

**NUMERICAL
COMPUTATION
of
INTERNAL
and
EXTERNAL FLOWS**

Volume 1

*Fundamentals of Numerical
Discretization*

C. HIRSCH

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NUMERICAL COMPUTATION of INTERNAL and EXTERNAL FLOWS

Volume 1: Fundamentals of Numerical Discretization

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This is the first of two volumes which together describe comprehensively the theory and practice of the numerical computation of internal and external flows. In this volume, the author explains the use of basic computational methods to solve problems in fluid dynamics, comparing these methods so that the reader can see which would be the most appropriate to use for a particular problem.

The book is divided into four parts. In the first part, mathematical models are introduced. In the second part, the various numerical methods are described, while in the third and fourth parts the workings of these methods are investigated in some detail.

Volume 2 will be concerned with the applications of numerical methods to flow problems, and together the two volumes will provide an excellent reference for practitioners and researchers working in computational fluid mechanics and dynamics.

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Nomenclature

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