimensions, such as metric g_{ij}, left connection Γ_{ijk} and right connection $\tilde{\Gamma}_{ijk}$, Riemannian tensor R_{ijkl}, Ricci tensor R_{ij} and scalar curvature R.

The results of this study indicate that to develop the distortions of Riemannian geometry, we have to distort the ordinary algebra into star algebra, and we have to apply it on spherical surfaces to get the non-commutative gravity elements. Nowadays, this subject experiences further developments in different senses. For exemple we cite Riemannian super-geometry of noncommutative super surfaces [16].

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