

Texts in
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13

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An Introduction to Partial Differential Equations



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Partial differential equations are fundamental to the modeling of natural phenomena; they arise in every field of science. Consequently, the desire to understand the solutions of these equations has always had a prominent place in the efforts of mathematicians; it has inspired such diverse fields as complex function theory, functional analysis, and algebraic topology.

Unfortunately, in the standard graduate curriculum, the subject of partial differential equations is seldom taught with the same thoroughness as algebra or integration theory. The present book is aimed at rectifying this situation. It is based on a four-semester course taught at Virginia Polytechnic Institute and State University. The goal of this course was to provide the background necessary to initiate work on a PhD thesis in partial differential equations. The level of the book is aimed at beginning graduate students. Prerequisites include a truly advanced calculus course and basic complex variables, but no knowledge is required of Lebesgue integration theory or functional analysis.

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