

# NUMERICAL RECIPES in Fortran 77

Second Edition

The Art of Scientific Computing

William H. Press

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